Chin. Ann. Math.
32B(1), 2011, 15–26
DOI: 10.1007/s11401-010-0627-y

Chinese Annals of Mathematics, Series B

On a Ginzburg-Landau Type Energy with Discontinuous Constraint for High Values of Applied Field

Hassen AYDI[1]

Abstract In the presence of applied magnetic fields H such that $|\ln\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$, the author evaluates the minimal Ginzburg-Landau energy with discontinuous constraint. Its expression is analogous to the work of Sandier and Serfaty.

Keywords Ginzburg-Landau functional, Mixed phase, Discontinuous constraint
2000 MR Subject Classification 35J20, 35J20, 35J25, 35B40

1 Introduction and Main Results

The energy of an inhomogeneous superconducting sample is given by the functional (see [2, 8])

$$\mathcal{G}_{\varepsilon,H}(\psi,A)=\int_\Omega\Big(|(\nabla-\mathrm{i}A)\psi|^2+\frac{1}{2\varepsilon^2}(p(x)-|\psi|^2)^2+|\mathrm{curl}\,A-H|^2\Big)\,\mathrm{d}x. \tag{1.1}$$

Ω an open, smooth and simply connected subset of $\mathbb{R}^2$. We take S_1 an open smooth set such that $\overline{S}_1\subset\Omega$, $S_2=\Omega\setminus\overline{S}_1$. In this paper, the function p is a step function defined as

$$p(x)=\begin{cases}1, & \text{if } x\in S_1,\\ a, & \text{if } x\in S_2,\end{cases} \tag{1.2}$$

where $a\in\mathbb{R}_+\setminus\{1\}$ is a given constant. Then, if (ψ,A) is a minimizer of (1.1), it holds that

$$\mathcal{G}_{\varepsilon,H}(\psi,H)=\mathcal{G}_{\varepsilon,0}(u_\varepsilon,0)+\mathcal{F}_{\varepsilon,H}\Big(\frac{\psi}{u_\varepsilon},A\Big),$$

and the configuration $(\frac{\psi}{u_\varepsilon},A)$ is a minimizer of the functional $\mathcal{F}_{\varepsilon,H}$ introduced below,

$$\mathcal{F}_{\varepsilon,H}(\varphi,A)=\int_\Omega\Big(u_\varepsilon^2|(\nabla-\mathrm{i}A)\varphi|^2+\frac{u_\varepsilon^4}{2\varepsilon^2}(1-|\varphi|^2)^2+|\mathrm{curl}\,A-H|^2\Big)\mathrm{d}y, \tag{1.3}$$

where u_ε is the minimizer over $H^1(\Omega,\mathbb{R})$ of

$$J(u)=\int_\Omega\Big(|\nabla u|^2+\frac{1}{2\varepsilon^2}(p(y)-|u|^2)^2\Big)\mathrm{d}y. \tag{1.4}$$

Manuscript received November 19, 2009. Published online December 28, 2010.
[1]Université de Monastir, Institut Supérieur d'Informatique de Mahdia, Km 4, Réjiche, 5121 Mahdia, Tunisie. E-mail: hassen.aydi@isima.rnu.tn

The vortex nucleation for minimizers of $\mathcal{F}_{\varepsilon,H}$ for applied magnetic fields comparable to the first critical field was done firstly by Kachmar (for more details see ([4, 5])), and afterwards by Aydi-Kachmar [1]. In this work, we let H be such that $|\log\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$ as $\varepsilon \to 0$ and our goal is to evaluate

$$\min_{H^1\times H^1}\left(\mathcal{F}_{\varepsilon,H}(\varphi, A)\right).$$

First, we state the following result (see [1]).

Theorem 1.1 (see [1]) *Given $\lambda > 0$, assume that*

$$\lim_{\varepsilon\to 0}\frac{H}{|\ln\varepsilon|} = \lambda.$$

Then if $(\varphi_\varepsilon, A_\varepsilon)$ is a minimizer of (1.3)*, then, denoting by*

$$h_\varepsilon = \operatorname{curl} A_\varepsilon, \quad \mu_\varepsilon = h_\varepsilon + \operatorname{curl}(\mathrm{i}\varphi_\varepsilon, (\nabla - \mathrm{i}A_\varepsilon)\varphi_\varepsilon)$$

the "induced magnetic field" and "vorticity measure" respectively, the following convergences hold,

$$\frac{\mu_\varepsilon}{H} \to \mu_*, \quad \text{in } \left(C^{0,\gamma}(\Omega)\right)^* \text{ for all } \gamma\in(0,1), \tag{1.5}$$

$$\frac{h_\varepsilon}{H} \to h_{\mu_*}, \quad \text{weakly in } H^1_1(\Omega) \text{ and strongly in } W^{1,p}(\Omega), \quad \forall\, p < 2. \tag{1.6}$$

Again

$$\frac{\mathcal{F}_{\varepsilon,H}(\varphi_\varepsilon, A_\varepsilon)}{H^2} \to E_\lambda(\mu_*)$$

in the sense of Γ-convergence. Here $E_\lambda(\mu_*)$ is by definition

$$E_\lambda(\mu_*) = \frac{1}{\lambda}\int_\Omega p(x)|\mu_*|\,\mathrm{d}x + \int_\Omega\left(\frac{1}{p(x)}|\nabla h_{\mu_*}|^2 + |h_{\mu_*} - 1|^2\right)\mathrm{d}x \tag{1.7}$$

and $\mu_* = -\operatorname{div}(\frac{\nabla h_*}{p}) + h_*$ is the unique minimizer of E_λ.

In [9], Sandier-Serfaty obtained that, for the classic Ginzburg-Landau energy denoted by G given by

$$G(\psi, A) = \int_\Omega\left(|(\nabla - \mathrm{i}A)\psi|^2 + \frac{1}{2\varepsilon^2}(1 - |\psi|^2)^2 + |\operatorname{curl} A - H|^2\right)\mathrm{d}x, \tag{1.8}$$

if $|\log\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$, we have

$$G(\psi_\varepsilon, A_\varepsilon) = \min_{H^1\times H^1} G(\psi, A) \simeq H|\Omega|\log\frac{1}{\varepsilon\sqrt{H}}(1 + o(1)), \tag{1.9}$$

as $\varepsilon \to 0$. Our motivation now is to evaluate the analogous minimal energy $\mathcal{F}_{\varepsilon,H}(\varphi_\varepsilon, A_\varepsilon)$. Our main result is the following theorem (in the same spirit as (1.9)).

Theorem 1.2 *Assume, as $\varepsilon \to 0$, that $|\ln\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$. Then, letting $(\varphi_\varepsilon, A_\varepsilon)$ minimize* (1.3)*, we have*

$$\mathcal{F}_{\varepsilon,H}(\varphi_\varepsilon, A_\varepsilon) \sim H\log\frac{1}{\varepsilon\sqrt{H}}(1 + o(1))\int_\Omega p(x)\mathrm{d}x. \tag{1.10}$$

A consequence of this result is the following corollary.

Corollary 1.1 *With* $|\ln\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$*, we have*

$$\lim_{\varepsilon\to 0}\frac{\mathcal{F}_{\varepsilon,H}(\varphi_\varepsilon, A_\varepsilon)}{H^2} = 0. \tag{1.11}$$

Then $h_{\mu_*} = 1$*, and so* $\mu_* = \mathrm{d}x$.

Proof It is clear with the above assumption on the applied field H, that $H\ln\frac{1}{\varepsilon\sqrt{H}} \ll H^2$, so taking it in (1.10) leads to (1.11). We know again that

$$\int_\Omega\Big(\frac{|\nabla h|^2}{u_\varepsilon^2} + |h-H|^2\Big)\mathrm{d}x \le \mathcal{F}_{\varepsilon,H}(\varphi_\varepsilon, A_\varepsilon) = o(H^2).$$

We use the uniform boundedness of u_ε, $\min(1,\sqrt{a}\,) < u_\varepsilon < \max(1,\sqrt{a}\,)$ in Ω for a small ε (this inequality is stated in Theorem 2.1 below), it is evident that $\frac{h}{H}$ tends strongly to $h_* = 1$ in H^1, so that $\mu_* = \mathrm{d}x$.

Remark 1.1 Remark that (1.10) is analogous to what done by Sandier-Serfaty given by (1.9). In the case $\lambda = +\infty$, i.e, for large H, Corollary 1.1 says that $\mu_* = 1$ which means that there is a uniform density of vortices in all Ω independently of a. This is in contrast with [1] where for a wide range of applied fields ($H = \lambda|\ln\varepsilon|(1+o(1))$) such that λ is chosen convenably and when a is sufficiently small, vortices exist and are pinned in S_2.

Sketch of the Proof of Theorem 1.2 The proof of Theorem 1.2 is obtained by getting first an upper bound on the minimal energy of $\mathcal{F}_{\varepsilon,H}$ (see Proposition 3.1, proved in Section 3), and then a lower bound (see Corollary 4.1, proved in Section 4).

The upper bound is done by construction of a test configuration which goes with the same idea of [10]. On the other hand, for such large applied fields, the problem of minimizing $\mathcal{F}_{\varepsilon,H}$ reduces to that of minimizing it on any subdomain, in other words, the minimization problem becomes local. Thus, we may perform blow-ups which yield the right lower bound.

Remark 1.2 (1) The letters $C, \widetilde{C}, M$, etc. denote positive constants independent of ε.

(2) For $n \in \mathbb{N}$ and $X \subset \mathbb{R}^n$, $|X|$ denotes the Lebesgue measure of X. $B(x,r)$ denotes the open ball in $\mathbb{R}^n$ of radius r and center x.

(3) $\mathcal{F}_{\varepsilon,H}(\varphi, A, U)$ means that the energy density of (φ, A) is integrated only on $U \subset \Omega$.

(4) Again, we define

$$G_a(\psi, A, U) = \int_U\Big(a\,|(\nabla - \mathrm{i}A)\psi|^2 + \frac{a^2}{2\varepsilon^2}(1-|\psi|^2)^2 + |\mathrm{curl}\,A - H|^2\Big)\,\mathrm{d}x. \tag{1.12}$$

(5) For two positive functions $a(\varepsilon)$ and $b(\varepsilon)$, we write $a(\varepsilon) \ll b(\varepsilon)$ as $\varepsilon \to 0$ to mean that $\lim\limits_{\varepsilon\to 0}\frac{a(\varepsilon)}{b(\varepsilon)} = 0$.

2 Preliminary Analysis of Minimizers

2.1 The case without applied magnetic field

This section is devoted to an analysis for minimizers of (1.1) when the applied magnetic field $H = 0$. We follow closely similar results obtained in [6].

We keep the notation introduced in Section 1. Upon taking $A = 0$ and $H = 0$ in (1.1), one is led to introduce the functional

$$\mathcal{G}_\varepsilon(u) := \int_\Omega \Big(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(p(x) - u^2)^2 \Big)\, \mathrm{d}x, \tag{2.1}$$

defined for functions in $H^1(\Omega;\mathbb{R})$.

We introduce

$$C_0(\varepsilon) = \inf_{u \in H^1(\Omega;\mathbb{R})} \mathcal{G}_\varepsilon(u). \tag{2.2}$$

The next theorem is an analogue of [6, Theorem 1.1].

Theorem 2.1 *Given $a \in \mathbb{R}_+ \setminus \{1\}$, there exists ε_0 such that for all $\varepsilon \in]0, \varepsilon_0[$, the functional* (2.1) *admits in $H^1(\Omega;\mathbb{R})$ a minimizer $u_\varepsilon \in C^2(\overline{S_1}) \cup C^2(\overline{S_2})$ such that*

$$\min(1, \sqrt{a}\,) < u_\varepsilon < \max(1, \sqrt{a}\,), \quad \text{in } \overline{\Omega}.$$

If $H = 0$, minimizers of (1.1) *are gauge equivalent to the state $(u_\varepsilon, 0)$.*

We state also some estimates, taken from [6, Proposition 5.1], that describe the decay of u_ε away from the boundary of S_1.

Lemma 2.1 *Let $k \in \mathbb{N}$. There exist positive constants ε_0, δ and C such that, for all $\varepsilon \in]0, \varepsilon_0]$,*

$$\Big\| (1 - u_\varepsilon) \exp\Big(\frac{\delta \mathrm{dist}(x, \partial S_1)}{\varepsilon} \Big) \Big\|_{H^k(S_1)} + \Big\| (\sqrt{a} - u_\varepsilon) \exp\Big(\frac{\delta \mathrm{dist}(x, \partial S_1)|}{\varepsilon} \Big) \Big\|_{H^k(S_2)} \leq \frac{C}{\varepsilon^k}. \tag{2.3}$$

Finally, we mention without proof that the energy $C_0(\varepsilon)$ (cf. (2.2)) has the order of ε^{-1}, and we refer to the methods in [6, Section 6] which permit to obtain the leading order asymptotic expansion

$$C_0(\varepsilon) = \frac{c_1(a)}{\varepsilon} + c_2(a) + o(1), \quad \varepsilon \to 0,$$

where $c_1(a)$ and $c_2(a)$ are positive explicit constants.

2.2 The case with magnetic field

This section is devoted to a preliminary analysis of the minimizers of (1.1) when $H \neq 0$. The main point that we shall show is how to extract the singular term $C_0(\varepsilon)$ (see (2.2)) from the energy of a minimizer.

Notice that the existence of minimizers is standard starting from a minimizing sequence (see e.g., [3]). A standard choice of gauge permits one to assume that the magnetic potential satisfies

$$\operatorname{div} A = 0 \quad \text{in } \Omega, \quad \nu \cdot A = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where ν is the outward unit normal vector of $\partial\Omega$.

With this choice of gauge, one is able to prove (since the boundaries of Ω and S_1 are smooth) that a minimizer (ψ, A) is in $C^1(\overline{\Omega};\mathbb{C}) \times C^1(\overline{\Omega};\mathbb{R}^2)$. One has also the following regularity (see [6, Appendix A]),

$$\psi \in C^2(\overline{S}_1;\mathbb{C}) \cup C^2(\overline{S}_2;\mathbb{C}), \quad A \in C^2(\overline{S}_1;\mathbb{R}^2) \cup C^2(\overline{S}_2;\mathbb{R}^2).$$

The next lemma is inspired from the work of Lassoued-Mironescu [7].

Lemma 2.2 *Let* (ψ, A) *be a minimizer of* (1.1). *Then* $0 \le |\psi| \le u_\varepsilon$ *in* Ω, *where* u_ε *is the positive minimizer of* (2.1). *Moreover, putting* $\varphi = \frac{\psi}{u_\varepsilon}$, *then the energy functional* (1.1) *splits in the form*

$$\mathcal{G}_{\varepsilon,H}(\psi, A) = C_0(\varepsilon) + \mathcal{F}_{\varepsilon,H}(\varphi, A), \tag{2.5}$$

where $C_0(\varepsilon)$ *has been introduced in* (2.2) *and the functional* $\mathcal{F}_{\varepsilon,H}$ *is defined in* (1.3) *by*

$$\mathcal{F}_{\varepsilon,H}(\varphi, A) = \int_\Omega \Big(u_\varepsilon^2|(\nabla - \mathrm{i}A)\varphi|^2 + \frac{1}{2\varepsilon^2}u_\varepsilon^4(1 - |\varphi|^2)^2 + |\mathrm{curl}\, A - H|^2\Big)\mathrm{d}x.$$

3 Upper Bound of the Energy

3.1 Main result

The objective of this section is to establish the following upper bound.

Proposition 3.1 *Assume that* $|\ln\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$. *Then, let* $(\varphi_\varepsilon, A_\varepsilon)$ *minimize* $\mathcal{F}_{\varepsilon,H}$. *For any small* ε,

$$\mathcal{F}_{\varepsilon,H}(\varphi_\varepsilon, A_\varepsilon, \Omega) \le H\Big(\ln\frac{1}{\varepsilon\sqrt{H}} + C\Big)\int_\Omega p(y)\,\mathrm{d}y.$$

With this assumption on the applied field H, the following is evident.

Corollary 3.1 *If* $|\ln\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$, *then when* $\varepsilon \to 0$,

$$\min_{H^1\times H^1} \mathcal{F}_{\varepsilon,H}(\varphi, A, \Omega) \le H\ln\frac{1}{\varepsilon\sqrt{H}}(1 + o(1))\int_\Omega p(y)\,\mathrm{d}y.$$

3.2 Proof of Proposition 3.1

The proof of Proposition 3.1 relies on a construction of a test configuration. Let us take $|\ln\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$ and let

$$\lambda = \sqrt{\frac{H}{2\pi}}.$$

Step 1 Let $L_\varepsilon = \lambda\mathbb{Z} \times \lambda\mathbb{Z}$ and h be the solution in $\mathbb{R}^2$ of

$$-\Delta h + h = 2\pi\sum_{a\in L_\varepsilon}\delta_a.$$

It is thus periodic with respect L_ε. Then, if we choose the origin carefully and take K_ε to be the unit cell of L_ε defined as

$$K_\varepsilon = \Big(-\frac{1}{2\lambda}, \frac{1}{2\lambda}\Big) \times \Big(-\frac{1}{2\lambda}, \frac{1}{2\lambda}\Big),$$

then h is also a solution of $-\Delta h + h = 2\pi\delta_0$ in K_ε and $\partial_\nu h = 0$ on ∂K_ε. Again we define an induced magnetic potential A by taking simply

$$\mathrm{curl}\, A = h.$$

We now turn to define an order parameter φ which we take in the form

$$\varphi = \rho\, \mathrm{e}^{\mathrm{i}\phi}, \tag{3.1}$$

where ρ is defined on Ω by

$$\rho(x) = \begin{cases} 0, & \text{if } |x-a| \le \varepsilon \ \text{ for some } a \in L_\varepsilon, \\ 1, & \text{if } \varepsilon < |x-a| < 2\varepsilon \ \text{ for some } a \in L_\varepsilon, \\ \dfrac{|x-a|}{\varepsilon} - 1, & \text{otherwise.} \end{cases} \tag{3.2}$$

The phase ϕ is defined (modulo 2π) by the relation

$$\nabla\phi - A = -\frac{1}{u_\varepsilon^2}\nabla^\perp h, \quad \text{in } \mathbb{R}^2 \setminus L_\varepsilon. \tag{3.3}$$

Let $g_{\varepsilon,H}$ be the energy density given as

$$g_{\varepsilon,H}(y) = \Big(|(\nabla - \mathrm{i}A)\varphi|^2 + \frac{1}{2\varepsilon^2}(1-|\varphi|^2)^2 + |\mathrm{curl}\, A - H|^2\Big)(y).$$

Proceeding as in [11, Chapter 8], we may define for each $x \in K_\varepsilon$ a translated lattice L_ε^x and a corresponding test configuration (φ^x, A^x) with energy density $g_{\varepsilon,H}(y-x)$. We find then

$$G(\varphi^x, A^x, S_1) \le \frac{|S_1|}{|K_\varepsilon|} G(\varphi, A, K_\varepsilon). \tag{3.4}$$

Similarly to this, we get again

$$G_a(\varphi^x, A^x, S_2) \le \frac{|S_2|}{|K_\varepsilon|}\, G_a(\varphi, A, K_\varepsilon). \tag{3.5}$$

Step 2 By definition of the functional $\mathcal{F}_{\varepsilon,H}$ given in (1.3)

$$\mathcal{F}_{\varepsilon,H}(\varphi^x, A^x, \Omega) = \int_\Omega \Big(u_\varepsilon^2|(\nabla - \mathrm{i}A^x)\varphi^x|^2 + \frac{u_\varepsilon^4}{2\varepsilon^2}(1-|\varphi^x|^2)^2 + |\mathrm{curl}\, A^x - H|^2\Big)\,\mathrm{d}y. \tag{3.6}$$

Recall that u_ε^2 converges uniformly to the function p in Ω, so we can write for a small ε,

$$\begin{aligned} \mathcal{F}_{\varepsilon,H}(\varphi^x, A^x, \Omega) = &\int_{S_1} \Big(|(\nabla - \mathrm{i}A^x)\varphi^x|^2 + \frac{1}{2\varepsilon^2}(1-|\varphi^x|^2)^2 + |\mathrm{curl}\, A^x - H|^2\Big)\,\mathrm{d}y \\ &+ \int_{S_2} \Big(a|(\nabla - \mathrm{i}A^x)\varphi^x|^2 + \frac{a^2}{2\varepsilon^2}(1-|\varphi^x|^2)^2 + |\mathrm{curl}\, A^x - H|^2\Big)\,\mathrm{d}y + o_\varepsilon(1) \\ = &\; G(\varphi^x, A^x, S_1) + G_a(\varphi^x, A^x, S_2) + o_\varepsilon(1). \end{aligned} \tag{3.7}$$

We return to (3.4)–(3.5),

$$\mathcal{F}_{\varepsilon,H}(\varphi^x, A^x, \Omega) \le \frac{|S_1|}{|K_\varepsilon|}\, G(\varphi, A, K_\varepsilon) + \frac{|S_2|}{|K_\varepsilon|}\, G_a(\varphi, A, K_\varepsilon) + o_\varepsilon(1). \tag{3.8}$$

Step 3 Let us estimate the right-hand side of (3.8), for example $G_a(\varphi, A, K_\varepsilon)$ (the other case $G(\varphi, A, K_\varepsilon)$ will be done similarly). First, by the definition of the configuration (φ, A) given in Step 1, it is evident that

$$G_a(\varphi, A, K_\varepsilon) \le \int_{K_\varepsilon \setminus B_\varepsilon} a\, |\nabla h(x)|^2 + \int_{K_\varepsilon} |h(x) - H|^2 \mathrm{d}x + C, \tag{3.9}$$

where $B_\varepsilon = B(0,\varepsilon)$. We take the constant a aside. Use the change of variables $y = \lambda x$. Then

$$\int_{K_\varepsilon \setminus B_\varepsilon} |\nabla h|^2 \mathrm{d}x + \int_{K_\varepsilon} |h(x) - H|^2 \mathrm{d}x = \int_{K \setminus B_{\lambda\varepsilon}} |\nabla \widehat{h}|^2 \mathrm{d}y + \frac{2\pi}{H} \int_K |\widehat{h}(y)|^2 \,\mathrm{d}y, \tag{3.10}$$

where $\widehat{h}(y) = h(x) - H$ and $K = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$. Now, we put

$$g(y) = \widehat{h}(y) + \ln |y|. \tag{3.11}$$

We show that g is bounded in $W^{1,q}(K)$ independently of ε for any $q > 0$. First, since $\widehat{h}$ satisfies

$$\begin{cases} -\lambda^2 \Delta \widehat{h}(y) + \widehat{h}(y) + H = 2\pi \delta_0(\frac{y}{\lambda}), & \text{in } K, \\ \partial_\nu \widehat{h} = 0, & \text{on } \partial K, \end{cases}$$

g is a solution of

$$\begin{cases} -\lambda^2 \Delta g(y) + g(y) + H - \ln |y| = 0, & \text{in } K, \\ \partial_\nu g = \partial_\nu \ln |y|, & \text{on } \partial K. \end{cases} \tag{3.12}$$

Multiply this equation by g and integrate convenably

$$\int_K |\nabla g|^2 \,\mathrm{d}y + \frac{1}{\lambda^2} \int_K (g^2(y) \,\mathrm{d}y + H\, g(y) \,\mathrm{d}y - \ln |y|\, g(y) \,\mathrm{d}y) = \int_{\partial K} g \,\partial_\nu \ln |y| \,\mathrm{d}y. \tag{3.13}$$

Since $\int_K \widehat{h}(y)\mathrm{d}y = 0$, from (3.11) we have

$$\int_K g(y)\mathrm{d}y = \int_K \ln |y| \,\mathrm{d}y \le C.$$

Therefore, using the Cauchy-Schwartz inequality in (3.13), we have

$$C \int_K |\nabla g|^2 \,\mathrm{d}y \le \frac{1}{\lambda^2} \Big(CH + \int_K g^2(y) \,\mathrm{d}y + C \Big(\int_K g^2(y) \Big)^{\frac{1}{2}} \Big) + C \Big(\int_{\partial K} g^2 \Big)^{\frac{1}{2}}, \tag{3.14}$$

where C is an arbitrary positive constant. Because the mean value of g in K is uniformly bounded in ε, then we deduce from the Poincaré's inequality that

$$|g|^2_{L^2(K)} \le C(1 + |\nabla g|^2_{L^2(K)}). \tag{3.15}$$

Recalling that $\lambda^2 = \frac{H}{2\pi} \gg 1$, so bounding the L^2 norm of the trace of g by the H^1 norm and using (3.15), the inequality (3.14) becomes

$$\int_K |\nabla g|^2 \,\mathrm{d}y \le C, \quad \text{hence} \quad |g|_{H^1(K)} \le C. \tag{3.16}$$

We return to (3.12) to deduce that g is bounded in $W^{1,q}(K)$ independently of ε for any $q > 0$. Together with (3.11), this implies that

$$\int_{K \setminus B_{\lambda\varepsilon}} |\nabla \widehat{h}|^2 \mathrm{d}y \le C + \int_{K \setminus B_{\lambda\varepsilon}} |\nabla \ln |y||^2 \,\mathrm{d}y \le \Big(C + 2\pi \,\ln \frac{1}{\lambda\varepsilon} \Big), \tag{3.17}$$

and also $\frac{2\pi}{H} \int_K |\widehat{h}(y)|^2 \,\mathrm{d}y \le C$.

Combining all the above in (3.10) together with (3.9), the desired control on $G_a(\varphi, A, K_\varepsilon)$ becomes

$$G_a(\varphi, A, K_\varepsilon) \le a\Big(2\pi \ln \frac{1}{\lambda\varepsilon} + C\Big).$$

Similarly, we can find that

$$G(\varphi, A, K_\varepsilon) \le \Big(2\pi \ln \frac{1}{\lambda\varepsilon} + C\Big).$$

Combining the two above inequalities in (3.8), we have

$$\begin{aligned}\mathcal{F}_{\varepsilon,H}(\varphi^x, A^x, \Omega) &\le \frac{|S_1| + a|S_2|}{|K_\varepsilon|}\Big(2\pi \log \frac{1}{\lambda\varepsilon} + C\Big) + o_\varepsilon(1)\\ &\le H\Big(\int_\Omega p(y)\mathrm{d}y\Big)\Big(\ln \frac{1}{\varepsilon\sqrt{H}} + C\Big),\end{aligned}$$

since $|K_\varepsilon| = \lambda^{-2} = \frac{2\pi}{H}$. This completes the proof of Proposition 3.1.

4 Lower Bound of the Energy

We now wish to compute a lower bound for $\mathcal{F}_{\varepsilon,H}(\varphi, A, \Omega)$ which matches the upper bound of the previous section.

In what follows, we denote $B_\alpha^x = B(x, \frac{1}{\alpha})$ and we often omit the subscript ε, where x is the center of the blow-up.

Proposition 4.1 *Assume that* $|\ln\varepsilon| \ll H \ll \frac{1}{\varepsilon^2}$ *and* $(\varphi_\varepsilon, A_\varepsilon)$ *minimizes* $\mathcal{F}_{\varepsilon,H}$. *Then, for any* $K > 0$, *there exists* $1 \ll \alpha \ll \frac{1}{\varepsilon}$ *such that for every* $x \in \Omega$ *such that* $B_\alpha^x \subset \Omega$, *we have*

$$\mathcal{F}_{\varepsilon,H}(\varphi_\varepsilon, A_\varepsilon, B_\alpha^x) \ge H \ln \frac{1}{\varepsilon\sqrt{H}}(1 - o(1)) \int_{B_\alpha^x} \gamma_K(y)p(y)\mathrm{d}y, \tag{4.1}$$

where $\gamma_K(x)$ *is equal to a constant* γ_K^1 *if* $x \in S_1$ *and* γ_K^2 *if* $x \in S_1$, *where for each* $i = 1, 2$, $\gamma_K^i \to 1$ *if* $K \to +\infty$.

As a consequence of this, the appropriate lower bound is given by the following result.

Corollary 4.1 *Under the hypotheses of Proposition* 4.1, *we have*

$$\mathcal{F}_{\varepsilon,H}(\varphi_\varepsilon, A_\varepsilon, \Omega) \ge H \ln \frac{1}{\varepsilon\sqrt{H}}(1 - o(1)) \int_\Omega p(y)\mathrm{d}y. \tag{4.2}$$

Proof We investigate (4.1) with respect to x. Letting U be any open subdomain of Ω and using Fubini's theorem, referring to [11, Chapter 8, p. 163], we have

$$\begin{aligned}\int_{x\in U} \mathcal{F}_{\varepsilon,H}(\varphi, A, U\cap B_\alpha^x) &= \int_{x\in U\cap S_1} \mathcal{F}_{\varepsilon,H}(\varphi, A, U\cap S_1\cap B_\alpha^x)\\ &\quad + \int_{x\in U\cap S_2} \mathcal{F}_{\varepsilon,H}(\varphi, A, U\cap S_2\cap B_\alpha^x)\\ &\le \frac{\pi}{\alpha^2}[\mathcal{F}_{\varepsilon,H}(\varphi, A, U\cap S_1) + \mathcal{F}_{\varepsilon,H}(\varphi, A, U\cap S_2)].\end{aligned}$$

Again similarly as in [11, Chapter 8, p. 163], we deduce by using (4.1), Fatou's lemma and the appropriate expression of $p(x)$ and $\gamma_K(x)$ that

$$\liminf_{\varepsilon\to 0} \frac{\mathcal{F}_{\varepsilon,H}(\varphi, A, U)}{H \ln \frac{1}{\varepsilon\sqrt{H}}} \ge \gamma_K^1\, |U\cap S_1| + \gamma_K^2\, a|U\cap S_2|.$$

Letting $K \to +\infty$, we get $\liminf\limits_{\varepsilon\to 0} \frac{\mathcal{F}_{\varepsilon,H}(\varphi,A,U)}{H\ln\frac{1}{\varepsilon\sqrt{H}}} \geq \int_U p(y)\mathrm{d}y$, since for each $i=1,2$, $\gamma_K^i \to 1$. The fact that U is arbitrary completes the proof of Corollary 4.1.

4.1 Proof of Proposition 4.1

First, we start with a preliminary rescaling formula. Its proof is straightforward and we omit it.

Lemma 4.1 *Given* (φ, A) *and* Ω, *assume* $0\in\Omega$. *Define* $(\varphi_\alpha,\, A_\alpha)$ *and*

$$\varphi_\alpha(\alpha x) = \varphi(\alpha), \quad \alpha A_\alpha(\alpha x) = A(x), \quad \Omega_\alpha = \alpha\Omega. \tag{4.3}$$

Then, for any H, *we have* $\mathcal{F}_{\varepsilon,H}(\varphi,A,\Omega) = \mathcal{F}^\alpha_{\varepsilon,H}(\varphi_\alpha,A_\alpha,\Omega_\alpha)$ *where*

$$\begin{aligned}\mathcal{F}^\alpha_{\varepsilon,H}(\varphi_\alpha,A_\alpha,\Omega_\alpha) = \int_{\Omega_\alpha}\Big(&u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla - \mathrm{i}A_\alpha)\varphi_\alpha|^2 + \alpha^2\Big|\mathrm{curl}\,A_\alpha - \frac{H}{\alpha^2}\Big|^2 \\ &+ u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi_\alpha|^2)^2}{2\alpha^2\varepsilon^2}\Big)\,\mathrm{d}y.\end{aligned} \tag{4.4}$$

The proof of Proposition 4.1 is achieved by blowing up at the scale α. Define $(\varphi_\alpha,\, A_\alpha)$ as in (4.3), but take the origin at x. Using Lemma 4.1 again with the origin at x, and dropping the ε subscripts, the left-hand side of (4.1) is equal to

$$\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla - \mathrm{i}A_\alpha)\varphi_\alpha|^2 + \alpha^2\Big|\mathrm{curl}\,A_\alpha - \frac{H}{\alpha^2}\Big|^2 + u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi_\alpha|^2)^2}{2\alpha^2\varepsilon^2}\Big)\,\mathrm{d}y.$$

Thus, if we choose $\varphi' = \varphi_\alpha$, $A' = A_\alpha$, $\varepsilon' = \alpha\varepsilon$ and $H' = \frac{H}{\alpha^2}$, the inequality (4.1) that we wish to prove is equivalent to

$$\begin{aligned}&\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla - \mathrm{i}A')\varphi'|^2 + \alpha^2|\mathrm{curl}\,A' - H'^2|^2 + u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y \\ &\geq H'\ln\frac{1}{\varepsilon\sqrt{H}}(1-o(1))\int_{B_1}\gamma_K(y)p(y)\mathrm{d}y.\end{aligned}$$

Now for any $\varepsilon > 0$, we choose α such that

$$H' = K|\ln\varepsilon'|. \tag{4.5}$$

Proceeding as in [11, Chapter 8, p. 161], this is possible and we find that (4.5) can be verified and then corresponding α, ε' verify

$$1 \ll \alpha \ll \frac{1}{\varepsilon}, \quad \varepsilon' \ll 1, \quad \ln\frac{1}{\varepsilon\sqrt{H}} \simeq |\ln\varepsilon'|.$$

The inequality that we wish to prove becomes

$$\begin{aligned}&\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla - \mathrm{i}A')\varphi'|^2 + \alpha^2|\mathrm{curl}\,A' - H'^2|^2 + u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y \\ &\geq H'|\ln\varepsilon'|(1-o(1))\int_{B_1}\gamma_K(y)p(y)\mathrm{d}y.\end{aligned} \tag{4.6}$$

There are two cases, depending on the blow-up origin x. Either

$$\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla-\mathrm{i}A')\varphi'|^2+\alpha^2|\mathrm{curl}\,A'-H'^2|^2+u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y\gg H'^2,$$

as $\varepsilon\to 0$, and then, the inequality (4.6) is clearly satisfied, or

$$\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla-\mathrm{i}A')\varphi'|^2+\alpha^2|\mathrm{curl}\,A'-H'^2|^2+u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y\leq CH'^2. \tag{4.7}$$

We know that u_ε^2 converges uniformly to the function p in Ω and $\alpha\gg 1$. Hence for a small ε,

$$\begin{aligned}&\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla-\mathrm{i}A')\varphi'|^2+\alpha^2|\mathrm{curl}\,A'-H'^2|^2+u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y\\&=\int_{B_1\cap S_1}\Big(|(\nabla-\mathrm{i}A')\varphi'|^2+|\mathrm{curl}\,A'-H'^2|^2+\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y\\&\quad+\int_{B_1\cap S_2}\Big(a|(\nabla-\mathrm{i}A')\varphi'|^2+|\mathrm{curl}\,A'-H'^2|^2+a^2\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y+o_\varepsilon(1)\\&=G(\varphi',A',B_1\cap S_1)+G_a(\varphi',A',B_1\cap S_2)+o_\varepsilon(1).\end{aligned} \tag{4.8}$$

Going back to (4.7), we have

$$G(\varphi',A',B_1\cap S_1)\leq CH'^2\quad\text{and}\quad G_a(\varphi',A',B_1\cap S_2)\leq CH'^2.$$

Here, we have reduced to the case of configurations with a relatively small energy, for which all the analysis of Sandier-Serfaty [10] will apply on the appropriate domains $B_1\cap S_1$ and $B_1\cap S_2$. In this case, replacing ε by ε' and H by H', the hypotheses (see [10, Theorem 1]) are satisfied and we deduce (here K plays the role of λ in [10])

$$\begin{aligned}&\liminf_{\varepsilon'\to 0}\frac{G(\varphi',A',B_1\cap S_1)}{H'^2}\\&\geq P_K(\mu_1^*)=\frac{1}{K}\int_{B_1\cap S_1}|\mu_1^*|\,\mathrm{d}y+\int_{B_1\cap S_1}(|\nabla h_{\mu_1^*}|^2+|h_{\mu_1^*}-1|^2)\,\mathrm{d}y,\end{aligned} \tag{4.9}$$

where again from [10], the limit measure $\mu_1^*=-\Delta h_1^*+h_1^*$ is equal to $(1-\frac{1}{2K})\mathbf{1}_{W_K^1}$ and the subdomain W_K^1 is the coincidence set $\{x\in B_1\cap S_1,\ h_1^*(x)=1-\frac{1}{2K}\}$. Similarly as in [10], we can have

$$\begin{aligned}&\liminf_{\varepsilon'\to 0}\frac{G_a(\varphi',A',B_1\cap S_2)}{H'^2}\\&\geq Q_K(\mu_2^*)=\frac{1}{K}\int_{B_1\cap S_2}a|\mu_2^*|\,\mathrm{d}y+\int_{B_1\cap S_2}(a|\nabla h_{\mu_2^*}|^2+|h_{\mu_2^*}-1|^2)\,\mathrm{d}y,\end{aligned} \tag{4.10}$$

where $\mu_2^*=-\Delta h_2^*+h_2^*=(1-\frac{1}{2K})\mathbf{1}_{W_K^2}$ and again W_K^2 is equal to the set

$$\Big\{x\in B_1\cap S_2,\ h_2^*(x)=1-\frac{1}{2K}\Big\}.$$

Combining (4.9) together with (4.10) in (4.8), we get

$$\begin{aligned}&\liminf_{\varepsilon'\to 0}\frac{1}{H'^2}\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla-\mathrm{i}A')\varphi'|^2+|\mathrm{curl}\,A'-H'^2|^2+u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y\\&\geq P_K(\mu_1^*)+Q_K(\mu_2^*).\end{aligned} \tag{4.11}$$

By definition of the functionals P_K and Q_K, it follows that

$$P_K(\mu_1^*) \geq \frac{1}{K}\Big|1-\frac{1}{2K}\Big||W_K^1| \quad \text{and} \quad Q_K(\mu_2^*) \geq a\frac{1}{K}\Big|1-\frac{1}{2K}\Big||W_K^2|.$$

Note that $|W_K^1|$ and $|W_K^2|$ tend respectively to $|B_1 \cap S_1|$ and $|B_1 \cap S_2|$ when K tends to $+\infty$. Therefore, for any $x \in \Omega$,

$$\begin{aligned}&\liminf_{\varepsilon'\to 0} \frac{1}{H'^2}\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla - \mathrm{i}A')\varphi'|^2 + |\mathrm{curl}\, A' - H'^2|^2 + u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y \\ &\geq \frac{1}{K}\Big|1-\frac{1}{2K}\Big|(|W_K^1| + a|W_K^2|). \end{aligned} \tag{4.12}$$

Taking the fact that $H'^2 = K\frac{H}{\alpha^2}\ln\frac{1}{\varepsilon\sqrt{H}}$ in (4.12), we obtain

$$\begin{aligned}&\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla - \mathrm{i}A')\varphi'|^2 + |\mathrm{curl}\, A' - H'^2|^2 + u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y \\ &\geq \frac{H}{\alpha^2}\Big|1-\frac{1}{2K}\Big|(|W_K^1| + a|W_K^2|)\ln\frac{1}{\varepsilon\sqrt{H}}. \end{aligned} \tag{4.13}$$

Let us take

$$\gamma_K(y) = \begin{cases} \gamma_K^1 = \Big|1-\dfrac{1}{2K}\Big|\dfrac{|W_K^1|}{|B_1\cap S_1|}, & \text{if } y \in S_1, \\ \gamma_K^2 = \Big|1-\dfrac{1}{2K}\Big|\dfrac{|W_K^2|}{|B_1\cap S_2|}, & \text{if } y \in S_2. \end{cases}$$

Remark that each γ_K^i tends to 1 when K tends to $+\infty$. We can then write

$$\begin{aligned}&\int_{B_1}\Big(u_\varepsilon^2\Big(\frac{y}{\alpha}\Big)|(\nabla - \mathrm{i}A')\varphi'|^2 + |\mathrm{curl}\, A' - H'^2|^2 + u_\varepsilon^4\Big(\frac{y}{\alpha}\Big)\frac{(1-|\varphi'|^2)^2}{2\varepsilon'^2}\Big)\,\mathrm{d}y \\ &\geq \frac{H}{\alpha^2}\ln\frac{1}{\varepsilon\sqrt{H}}\int_{B_1}\gamma_K(y)p(y)\mathrm{d}y = H\log\frac{1}{\varepsilon\sqrt{H}}\int_{B_\alpha^x}\gamma_K(y)p(y)\mathrm{d}y. \end{aligned}$$

Since $1 \ll \alpha$, (4.1) is satisfied for every choice of blow-up origin x. Proposition 4.1 is then proved.

Acknowledgement The author wishes to thank Ayman Kachmar for helpful discussions and constructive comments during the course of this work.

References

[1] Aydi, H. and Kachmar, A., Magnetic vortices for a Ginzburg-Landau type energy with discontinuous constraint, II, *Comm. Pure. Appl. Anal.*, **8**(3), 2009, 977–998.

[2] Chapman, S. J., Du, Q. and Gunzburger, M. D., A Ginzburg-Landau type model of superconducting/normal junctions including Josephson junctions, *European J. Appl. Math.*, **6**, 1996, 97–114.

[3] Giorgi, T., Superconductors surrounded by normal materials, *Proc. Roy. Soc. Edinburgh Sec. A*, **135**, 2005, 331–356.

[4] Kachmar, A., Magnetic Ginzburg-Landau functional with disconinuous constraint, *C. R. Math. Acad. Sci. Paris*, **346**, 2008, 297–300.

[5] Kachmar, A., Magnetic vortices for a Ginzburg-Landau type energy with disconinuous constraint, *ESAIM: COCV*, accepted.

[6] Kachmar, A., On the perfect superconducting solution for a generalized Ginzburg-Landau equation, *Asymptot. Anal.*, **54**, 2007, 125–164.

[7] Lassoued, L., and Mironescu, P., Ginzburg-Landau type energy with discontinuous constraint, *J. Anal. Math.*, **77**, 1999, 1–26.

[8] Rubinstein, J., Six Lectures on Superconductivity, Boundaries, Interfaces and Transitions, M. Delfour (ed.), CRM Proc. Lecture Notes, **13**, A. M. S., Providence, RI, 1998, 163–184.

[9] Sandier, E. and Serfaty, S., On the energy of type-II superconductors in the mixed phaseA, *Review. Math. Phys*, **12**(9), 2000, 1219–1257.

[10] Sandier, E. and Serfaty, S., A rigorous derivation of a free-boundary problem arising in superconductivity, *Annales Scientifiques de l'ENS*, **33**, 2000, 561–592.

[11] Sandier, E. and Serfaty, S., Vortices for the magnetic Ginzburg-Landau model, Progress in Nonlinear Differential Equations and Their Applications, **70**, Birkhäuser, Boston, 2007.