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The Inverse Mean Curvature Flow in Rotationally Symmetric Spaces

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Abstract In this paper, the motion of inverse mean curvature flow which starts from a closed star-sharped hypersurface in special rotationally symmetric spaces is studied. It is proved that the flow converges to a unique geodesic sphere, i.e., every principle curvature of the hypersurfaces converges to a same constant under the flow.

Keywords Asymptotic behavior, Inverse mean curvature flow, Hyperbolic space
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1 Introduction

The inverse mean curvature flow space was studied in Euclidean space or asymptotically flat Riemannian spaces. Huisken and Ilmanen used it to prove the Penrose inequality for asymptotically flat 3-manifolds. A classical solution of inverse mean curvature in Euclidean space is a smooth family $X : M^n \times [0, T] \to \mathbb{R}^{n+1}$ of closed hypersurfaces satisfying

$$\frac{\partial}{\partial t}X(p,t) = \frac{1}{H}\overrightarrow{\nu}(p,t), \quad p \in M^n,\ 0 \leq t \leq T,$$

where $H(p,t) = \mathrm{div}(\nu) > 0$ and $\overrightarrow{\nu}(p,t)$ is the outward unit normal vector of the surface $X(\,\cdot\,,t)(M^n)$ at the point $X(p,t)$ and div is the divergence of $X(M,t)$. Gerhardt [4] proved that for a smooth, closed, star-sharped initial hypersurface of strictly positive mean curvature, the evolution equation has a unique smooth solution for all times, moreover the rescaled surfaces

$$\widetilde{X}(t) \triangleq \mathrm{e}^{-\frac{t}{n}} \cdot X(t)$$

converge exponentially fast to a unique sphere. On the other hand, Huisken and Ilmanen [5] proved higher regularity properties of inverse mean curvature flow in Euclidean space.

However, the results do not close relate to the ambient space, namely, Euclidean space could be perturbed in some fashion. In this paper, we discuss that the ambient space is a rotationally symmetric space with nonpositive sectional curvature and Euclidean volume growth. We also discuss the case that the ambient space is hyperbolic space whose flow is different from Euclidean space to a certain extent.

Let N^{n+1} be a rotationally symmetric space, whose metric is

$$\overline{g} = \mathrm{d}r^2 + \lambda^2(r)\sigma_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{1.1}$$

under the geodesic polar coordinates, where $\sigma \triangleq \sigma_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the canonical metric of $\mathbb{S}^n$, $\lambda \in C^\infty(\mathbb{R}_+), \lambda(0) = 0, \lambda'(0) = 1$ and $\lambda(r) > 0$ for any $r > 0$. Correspondingly, we introduce

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the local tangent vector fields of $\mathbb{S}^n$ and N^{n+1}

$$\Big\{\frac{\partial}{\partial x_i}\Big\}_{i=1}^n \quad \text{and} \quad \Big\{\frac{\partial}{\partial x_i}\Big\}_{i=1}^n \cup \Big\{\frac{\partial}{\partial r}\Big\},$$

respectively.

In this paper, we only consider the manifolds with nonpositive sectional curvature, which is equivalent to $\lambda'' \geq 0$. Furthermore, if the manifold N also has Euclidean volume growth, which means that λ' is uniformly bounded, we prove the following theorem.

Theorem 1.1 *Let N^{n+1} be a rotationally symmetric space with nonpositive sectional curvature, M_0 be a smooth closed, star-shaped hypersurface of N^{n+1}, which is given as an embedding*

$$X_0 : \mathbb{S}^n \mapsto N^{n+1},$$

whose mean curvature H is positive. Then the evolution equation

$$\dot{X} = \frac{1}{H}\overrightarrow{\nu}, \quad X(0) = X_0 \tag{1.2}$$

on $\mathbb{S}^n \times \mathbb{R}_+$, where ν is the outward unit normal vector of the surface $X(t)$, $H = H(t)$ is the mean curvature of $X(t)$ and $\dot{X} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\partial}{\partial r} + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\partial}{\partial x^i}$ if we use the geodesic polar coordinates in N^{n+1}, has a unique smooth solution for all times.

Case 1 *If λ' is uniformly bounded, then the rescaled surfaces*

$$\widetilde{X}(t) = \mathrm{e}^{-\frac{t}{n}} X(t)$$

converge to a uniquely determinate sphere of radius $\frac{1}{\lambda'(\infty)}\Big(\frac{\mathrm{Area}(M_0)}{|\mathbb{S}^n|}\Big)^{\frac{1}{n}}$, where $\lambda'(\infty) \triangleq \lim_{r\to\infty} \lambda'(r)$, $\mathrm{Area}(M_0)$ is the area of M_0 in N^{n+1}, and $|\mathbb{S}^n|$ is Lebesgue measure of n-sphere in Euclidean space.

Case 2 *If $\lambda(r) = \sinh(r)$, the rescaled surfaces*

$$\widetilde{X}(t) = \frac{n}{t} X(t)$$

converge to a uniquely determinate sphere of radius 1.

2 A Reformulation of the Problem

It is well-known the existence of the short time solution to (1.2) (see [3] for example). Let $[0, T)$ be its maximum interval, that is, $H(t) > 0$ for all $t \in [0, T)$. Considering this embedding

$$X_t : \mathbb{S}^n \mapsto M_t^n \hookrightarrow N^{n+1}, \quad \forall\, t \in [0, T).$$

Let D be the Levi-Civata connection on $\mathbb{S}^n$. We always regard $r(x, t)$ as a function on $\mathbb{S}^n$, for fixed t. Then derivative of $r(x, t)$ in the direction of $\frac{\partial}{\partial x_i}$ can be written as $D_i r$. Then, we have local coordinate vector fields of $M(t)$

$$e_i \triangleq X_{t*}\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial X_t}{\partial x^i} = D_i r \frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}, \quad 1 \leq i \leq n, \tag{2.1}$$

for fixed t and the outward unit normal vector of $M(t)$

$$\nu = \frac{1}{v}\Big(\frac{\partial}{\partial r} - \lambda^{-2}(r) D^j r \frac{\partial}{\partial x^j}\Big), \tag{2.2}$$

where $D^j r = \sigma^{ij} D_i r$, (σ^{ij}) is the inverse matrix of (σ_{ij}), $Dr \in \Gamma(T\mathbb{S}^n)$, $v = \sqrt{1+\lambda^{-2}(r)|Dr|^2}$, $|\bullet|$ is with respect to the metric of $\mathbb{S}^n$: $\sigma_{ij}\mathrm{d}x^i\mathrm{d}x^j$.

Remark 2.1 In this paper, we use different norms $|\bullet|$ with respect to the metric from tangent space where $\bullet$ belongs; D^j always denotes $\sigma^{ij}D_i$.

Naturally, we have the metric $g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ of $M(t)$ induced from N, where

$$g_{ij} = \overline{g}(e_i, e_j) = D_i r D_j r + \lambda^2(r)\sigma_{ij}. \tag{2.3}$$

From (1.2) and (2.2), we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{Hv} \quad \text{and} \quad \dot{x}^i = -\frac{D^i r}{\lambda^2 H v}. \tag{2.4}$$

Then

$$\frac{\partial r}{\partial t} = \frac{\mathrm{d}r}{\mathrm{d}t} - D_j r \cdot \dot{x}^j = \frac{v}{H}. \tag{2.5}$$

Let D, ∇ and $\overline{\nabla}$ be the Levi-Civita connections of $\mathbb{S}^n$, M and N, respectively. $\widetilde{\Gamma}_{ij}^k$ denote the Christoffel symbols of $\mathbb{S}^n$ with respect to the tangent basis $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$, and $\Gamma_{\alpha\beta}^\gamma$ denote the Christoffel symbols of N^{n+1} with respect to the tangent basis $\{\frac{\partial}{\partial x_i}\}_{i=1}^n \cup \{\frac{\partial}{\partial r}\}$. Then we have

$$\Gamma_{ij}^k = \widetilde{\Gamma}_{ij}^k, \quad \Gamma_{ij}^0 = -\lambda\lambda'\sigma_{ij}, \quad \Gamma_{0i}^k = \frac{\lambda'}{\lambda}\delta_i^k, \quad \Gamma_{0i}^0 = \Gamma_{00}^k = \Gamma_{00}^0 = 0$$

for $i,j,k > 0$. $\overline{R}, \overline{\mathrm{Ric}}$ and R, Ric and $\widetilde{R}, \widetilde{\mathrm{Ric}}$ denote the curvature tensors and Ricci tensors of $N, M, \mathbb{S}^n$, respectively, where $\overline{R} = -\overline{\nabla}\circ\overline{\nabla}$, etc., and we write $\overline{g}_{ij} \triangleq \overline{g}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

From [7, Appendix A], we have

$$\Big\langle \overline{R}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial r}\Big)\frac{\partial}{\partial x^j}, \frac{\partial}{\partial r}\Big\rangle = -\frac{\lambda''}{\lambda}\overline{g}_{ij},$$
$$\Big\langle \overline{R}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big\rangle = \frac{1-(\lambda')^2}{\lambda^2}(\overline{g}_{ik}\overline{g}_{jl} - \overline{g}_{il}\overline{g}_{jk}),$$

and other components are equal to zero. A straightforward computation shows

$$\overline{\mathrm{Ric}}\Big(\nu, \frac{\partial}{\partial r}\Big) = \overline{g}^{ij}\Big\langle \overline{R}\Big(\frac{\partial}{\partial x^i}, \nu\Big)\frac{\partial}{\partial x^j}, \frac{\partial}{\partial r}\Big\rangle = -\frac{n}{v}\cdot\frac{\lambda''}{\lambda},$$

and

$$\overline{\mathrm{Ric}}(\nu,\nu) = -n\frac{\lambda''}{\lambda} - |Dr|^2\cdot\frac{n-1}{v^2\lambda^2}\cdot\frac{(\lambda')^2 - 1 - \lambda\lambda''}{\lambda^2}.$$

Let $h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ be the second fundamental form of M in N. Then

$$\begin{aligned}
h_{ij} &= -\langle \overline{\nabla}_{e_j} e_i, \nu\rangle \\
&= -\frac{1}{v}\Big\langle \frac{\partial}{\partial r} - \lambda^{-2}(r)D^k r\frac{\partial}{\partial x^k}, r_j\overline{\nabla}_{\frac{\partial}{\partial r}}\Big(r_i\frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}\Big) + \overline{\nabla}_{\frac{\partial}{\partial x^j}}\Big(r_i\frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}\Big)\Big\rangle \\
&= -\frac{1}{v}\Big\langle \frac{\partial}{\partial r} - \frac{1}{\lambda^2(r)}D^k r\frac{\partial}{\partial x^k}, r_j\overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^i} + r_{ij}\frac{\partial}{\partial r} + r_i\overline{\nabla}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial r} + \overline{\nabla}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}\Big\rangle \\
&= -\frac{1}{v}\Big\{ r_j\Big\langle \frac{\partial}{\partial r}, \overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^i}\Big\rangle + r_{ij} + \Big\langle \frac{\partial}{\partial r}, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}\Big\rangle - \frac{D^k r D_j r}{\lambda^2}\Big\langle \frac{\partial}{\partial x^k}, \overline{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^i}\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&- \frac{D^k r D_i r}{\lambda^2}\Big\langle \frac{\partial}{\partial x^k}, \overline{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial r}\Big\rangle - \frac{D^k r}{\lambda^2}\Big\langle \frac{\partial}{\partial x^k}, \overline{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}\Big\rangle\Big\} \\
= &-\frac{1}{v}\Big\{ r_{ij} + \Big\langle \frac{\partial}{\partial r}, \overline{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}\Big\rangle + \frac{D^k r D_j r}{\lambda^2}\Big\langle \frac{\partial}{\partial r}, \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}\Big\rangle \\
&+ \frac{D^k r D_i r}{\lambda^2}\Big\langle \frac{\partial}{\partial r}, \overline{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}\Big\rangle - \frac{D^k r}{\lambda^2}\Big\langle \frac{\partial}{\partial x^k}, \overline{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}\Big\rangle\Big\} \\
= &-\frac{1}{v}\Big\{ r_{ij} - \lambda(r)\lambda'(r)\sigma_{ij} - 2\frac{\lambda'(r)}{\lambda(r)} r_i r_j - \frac{D^k r}{\lambda^2}\widetilde{\Gamma}^l_{ij}\lambda^2\sigma_{kl}\Big\} \\
= &-\frac{1}{v}\Big\{ r_{i,j} - \lambda(r)\lambda'(r)\sigma_{ij} - 2\frac{\lambda'(r)}{\lambda(r)} r_i r_j\Big\},
\end{aligned} \tag{2.6}
$$

where $r_{i,j}$ denotes the second covariant derivative of r.

For the convenience, we define function

$$
\varphi(x,t) = \int_c^{r(x,t)} \frac{1}{\lambda(s)}\mathrm{d}s
$$

on $\mathbb{S}^n \times [0,T)$. To make the integral been meaningful, c is supposed to be an arbitrary fixed positive constant. Then we have

$$
Dr = \lambda \cdot D\varphi, \quad \frac{\partial r}{\partial t} = \lambda \cdot \frac{\partial \varphi}{\partial t}, \quad \varphi_{i,j} = \frac{1}{\lambda} r_{i,j} - \frac{\lambda'}{\lambda^2} r_i r_j. \tag{2.7}
$$

Moreover

$$
v = \sqrt{1 + |D\varphi|^2}, \quad g_{ij} = \lambda^2(\sigma_{ij} + \varphi_i\varphi_j), \quad g^{ij} = \lambda^{-2}\Big(\sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}\Big). \tag{2.8}
$$

So the second fundamental form of M could be expressed as

$$
h_{ij} = \frac{\lambda}{v}[\lambda'(\sigma_{ij} + \varphi_i\varphi_j) - \varphi_{i,j}], \quad h^i_j = g^{ik}h_{jk} = \frac{\lambda'}{\lambda v}\delta^i_j - \frac{1}{\lambda v}\widetilde{\sigma}^{ik}\varphi_{k,j}, \tag{2.9}
$$

where $\widetilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$ and $\varphi^i = \sigma^{ij}\varphi_j$.

Evolution equation (2.5) can also be rewritten as

$$
\frac{\partial \varphi}{\partial t} = \frac{v^2}{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}}. \tag{2.10}
$$

Since $\overline{\nabla}_{\dot{X}} e_i - \overline{\nabla}_{e_i}\dot{X} = X_*[\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}] = 0$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \overline{\nabla}_{\frac{\partial}{\partial t}}\langle e_i, e_j\rangle = \langle \overline{\nabla}_{\dot{X}} e_i, e_j\rangle + \langle e_i, \overline{\nabla}_{\dot{X}} e_j\rangle \\
&= \langle \overline{\nabla}_{e_i}\dot{X}, e_j\rangle + \langle e_i, \overline{\nabla}_{e_j}\dot{X}\rangle \\
&= \Big\langle \overline{\nabla}_{e_i}\Big(\frac{\nu}{H}\Big), e_j\Big\rangle + \Big\langle e_i, \overline{\nabla}_{e_j}\Big(\frac{\nu}{H}\Big)\Big\rangle \\
&= \frac{1}{H}\langle \overline{\nabla}_{e_i}\nu, e_j\rangle + \frac{1}{H}\langle e_i, \overline{\nabla}_{e_j}\nu\rangle \\
&= \frac{2}{H}h_{ij},
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t}\nu &= \Big\langle \frac{\partial}{\partial t}\nu, e_i\Big\rangle g^{ij}e_j = -\langle \nu, \overline{\nabla}_{e_i}\dot{X}\rangle g^{ij}e_j = -\Big\langle \nu, \overline{\nabla}_{e_i}\Big(\frac{\nu}{H}\Big)\Big\rangle g^{ij}e_j \\
&= -\overline{\nabla}_{e_i}\Big(\frac{1}{H}\Big)g^{ij}e_j = \nabla_M\Big(-\frac{1}{H}\Big).
\end{aligned} \tag{2.12}
$$

Finally, we give a comparison theorem which is used several times in this paper. Suppose that (M^m, g) is an arbitrary Riemannian manifold, ∇ is Levi-Civita connection. Let

$$\mathfrak{L}[u] \equiv \frac{\partial u}{\partial t} - F(x, t, u, \nabla_i u, \nabla^2_{ij} u),$$

where $x = (x^1, \cdots, x^m)$ is a local coordinate system of M, $\nabla_i = \nabla_{\frac{\partial}{\partial x^i}}$, $\nabla^2_{ij} u = \nabla\nabla u(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, and the function F contains the metric $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

If $v, w \in C^{2,1}(M \times [0, \tau))$, then we define set

$$G_{v,w} = \{(x, t, u, p_i, p_{ij}) \mid (x, t) \in M \times [0, \tau),\ u \in \langle v, w\rangle,\ p_i \in \langle \nabla_i v,\ \nabla_i w\rangle,\ p_{ij} \in \langle \nabla^2_{ij} v,\ \nabla^2_{ij} w\rangle\},$$

where $\langle v, w\rangle = \langle v(x, t), w(x, t)\rangle$, $\langle a, b\rangle$ denotes the interval between a, b. That is, when $a \le b$, $\langle a, b\rangle = [a, b]$, and when $a \ge b$, $\langle a, b\rangle = [b, a]$.

We need the following theorem, which is a generalization of the result in [10, Theorem 2.4.4]. The proof is similar.

Theorem 2.1 *If (M^m, g, ∇) is a closed Riemannian manifold, and*

(1) $v, w \in C^{2,1}(M \times [0, \tau))$*;*

(2) *there exists domain E of $M \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, s.t. $E \supset \overline{G}_{v,w}$, $F(x, t, u, p_i, p_{ij})$ is continuous and has one order partial derivative with respect to p_{ij} in E;*

(3) *matrix $(\frac{\partial F(x,t,u,p_i,p_{ij})}{\partial p_{kl}})$ is positive definite in $G_{v,w}$;*

(4) $\mathfrak{L}[v] \ge (\le)\, 0, \mathfrak{L}[w] < (>)\, 0$ *in $M \times [0, \tau)$, and $v > (<)\, w$ in M.*

Then we have $v > (<)\, w$ in $M \times [0, \tau]$.

Remark 2.2 When we use this comparison theorem, we often compare with function w which satisfies

$$\frac{\partial w}{\partial t} < (>) f(t, w, 0, 0),$$

where $F(x, t, u, p_i, p_{ij}) \ge (\le) f(t, u, p_i, p_{ij})$ for any $x \in M$, correspondingly we consider operator

$$\widetilde{\mathfrak{L}}[u] \equiv \frac{\partial u}{\partial t} - f(t, u, \nabla_i u, \nabla^2_{ij} u).$$

3 Evolutions

Let $\phi \triangleq \lambda(r)\langle \frac{\partial}{\partial r}, \nu\rangle = \frac{\lambda(r)}{v}$, which could be seen as "support function" of M in N, and $\psi \triangleq \frac{1}{H\phi}$. Curvature tensors of N are $\overline{R}_{ijkl} \triangleq \langle \overline{R}(e_i, e_j)e_k, e_l\rangle$, and $\overline{R}_{0jkl} = \overline{R}_{\nu jkl} \triangleq \langle \overline{R}(\nu, e_j)e_k, e_l\rangle$. We sometimes write $\triangle = \triangle_M$, $\nabla = \nabla_M$, $\nabla_i = \nabla_{e_i}$, grad= grad_M, for short.

Evolution equations are very helpful for estimates of asymptotic behavior and long time existence (see [1, 2, 12] for instance). Now let us compute several evolution equations, which will be in the following chapters.

Lemma 3.1 (i)
$$\frac{\partial}{\partial t} h_{ij} = \frac{\triangle h_{ij}}{H^2} - \frac{2}{H^3}\nabla_i H \nabla_j H + \frac{|A|^2}{H^2} h_{ij} - \frac{1}{H}\langle \overline{R}_{i\nu} e_j, \nu\rangle + \frac{1}{H^2}[-g^{kl}\nabla_k(\overline{R}_{0jil}) + g^{kl}\nabla_i(\overline{R}_{0lkj}) - \overline{R}_{kilp} h^p_j g^{kl} + \overline{R}_{ikjp} h^p_l g^{kl}],$$

(ii) $$\frac{\partial H}{\partial t} = \frac{\triangle H}{H^2} - \frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H} - \frac{1}{H}\overline{\mathrm{Ric}}(\nu, \nu),$$

(iii) $$\frac{\partial \phi}{\partial t} = \frac{1}{H^2}(\triangle \phi + |A|^2 \phi) + \frac{1}{H^2}\Big(\phi \overline{\mathrm{Ric}}(\nu, \nu) - \lambda \overline{\mathrm{Ric}}\Big(\nu, \frac{\partial}{\partial r}\Big)\Big),$$

(iv) $$\frac{\partial \psi}{\partial t} = \phi^2\psi^2 \triangle \psi + 2\phi\psi^2 \nabla\phi \cdot \nabla\psi - n\psi^3\phi^2 \frac{\lambda''}{\lambda}.$$

Proof (i) By definition of h_{ij}, we deduce

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= -\overline{\nabla}_{\frac{\partial}{\partial t}}\langle\overline{\nabla}_{e_i}e_j,\nu\rangle\\&= -\langle\overline{\nabla}_{\dot{X}}\overline{\nabla}_{e_i}e_j,\nu\rangle - \Big\langle\overline{\nabla}_{e_i}e_j,\frac{\partial\nu}{\partial t}\Big\rangle\\&= -\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}\dot{X},\nu\rangle - \frac{1}{H}\langle\overline{R}_{i\nu}e_j,\nu\rangle + \langle\overline{\nabla}_{e_i}e_j,e_k\rangle\nabla_{e_k}\Big(\frac{1}{H}\Big)\\&= -\Big\langle\overline{\nabla}_{e_i}\Big(\frac{1}{H}\overline{\nabla}_{e_j}\nu + \nu\overline{\nabla}_{e_j}H^{-1}\Big),\nu\Big\rangle - \frac{1}{H}\langle\overline{R}_{i\nu}e_j,\nu\rangle + \langle\nabla_{e_i}e_j,e_k\rangle\nabla_{e_k}\Big(\frac{1}{H}\Big)\\&= \frac{1}{H}\langle\overline{\nabla}_{e_i}\nu,\overline{\nabla}_{e_i}\nu\rangle - \nabla\nabla\Big(\frac{1}{H}\Big)(e_i,e_j) - \frac{1}{H}\langle\overline{R}_{i\nu}e_j,\nu\rangle\\&= \frac{1}{H}h_{ik}h_{jl}g^{kl} - \nabla\nabla\Big(\frac{1}{H}\Big)(e_i,e_j) - \frac{1}{H}\langle\overline{R}_{i\nu}e_j,\nu\rangle;\end{aligned} \tag{3.1}$$

from Simon's equation in [8] (see also [11]),

$$\begin{aligned}\triangle h_{ij} &= \nabla\nabla H(e_i,e_j) + g^{kl}g^{pq}(\overline{R}_{kilp}h_{qj} + \overline{R}_{kijp}h_{ql}) + Hh_i^q h_{qj} - h_{ij}|A|^2\\&\quad + g^{kl}\nabla_k(\overline{R}_{0jil}) + g^{kl}\nabla_i(\overline{R}_{0ljk}).\end{aligned}$$

Combining above two formulas, we obtain (i).

(ii) Recalling (2.9) and contracting the above equation (i), then the desired equation follows easily.

(iii) Let X be the local tangent field in N. We have

$$\overline{\nabla}_X\frac{\partial}{\partial r} = \frac{\lambda'}{\lambda}\Big(X - \Big\langle\frac{\partial}{\partial r},X\Big\rangle\frac{\partial}{\partial r}\Big) \tag{3.2}$$

and

$$\overline{\nabla}_{\frac{\partial}{\partial t}}\Big(\lambda\frac{\partial}{\partial r}\Big) = \lambda'\frac{1}{Hv}\frac{\partial}{\partial r} + \lambda\frac{\lambda'}{\lambda}\Big(\frac{\nu}{H} - \frac{1}{H}\Big\langle\frac{\partial}{\partial r},\nu\Big\rangle\frac{\partial}{\partial r}\Big) = \lambda'\frac{\nu}{H}.$$

Combining (2.10), we get

$$\frac{\partial}{\partial t}\phi = \overline{\nabla}_{\frac{\partial}{\partial t}}\Big\langle\lambda\frac{\partial}{\partial r},\nu\Big\rangle = \frac{\lambda'}{H} - \lambda\Big\langle\frac{\partial}{\partial r},\mathrm{grad}\Big(\frac{1}{H}\Big)\Big\rangle. \tag{3.3}$$

On the other hand, we have

$$\triangle\phi = \triangle\Big(\lambda\Big\langle\frac{\partial}{\partial r},\nu\Big\rangle\Big) = \frac{1}{v}\triangle\lambda(r) + 2\Big\langle\mathrm{grad}\lambda,\mathrm{grad}\Big(\Big\langle\frac{\partial}{\partial r},\nu\Big\rangle\Big)\Big\rangle + \lambda\triangle\Big\langle\frac{\partial}{\partial r},\nu\Big\rangle.$$

Hence, we need to calculate above 3 terms on the right-hand side. Since N^{n+1} is a rotationally symmetric space, we have

$$\overline{\triangle}r = \triangle_N r = n\frac{\lambda'}{\lambda} \quad \text{and} \quad \overline{\nabla}^2 r(X,X) = \frac{\lambda'}{\lambda}\Big\{\langle X,X\rangle - \Big\langle X,\frac{\partial}{\partial r}\Big\rangle^2\Big\}. \tag{3.4}$$

And $\nabla_M r = (\frac{\partial}{\partial r})^\top$, where $\top$ represents the projection from T_pN to T_pM for any $p \in M$; sometimes we write $\partial_r = \frac{\partial}{\partial r}$ for short. If we choose normal coordinates $\{X_i\}_{i=1}^n$ on M, then

$$\begin{aligned}\triangle\lambda(r) &= \lambda'\triangle r + \lambda''|\nabla_M r|^2\\&= \lambda'\sum_{i=1}^n(\nabla_{X_i}\nabla_{X_i}r - (\nabla_{X_i}X_i)r) + \lambda''|\partial_r^\top|^2\end{aligned}$$

$$
\begin{aligned}
&= \lambda'\Big(\sum_{i=1}^{n}(\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}r - (\overline{\nabla}_{X_i}X_i)r) - H\Big\langle \nu, \frac{\partial}{\partial r}\Big\rangle\Big) + \lambda''|\partial_r^\top|^2 \\
&= \lambda'\Big(\overline{\triangle} r - \overline{\nabla}\overline{\nabla} r(\nu,\nu) - \frac{H}{v}\Big) + \lambda''|\partial_r^\top|^2 \\
&= \Big(\lambda'' - \frac{(\lambda')^2}{\lambda}\Big)\cdot|\partial_r^\top|^2 + \lambda'\Big(n\frac{\lambda'}{\lambda} - \frac{H}{v}\Big),
\end{aligned}
\tag{3.5}
$$

where the last equality comes from (3.4) and

$$
\begin{aligned}
\Big\langle \mathrm{grad}\lambda, \mathrm{grad}\Big(\Big\langle\frac{\partial}{\partial r},\nu\Big\rangle\Big)\Big\rangle &= \sum_{i=1}^{n}\lambda' X_i(r)X_i\Big\langle \frac{\partial}{\partial r},\nu\Big\rangle \\
&= \sum_{i=1}^{n}\lambda'\Big\langle X_i,\frac{\partial}{\partial r}\Big\rangle\Big\{\Big\langle\overline{\nabla}_{X_i}\frac{\partial}{\partial r},\nu\Big\rangle + \Big\langle\frac{\partial}{\partial r},\overline{\nabla}_{X_i}\nu\Big\rangle\Big\} \\
&= \sum_{i=1}^{n}\lambda'\langle X_i,\partial_r^\top\rangle\Big\{-\frac{\lambda'}{\lambda}\langle\partial_r^\top,X_i\rangle\langle\partial_r,\nu\rangle + h(\partial_r^\top,X_i)\Big\} \\
&= \lambda' h(\partial_r^\top,\partial_r^\top) - \frac{(\lambda')^2}{\lambda}\Big\langle\frac{\partial}{\partial r},\nu\Big\rangle|\partial_r^\top|^2,
\end{aligned}
\tag{3.6}
$$

where the third equality come from (3.2).

Since

$$
\triangle\Big\langle\frac{\partial}{\partial r},\nu\Big\rangle = \langle\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}\partial_r,\nu\rangle + 2\langle\overline{\nabla}_{X_i}\partial_r,\overline{\nabla}_{X_i}\nu\rangle + \langle\partial_r,\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}\nu\rangle,
\tag{3.7}
$$

$$
\begin{aligned}
\langle\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}\partial_r,\nu\rangle &= \Big\langle\overline{\nabla}_{X_i}\Big(\frac{\lambda'}{\lambda}(X_i - \langle\partial_r,X_i\rangle\partial_r)\Big),\nu\Big\rangle \\
&= \frac{\lambda'}{\lambda}\langle\overline{\nabla}_{X_i}X_i,\nu\rangle - \Big(\frac{\lambda'}{\lambda}\Big)'\overline{\nabla}_{X_i}r\langle\partial_r,X_i\rangle\langle\partial_r,\nu\rangle - \frac{\lambda'}{\lambda}\overline{\nabla}_{X_i}\langle\partial_r,X_i\rangle\langle\partial_r,\nu\rangle \\
&\quad - \frac{\lambda'}{\lambda}\langle\partial_r,X_i\rangle\langle\overline{\nabla}_{X_i}\partial_r,\nu\rangle \\
&= -H\frac{\lambda'}{\lambda} - \Big(\frac{\lambda'}{\lambda}\Big)'|\partial_r^\top|\langle\partial_r,\nu\rangle - \frac{\lambda'}{\lambda}(\langle\overline{\nabla}_{X_i}\partial_r,X_i\rangle + \langle\partial_r,\overline{\nabla}_{X_i}X_i\rangle)\langle\partial_r,\nu\rangle \\
&\quad + \Big(\frac{\lambda'}{\lambda}\Big)^2|\langle\partial_r,X_i\rangle|^2\langle\partial_r,\nu\rangle \\
&= -H\frac{\lambda'}{\lambda} - \Big(2\Big(\frac{\lambda'}{\lambda}\Big)^2 - \frac{\lambda''}{\lambda}\Big)|\partial_r^\top|\langle\partial_r,\nu\rangle - \frac{\lambda'}{\lambda}\cdot\frac{\lambda'}{\lambda}(n - |\partial_r^\top|^2)\langle\partial_r,\nu\rangle \\
&\quad + H\frac{\lambda'}{\lambda}|\langle\partial_r,\nu\rangle|^2 \\
&= -H\frac{\lambda'}{\lambda} - n\Big(\frac{\lambda'}{\lambda}\Big)^2\langle\partial_r,\nu\rangle + H\frac{\lambda'}{\lambda}|\langle\partial_r,\nu\rangle|^2 + \Big(3\Big(\frac{\lambda'}{\lambda}\Big)^2 - \frac{\lambda''}{\lambda}\Big)|\partial_r^\top|\langle\partial_r,\nu\rangle,
\end{aligned}
\tag{3.8}
$$

$$
2\langle\overline{\nabla}_{X_i}\partial_r,\overline{\nabla}_{X_i}\nu\rangle = 2\frac{\lambda'}{\lambda}\langle X_i - \langle\partial_r,X_i\rangle\partial_r,\overline{\nabla}_{X_i}\nu\rangle = 2H\frac{\lambda'}{\lambda} - 2\frac{\lambda'}{\lambda}h(\partial_r^\top,\partial_r^\top),
\tag{3.9}
$$

$$
\begin{aligned}
\langle\partial_r,\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}\nu\rangle &= \langle\partial_r,X_j\rangle\langle X_j,\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}\nu\rangle + \langle\partial_r,\nu\rangle\langle\nu,\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}\nu\rangle \\
&= \langle\partial_r,X_j\rangle\overline{\nabla}_{X_i}h(X_i,X_j) - \langle\partial_r,X_j\rangle\langle(\overline{\nabla}_{X_i}X_j)^\perp,\overline{\nabla}_{X_i}\nu\rangle \\
&\quad + \langle\partial_r,\nu\rangle\langle\nu,\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}\nu\rangle \\
&= \langle\partial_r,X_j\rangle(X_j(H) + \overline{\mathrm{Ric}}(\nu,X_j)) - \langle\partial_r,\nu\rangle|A|^2 \\
&= \langle\partial_r,\mathrm{grad}H\rangle + \overline{\mathrm{Ric}}(\nu,\partial_r) - \langle\nu,\partial_r\rangle\overline{\mathrm{Ric}}(\nu,\nu) - \langle\partial_r,\nu\rangle|A|^2,
\end{aligned}
\tag{3.10}
$$

combining (3.5)–(3.10), we reach at

$$\triangle\phi = H\lambda' + \lambda\langle\partial_r, \mathrm{grad}H\rangle - \frac{\lambda}{v}|A|^2 + \lambda\overline{\mathrm{Ric}}(\nu, \partial_r) - \frac{\lambda}{v}\overline{\mathrm{Ric}}(\nu, \nu).$$

Combining this with (3.3) yields (iii).

(iv) From (ii) and (iii), we have

$$\frac{\partial\psi}{\partial t} = -\frac{1}{H^2\phi}\frac{\partial H}{\partial t} - \frac{1}{H\phi^2}\frac{\partial\phi}{\partial t} = -\frac{1}{H^4\phi}\Big(\triangle H - \frac{2}{H}|\nabla H|^2\Big) - \frac{1}{H^3\phi^2}\Big(\triangle\phi - \lambda\overline{\mathrm{Ric}}\Big(\nu, \frac{\partial}{\partial r}\Big)\Big)$$

and

$$\triangle\psi = \frac{1}{H^2\phi}\Big(\triangle H - \frac{2}{H}|\nabla H|^2\Big) - \frac{1}{H\phi^2}\triangle\phi - \frac{2}{H\phi^3}|\nabla\phi|^2 + \frac{2}{H^2\phi^2}\nabla H\cdot\nabla\phi.$$

Then, we deduce

$$\frac{\partial\psi}{\partial t} = \frac{1}{H^2}\triangle\psi + 2\phi\psi^2\nabla\phi\cdot\nabla\psi + \lambda\psi^3\phi\overline{\mathrm{Ric}}\Big(\nu, \frac{\partial}{\partial r}\Big).$$

Note $\overline{\mathrm{Ric}}(\nu, \frac{\partial}{\partial r}) = -\frac{n}{v}\cdot\frac{\lambda''}{\lambda}$. (iv) follows immediately.

4 Rotationally Symmetric Spaces with Nonpositive Sectional Curvature and Euclidean Volume Growth

From (2.5), due to Theorem 2.1, we only need to solve corresponding ODE,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\lambda(s)}{n\lambda'(s)}.$$

Hence we obtain

$$\inf_{y\in\mathbb{S}^n}\ln\lambda(r(y,0)) \le \ln\lambda(r(x,t)) - \frac{t}{n} \le \sup_{y\in\mathbb{S}^n}\ln\lambda(r(y,0)), \quad \forall\, x\in\mathbb{S}^n,\ t\in[0,T). \tag{4.1}$$

If we differentiate (2.10) with respect to the operator $D^k\varphi D_k$, define $\omega = \frac{1}{2}|D\varphi|^2$, and $F = \frac{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}}{v^2}$, then we obtain

$$\frac{\partial\omega}{\partial t} + \frac{1}{F^2}\Big(-a^{ij}D_k(D_iD_j\varphi)D^k\varphi + a^iD_i\omega + 2\frac{n\lambda\lambda''}{v^2}\omega\Big) = 0,$$

where $a^{ij} = \frac{\widetilde{\sigma}^{ij}}{v^2}$, $a^i = \frac{\partial F}{\partial\varphi_i}$ and $F = H\frac{\lambda}{v}$.

Applying Ricci identity and the Gauss equation $\widetilde{R}_{ijkl} = \sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk}$, we deduce

$$\begin{aligned}&\frac{\partial\omega}{\partial t} + \frac{1}{F^2}\Big(-a^{ij}D_iD_j\omega + a^iD_i\omega + a^{ij}\sigma_{ij}|D\varphi|^2 - a^{ij}\varphi_i\varphi_j + \frac{n\lambda\lambda''}{v^2}|D\varphi|^2\Big)\\ &= -\frac{a^{ij}}{F^2}\sigma^{kl}D_iD_k\varphi\cdot D_jD_l\varphi \le 0,\end{aligned} \tag{4.2}$$

where the above inequality follows from this simply fact: if matrix A, B, X are symmetry, and $A>0, B>0$, i.e., A, B are both positive definite, then we have $\mathrm{tr}(AXBX)\ge 0$. In fact, there exist a reversible matrix P and diagonal matrices $\Lambda_1, \Lambda_2 > 0$, s.t. $A = P\Lambda_1P^{\mathrm{T}}, B = P\Lambda_2P^{\mathrm{T}}$. If we define $Y = P^{\mathrm{T}}XP$, then

$$\mathrm{tr}(AXBX) = \mathrm{tr}(P\Lambda_1P^{\mathrm{T}}XP\Lambda_2P^{\mathrm{T}}X) = \mathrm{tr}(\Lambda_1Y^{\mathrm{T}}\Lambda_2Y)\ge 0.$$

Lemma 4.1 *If θ is a constant and $(a^{ij})_{n\times n} \geq \theta(\sigma^{ij})_{n\times n}$, then $a^{ij}\sigma_{ij}|D\varphi|^2 - a^{ij}\varphi_i\varphi_j \geq (n-1)\theta|D\varphi|^2$.*

Proof Let $A = (a^{ij})_{n\times n}$, $\sigma = (\sigma_{ij})_{n\times n}$. Then there exist a reversible matrix P and real diagonal matrices $\widetilde{\Lambda} = \mathrm{diag}\{\mu_1, \cdots, \mu_n\}$, $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$, s.t. $A = P\widetilde{\Lambda}P^{\mathrm{T}}, \sigma^{-1} = P\Lambda^{-1}P^{\mathrm{T}}$.

By $(a^{ij})_{n\times n} \geq \theta(\sigma^{ij})_{n\times n}$, we have $\mu_i \geq \theta\lambda_i^{-1}$ for each i. Then

$$\begin{aligned}
a^{ij}\sigma_{ij}|D\varphi|^2 - a^{ij}\varphi_i\varphi_j &= \sum_i (\mu_i\lambda_i(D\varphi)^{\mathrm{T}}P\Lambda^{-1}P^{\mathrm{T}}D\varphi) - (D\varphi)^{\mathrm{T}}P\widetilde{\Lambda}P^{\mathrm{T}}D\varphi \\
&= \sum_i \mu_i\lambda_i\Big((D\varphi)^{\mathrm{T}}P\Lambda^{-1}P^{\mathrm{T}}D\varphi - \frac{1}{\lambda_i}(P^tD\varphi)_i^2\Big) \\
&\geq \theta\sum_i \Big((D\varphi)^{\mathrm{T}}P\Lambda^{-1}P^{\mathrm{T}}D\varphi - \frac{1}{\lambda_i}(P^{\mathrm{T}}D\varphi)_i^2\Big) \\
&= \theta(n-1)|D\varphi|^2. \qquad (4.3)
\end{aligned}$$

where $(P^{\mathrm{T}}D\varphi)_i$ is the i-th component of vector $P^{\mathrm{T}}D\varphi$.

Due to Theorem 2.1, $\omega \leq \sup\limits_{\mathbb{S}^n}\omega(x,0)$. Therefore, v is uniformly bounded, matrix $\widetilde{\sigma}$ is uniformly bounded, and a^{ij} is uniformly elliptic.

If we differentiate (2.10) with respect to t, we obtain

$$\frac{\partial}{\partial t}\frac{\partial\varphi}{\partial t} - \frac{a^{ij}}{F^2}D_iD_j\frac{\partial\varphi}{\partial t} + \frac{1}{F^2}\frac{\partial F}{\partial\varphi_i}\Big(\frac{\partial\varphi}{\partial t}\Big)_i = -\frac{1}{F^2}\frac{\partial F}{\partial\varphi}\frac{\partial\varphi}{\partial t} = -\frac{1}{F^2}\frac{n\lambda\lambda''}{v^2}\frac{\partial\varphi}{\partial t}. \qquad (4.4)$$

By Theorem 2.1, $\frac{\partial\varphi}{\partial t}$ is uniformly bounded. From estimates of [6, Section 5.5], $|\varphi|_{2,\alpha}$ is bounded.

Since $H = \frac{n\lambda'}{\lambda v} - \frac{1}{\lambda v}\widetilde{\sigma}^{ij}\varphi_{i,j}$, we have

$$H \leq \frac{n\lambda' + C}{\lambda v} \leq \frac{n\lambda' + C}{\lambda}, \qquad (4.5)$$

where C is a constant depending on the initial hypersurface M_0 and the dimension n.

In this paper, we always define C and θ to be generic positive constants which only depend on the initial hypersurface M_0 and the dimension n, otherwise, we will specify it.

In the rest of this section, we always suppose that λ' is bounded from above.

Then we have

$$\frac{\lambda^2}{1+\lambda\lambda''}H^2 \leq \frac{(n\lambda'+C)^2}{1+\lambda\lambda''} \leq C_1, \qquad (4.6)$$

where C_1 is a positive constant depending on the initial hypersurface M_0 and the dimension n.

Theorem 4.1 *If $\omega = \frac{1}{2}|D\varphi|^2$ is as above, then there exists $\theta > 0$ s.t. $\omega \leq C\cdot \mathrm{e}^{-\theta t}$.*

Proof Let $f(t) = \sup\limits_{x\in\mathbb{S}^n}\omega(x,t)$. Applying Theorem 2.1 to (4.2), we obtain

$$\exists\, C > 0, \quad \text{s.t.} \quad \frac{\mathrm{d}f}{\mathrm{d}t} + C\inf_{\mathbb{S}^n}\Big(\Big(\frac{1}{\lambda^2} + \frac{\lambda''}{\lambda}\Big)\frac{1}{H^2}\Big)f \leq 0.$$

Then by (4.5),

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{C}{C_1}f \leq 0;$$

integrating both sides yields the conclusion.

Combining Theorems 4.1, we use interpolation theorems to get

$$\exists\, C > 0,\ \theta_1 > 0, \quad \text{s.t.} \quad |D^2\varphi| \leq C\cdot\mathrm{e}^{-\theta_1 t}. \qquad (4.7)$$

Lemma 4.2 *The mean curvature $H(t)$ of $M(t)$ satisfy*

$$C_1 \le H(x,t)\cdot \mathrm{e}^{\frac{t}{n}} \le C_2,$$

where C_1 and C_2 are positive constants.

Proof From (4.5) and (4.1), the right-hand side inequality is satisfied clearly. Then let us prove the left-hand side one. Recall Lemma 3.1(iv), we have

$$\frac{\partial\psi}{\partial t} = \phi^2\psi^2\triangle\psi + 2\phi\psi^2\nabla\phi\cdot\nabla\psi - n\psi^3\phi^2\frac{\lambda''}{\lambda}.$$

By Theorem 2.1, we obtain

$$\psi \le \sup_{\mathbb{S}^n}\psi(x,0), \quad \text{i.e.,} \quad H\phi \ge \inf_{\mathbb{S}^n}(H\phi)(x,0).$$

By (4.1), we have

$$\phi \le \lambda(r) \le C\cdot \mathrm{e}^{\frac{t}{n}}.$$

Combining the above inequality, we get the conclusion.

Since (2.8) can be rewritten as

$$\frac{\partial\varphi}{\partial t} = \frac{v}{H\lambda},$$

by Lemma 4.2, it is clearly that the time of the solution to (2.8) is infinite. So the solution to (1.2) exists in $\mathbb{S}^n\times[0,+\infty)$.

We consider the rescaled surfaces

$$\widetilde{X}(t) = X(t)\mathrm{e}^{-\frac{t}{n}}, \quad \forall\, x\in\mathbb{S}^n,\ t\in[0,\infty), \tag{4.8}$$

Correspondingly, radial function $\widetilde{r}(x,t) = r(x,t)\mathrm{e}^{-\frac{t}{n}}$, and $\widetilde{g}_{ij},\widetilde{h}_{ij},\widetilde{v}$ denote the first, second fundament form and volume form of $\widetilde{X}$, respectively.

Since $1\le\lambda'<\infty$, $\widetilde{r}$ has an upper bound and a positive lower bound. Moreover, we claim the existence of limit of $\widetilde{r}$. We want to estimate the velocity of $\widetilde{r}$ with respect to the time t,

$$\frac{\partial\widetilde{r}}{\partial t} = \frac{\partial r}{\partial t}\mathrm{e}^{-\frac{t}{n}} - \frac{r}{n}\mathrm{e}^{-\frac{t}{n}} = \frac{v}{H}\mathrm{e}^{-\frac{t}{n}} - \frac{r}{n}\mathrm{e}^{-\frac{t}{n}} = \Big(\frac{\lambda v^2}{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}} - \frac{r}{n}\Big)\mathrm{e}^{-\frac{t}{n}}. \tag{4.9}$$

If we define $f(x,t) = \frac{\lambda v^2}{n\lambda'-\widetilde{\sigma}^{ij}\varphi_{i,j}} - \frac{\lambda}{n\lambda'}$, then by Theorem 4.1 and (4.7), it is not hard to see that $|f(x,t)|\le C\cdot\mathrm{e}^{(\frac{1}{n}-\theta)t}$ for a sufficient small $\theta>0$. Thus we have

$$\begin{aligned}\frac{\partial\widetilde{r}}{\partial t} &= \Big(f + \frac{\lambda}{n\lambda'} - \frac{r}{n}\Big)\mathrm{e}^{-\frac{t}{n}}\\ &= f\cdot\mathrm{e}^{-\frac{t}{n}} + \frac{1}{n\lambda'(r)}\Big(\int_0^r\lambda'(s)\mathrm{d}s - r\lambda'(r)\Big)\mathrm{e}^{-\frac{t}{n}}\\ &= f\cdot\mathrm{e}^{-\frac{t}{n}} - \frac{1}{n\lambda'(r)}\mathrm{e}^{-\frac{t}{n}}\int_0^r\int_s^r\lambda''(u)\mathrm{d}u\mathrm{d}s\\ &= f\cdot\mathrm{e}^{-\frac{t}{n}} - \frac{1}{n\lambda'(r)}\mathrm{e}^{-\frac{t}{n}}\int_0^r s\lambda''(s)\mathrm{d}s\end{aligned} \tag{4.10}$$

Since $\lambda''\ge0$ and $\widetilde{r}$ is bounded, from $\int_0^\infty|f|\mathrm{e}^{-\frac{t}{n}}\mathrm{d}t<\infty$ and $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{v^2}\frac{\partial r}{\partial t}$, we conclude

$$\int_0^\infty\frac{1}{n\lambda'(r)}\mathrm{e}^{-\frac{t}{n}}\int_0^r s\lambda''(s)\mathrm{d}s\,\mathrm{d}t<\infty.$$

Thus the limit of $\widetilde{r}$ exists for any fixed $x \in \mathbb{S}^n$.

If we differentiate $\widetilde{r}$ with respect to the operator D of $\mathbb{S}^n$, then we have

$$D\widetilde{r} = \mathrm{e}^{-\frac{t}{n}} Dr = \lambda \cdot D\varphi \cdot \mathrm{e}^{-\frac{t}{n}} \to 0, \quad t \to \infty.$$

Hence $\widetilde{r}$ converges to a positive constant κ, uniformly. The metric of $\widetilde{X}(t)$ is

$$\widetilde{g}_{ij} = \mathrm{e}^{-\frac{2t}{n}} r_i r_j + \lambda^2(\widetilde{r})\sigma_{ij} = \lambda^2(r(t))\mathrm{e}^{-\frac{2t}{n}} \varphi_i\varphi_j + \lambda^2(\widetilde{r})\sigma_{ij} \to \lambda^2(\kappa)\sigma_{ij}.$$

Moreover,

$$\widetilde{\varphi}_{i,j} = \frac{1}{\lambda(\widetilde{r})}\widetilde{r}_{i,j} - \frac{\lambda'(\widetilde{r})}{\lambda^2(\widetilde{r})}\widetilde{r}_i\widetilde{r}_j = \frac{1}{\lambda(\widetilde{r})}\frac{\lambda(r)}{e^{\frac{t}{n}}}\frac{r_{i,j}}{\lambda(r)} - \frac{\lambda'(\widetilde{r})}{\lambda^2(\widetilde{r})}\Big(\frac{\lambda(r)}{e^{\frac{t}{n}}}\Big)^2 \varphi_i\varphi_j.$$

Due to $\varphi_{i,j} \to 0$ $(t \to \infty)$, by (2.7), we have $\frac{r_{i,j}}{\lambda(r)} \to 0(t \to \infty)$. From Theorem 4.1, we deduce $\widetilde{\varphi}_{i,j} \to 0$ $(t \to \infty)$. Hence

$$\widetilde{h}^i_j = \frac{\lambda'(\widetilde{r})}{\lambda(\widetilde{r})\widetilde{v}}\delta^i_j - \frac{1}{\lambda(\widetilde{r})\widetilde{v}}\widetilde{\sigma^{ik}}\widetilde{\varphi}_{k,j} \to \frac{\lambda'(\kappa)}{\lambda(\kappa)}\delta^i_j.$$

So the rescaled surface converges to a sphere of radius of κ. Fortunately, we can determine this unique κ. By (2.11), we have

$$\frac{\partial}{\partial t}\sqrt{\det g} = \frac{\sqrt{\det g}}{2} g^{ij}\frac{\partial g_{ij}}{\partial t} = \frac{\sqrt{\det g}}{2} g^{ij}\frac{2}{H}h_{ij} = \sqrt{\det g},$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Area}(M_t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} \mathrm{d}\mu_t = \mathrm{Area}(M_t).$$

Thus, we get $\mathrm{Area}(M_t) = \mathrm{Area}(M_0)\cdot \mathrm{e}^t$. Since $\widetilde{r}$ converges to a positive constant κ, uniformly, there exists function $\varepsilon(x,t)$, s.t.

$$\lim_{t\to\infty}\max_{\mathbb{S}^n}|\varepsilon(x,t)| = 0 \quad \text{and} \quad r(x,t) = (\kappa + \varepsilon(x,t))\mathrm{e}^{\frac{t}{n}}.$$

For any $\delta > 0$, let $\lambda'(\infty) \triangleq \lim_{r\to\infty}\lambda'(r)$. When t is sufficiently large, we have $\lambda(\frac{\kappa}{2}\mathrm{e}^{\frac{t}{n}}) > (\lambda'(\infty) - \delta)(\frac{\kappa}{2}\mathrm{e}^{\frac{t}{n}})$. Then

$$\begin{aligned}\mathrm{Area}(M_0) = \mathrm{e}^{-t}\mathrm{Area}(M_t) &= \mathrm{e}^{-t}\int_{\mathbb{S}^n}\lambda^n(r)\mathrm{d}\sigma\\ &\geq \mathrm{e}^{-t}\cdot\lambda^n\Big(\Big(\kappa + \min_{\mathbb{S}^n}\varepsilon(x,t)\Big)\mathrm{e}^{\frac{t}{n}}\Big)\cdot|\mathbb{S}^n|\\ &\geq (\lambda'(\infty)-\delta)^n\Big(\kappa + \min_{\mathbb{S}^n}\varepsilon(x,t)\Big)^n\cdot|\mathbb{S}^n|,\end{aligned}$$

where $|\mathbb{S}^n|$ is Lebesgue measure of n-sphere in Euclidean space. Let t go to infinite. Then

$$\mathrm{Area}(M_0) \geq (\lambda'(\infty)-\delta)^n\kappa^n\cdot|\mathbb{S}^n|.$$

Since δ is arbitrary, we have

$$\mathrm{Area}(M_0) \geq (\lambda'(\infty))^n\kappa^n\cdot|\mathbb{S}^n|.$$

Similarly, we can get

$$\mathrm{Area}(M_0) \leq (\lambda'(\infty))^n\kappa^n\cdot|\mathbb{S}^n|.$$

Hence, we obtain

$$\kappa = \frac{1}{\lambda'(\infty)}\Big(\frac{\mathrm{Area}(M_0)}{|\mathbb{S}^n|}\Big)^{\frac{1}{n}}.$$

5 Hyperbolic Space

When N is the hyperbolic space (whose sectional curvature is -1), the metric is

$$\overline{g} = \mathrm{d}r^2 + \sinh^2(r)\sigma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

under the geodesic polar coordinates, that is, we choose $\lambda(r) = \sinh(r)$. Inheriting the notations before, $\phi = \sinh r\langle\frac{\partial}{\partial r}, \nu\rangle$, $\psi = \frac{1}{H\phi}$, $\widetilde{\sigma}_{ij} = \sigma_{ij} + \frac{1}{\sinh^2 r}r_i r_j$, $g_{ij} = \sinh^2 r \cdot \widetilde{\sigma}_{ij}$, $\varphi(x,t) = \int_c^{r(x,t)} \frac{1}{\sinh(s)}\mathrm{d}s$ and $\widetilde{\sigma}^{ij} = \sigma^{ij} - \frac{1}{\sinh^2 r}\frac{D^i r D^j r}{v^2}$. We obtain

$$\begin{aligned} h^i_j = g^{ik}h_{kj} &= \frac{\cosh r}{v\cdot\sinh r}\delta^i_j - \frac{1}{v\cdot\sinh r}\widetilde{\sigma}^{ik}\varphi_{k,j}, \\ &= \frac{\cosh r}{v\cdot\sinh r}\delta^i_j + \frac{\cosh r}{v^3\sinh^3 r}D^i r D_j r - \frac{\widetilde{\sigma}^{ik}r_{k,j}}{v\cdot\sinh^2 r}. \end{aligned} \tag{5.1}$$

Moreover, if we define $M_{ij} \triangleq H\cdot h_{ij}$ and $M^i_j \triangleq g^{ik}\cdot M_{kj}$, then we have the following result.

Lemma 5.1 (i) $\dfrac{\partial}{\partial t}h_{ij} = \dfrac{\triangle h_{ij}}{H^2} - \dfrac{2}{H^3}\nabla_i H\nabla_j H + \dfrac{|A|^2}{H^2}h_{ij} + \dfrac{n}{H}h_{ij}$,

(ii) $\dfrac{\partial H}{\partial t} = \dfrac{\triangle H}{H^2} - \dfrac{2}{H^3}|\nabla H|^2 - \dfrac{|A|^2}{H} + \dfrac{n}{H}$,

(iii) $\dfrac{\partial\phi}{\partial t} = \dfrac{1}{H^2}(\triangle\phi + |A|^2\phi)$,

(iv) $\dfrac{\partial\psi}{\partial t} = \phi^2\psi^2\triangle\psi + 2\phi\psi^2\nabla\phi\cdot\nabla\psi - n\psi^3\phi^2$,

(v) $\dfrac{\partial M_{ij}}{\partial t} = \dfrac{1}{H^2}\triangle M_{ij} - \dfrac{2}{H^2}\nabla_i H\cdot\nabla_j H + \dfrac{2n}{H^2}M_{ij} - \dfrac{2}{H^3}\langle\nabla H, \nabla M_{ij}\rangle$,

(vi) $\dfrac{\partial M^k_i}{\partial t} = \dfrac{1}{H^2}\triangle M^k_i - \dfrac{2}{H^3}\langle\nabla H, \nabla M^k_i\rangle - \dfrac{2}{H^2}g^{jk}\nabla_i H\cdot\nabla_j H + \dfrac{2n}{H^2}M^k_i - \dfrac{2}{H^2}M_{ij}M^{jk}$,

(vii) $\dfrac{\partial|A|^2}{\partial t} = \dfrac{1}{H^2}\triangle|A|^2 - \dfrac{4}{H^3}h^{ij}\nabla_i H\nabla_j H - \dfrac{2}{H^2}\nabla h^j_i\cdot\nabla h^i_j + 2\dfrac{n+|A|^2}{H^2}|A|^2 - \dfrac{4}{H}h^j_i h^k_j h^i_k$.

Proof Since N has constant sectional curvature -1, we have $\overline{R}_{ijkl} = -g_{ik}g_{jl} + g_{il}g_{jk}$, $\overline{R}_{i0j0} = -g_{ij}$ and $\overline{R}_{0ijk} = 0$. Hence by Lemma 3.1(i)–(iv), Lemma 5.1(i)–(iv) follow easily. Combining (i), (ii) and (2.11), it is not hard to show (v)–(vii).

By (4.1), we have

$$\inf_{y\in\mathbb{S}^n}\ln(\sinh(r(y,0))) \le \ln(\sinh(r(x,t))) - \frac{t}{n} \le \sup_{y\in\mathbb{S}^n}\ln(\sinh(r(y,0))), \quad \forall\, x\in\mathbb{S}^n,\ t\in[0,T). \tag{5.2}$$

Since $\phi = \sinh r\langle\frac{\partial}{\partial r}, \nu\rangle$, due to $|A|^2 \ge \frac{H^2}{n}$ and Theorem 2.1, we have

$$\phi \ge \mathrm{e}^{\frac{t}{n}}\cdot\min_{x\in\mathbb{S}^n}\phi(x,0). \tag{5.3}$$

Let $R_1 = \min\limits_{x\in\mathbb{S}^n}\phi(x,0)$, $R_2 = \max\limits_{x\in\mathbb{S}^n}\sinh r(x,0)$. By (5.2) and (5.3), we have

$$R_1 \le \phi\cdot\mathrm{e}^{-\frac{t}{n}} \le R_2. \tag{5.4}$$

From Lemma 3.1(iv), we define $f(t) = \max\limits_{\mathbb{S}^n}\psi(x,t)$. Then by (5.4) and Theorem 2.1, we deduce

$$\frac{\mathrm{d}f}{\mathrm{d}t} \le -nf^3R_1^2\mathrm{e}^{\frac{2t}{n}} \Rightarrow \psi^{-2} \ge \inf_{x\in\mathbb{S}^n}(H\phi)^2(x,0) + n^2R_1^2(\mathrm{e}^{\frac{2}{n}t} - 1).$$

Similarly, we have

$$\psi^{-2} \leq \sup_{x\in\mathbb{S}^n} (H\phi)^2(x,0) + n^2 R_2^2 (\mathrm{e}^{\frac{2}{n}t} - 1).$$

Therefore, we have

$$\begin{aligned}&\frac{1}{R_2}\sqrt{\mathrm{e}^{-\frac{2}{n}t}\inf_{x\in\mathbb{S}^n}(H\phi)^2(x,0) + n^2R_1^2(1-\mathrm{e}^{-\frac{2}{n}t})}\\ \leq H \leq &\frac{1}{R_1}\sqrt{\mathrm{e}^{-\frac{2}{n}t}\sup_{x\in\mathbb{S}^n}(H\phi)^2(x,0) + n^2R_2^2(1-\mathrm{e}^{-\frac{2}{n}t})}.\end{aligned} \tag{5.5}$$

In particular, when the initial surface is a ball, equality is attained.

Using Lemma 5.1(vi) and Theorem 2.1, we know that the largest eigenvalue of matrix of h^i_j is bounded from above. Since $H = \sum_{i=1}^n h^i_i$ is bounded, all the eigenvalues of matrix of h^i_j is bounded. From (2.11), we know that the solution to (1.2) exists in $\mathbb{S}^n \times [0, +\infty)$.

From (4.2), then by (5.5) and Theorem 2.1, we get $\exists\, \theta_3 > 0$, s.t. $\omega \leq \sup_{x\in\mathbb{S}^n} \omega(x,0)\mathrm{e}^{-\theta_3 t}$. Moreover, (4.4) becomes

$$\frac{\partial}{\partial t}\frac{\partial\varphi}{\partial t} - \frac{a^{ij}}{F^2}D_iD_j\Big(\frac{\partial\varphi}{\partial t}\Big) + \frac{1}{F^2}\frac{\partial F}{\partial\varphi_i}\Big(\frac{\partial\varphi}{\partial t}\Big)_i = -\frac{n}{H^2}\frac{\partial\varphi}{\partial t}. \tag{5.6}$$

Due to Theorem 2.1, $\frac{\partial\varphi}{\partial t}$ is uniformly bounded. Also using estimates of [7, Section 5.5], we get that $|\varphi|_{2,\alpha}$ is bounded. Due to interpolation theorems, we know

$$\exists\, C > 0,\ \theta_4 > 0, \quad \text{s.t.} \quad |D^2\varphi| \leq C \cdot \mathrm{e}^{-\theta_4 t}.$$

Since $\frac{\partial r}{\partial t} = \frac{\sinh(r)v^2}{n\cosh(r) - \widetilde{\sigma}^{ij}\varphi_{i,j}}$ and (5.1), we know $\exists\, 0 < \beta < \frac{1}{n}, C > 0$, s.t.

$$\Big|\frac{\partial r}{\partial t} - \frac{1}{n}\Big| \leq C \cdot \mathrm{e}^{-\beta t}$$

and

$$\begin{aligned}|h^i_j - \delta^i_j| &\leq \Big|\frac{\cosh r}{v\sinh r} - 1\Big|\delta^i_j + \frac{1}{v\sinh r}|\widetilde{\sigma}^{ik}\varphi_{k,j}|\\ &\leq \frac{1}{v}\Big|\frac{\cosh r}{\sinh r} - 1\Big| + \Big|\frac{1}{v} - 1\Big| + \frac{1}{v\sinh r}|\widetilde{\sigma}^{ik}\varphi_{k,j}| \leq C\cdot \mathrm{e}^{-(\frac{1}{n}+\beta)t}.\end{aligned}$$

Hence

$$|H| \leq n + C\cdot \mathrm{e}^{-(\frac{1}{n}+\beta)t}. \tag{5.7}$$

Let $f(t) = \max_{x\in\mathbb{S}^n} \omega(x,t)$ in (4.2). Using Theorem 2.1 and (5.7), we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{2n}{\max_{\mathbb{S}^n} H^2} f \leq 0, \quad \text{i.e.,} \quad \frac{\mathrm{d}f}{\mathrm{d}t} \leq -\frac{2nf}{n^2 + C\mathrm{e}^{-\beta t}}.$$

From the above differential inequality, we deduce

$$|D\varphi|^2 = 2\omega \leq 2f(t) \leq C\cdot\mathrm{e}^{-\frac{2}{n}t}. \tag{5.8}$$

From (5.1), we have

$$\Big|\frac{\sinh r}{\cosh r}h^i_j - \delta^i_j\Big| \leq \Big|\frac{1}{v} - 1\Big|\delta^i_j + \frac{1}{v\cosh r}|\widetilde{\sigma}^{ik}\varphi_{k,j}| \leq C\cdot\mathrm{e}^{-(\frac{1}{n}+\beta)t}, \tag{5.9}$$

$$|vh^i_j-\delta^i_j|\le\Big|\frac{\cosh r}{\sinh r}-1\Big|\delta^i_j+\frac{1}{\sinh r}|\widetilde{\sigma}^{ik}\varphi_{k,j}|\le C\cdot \mathrm{e}^{-(\frac{1}{n}+\beta)t}. \tag{5.10}$$

Let us consider rescaled surfaces

$$\widetilde{X}(t)=\frac{n}{t}X(t), \tag{5.11}$$

comparing to (4.8). Correspondingly, radial function $\widetilde{r}=\frac{nr}{t}$, and $\widetilde{g}_{ij},\widetilde{h}_{ij},\widetilde{v}$ denote the first, second fundament form and volume form of $\widetilde{X}$, respectively.

By (5.2), it is clearly $\widetilde{r}\to 1$ and $\widetilde{v}\to 1$ as $t\to\infty$. Moreover

$$\widetilde{g}_{ij}=\sinh^2(\widetilde{r})\sigma_{ij}+\widetilde{r}_i\widetilde{r}_j=\sinh^2(\widetilde{r})\sigma_{ij}+\sinh^2(r)\Big(\frac{n}{t}\Big)^2\varphi_i\varphi_j\to\sinh^2(1)\sigma_{ij}.$$

Due to

$$h^i_j=\frac{\cosh r}{v\sinh r}\delta^i_j+\frac{\cosh r}{v^3\sinh^3 r}D^irD_jr-\frac{\widetilde{\sigma}^{ik}\cdot r_{k,j}}{v\sinh^2 r},$$

then we have

$$\widetilde{h}^i_j=\frac{\cosh(\frac{n}{t}r)}{\widetilde{v}\sinh(\frac{n}{t}r)}\delta^i_j+\frac{\cosh(\frac{n}{t}r)}{\widetilde{v}^3\sinh^3(\frac{n}{t}r)}\Big(\frac{n}{t}\Big)^2D^irD_jr-\frac{\widetilde{\widetilde{\sigma}}^{ik}\cdot\frac{n}{t}\cdot r_{k,j}}{\widetilde{v}\sinh^2(\frac{n}{t}r)},$$

where $\widetilde{\widetilde{\sigma}}^{ik}$ is $\widetilde{\sigma}^{ik}$ in the version of $\widetilde{X}$. In order to prove

$$\widetilde{h}^i_j\to\frac{\cosh 1}{\sinh 1}\delta^i_j,$$

we need

$$|D^2r|\le C_1.$$

Since $\varphi_{i,j}=\frac{1}{\sinh r}r_{i,j}-\frac{\cosh r}{\sinh^2 r}r_ir_j$, $\omega\le\sup\limits_{x\in\mathbb{S}^n}\omega(x,0)\mathrm{e}^{-\theta_3t}$, $|D^2r|\le C_1$ is equivalent to $|\varphi_{i,j}|\le C_2\cdot\mathrm{e}^{-\frac{1}{n}t}$, where C_1,C_2 are two different constants.

From above, we know $\exists\, 0<\beta<\frac{1}{n},\ C>0$ s.t. $|D^2\varphi|<C\mathrm{e}^{-\beta t}$.

Let us consider function

$$G\triangleq|A|^2-\frac{2}{v}H+\frac{n}{v^2}=\sum_{i,j}(h^i_j-\delta^i_j)(h^j_i-\delta^j_i).$$

Theorem 5.1

$$\begin{aligned}\frac{\partial G}{\partial t}-\frac{\triangle G}{H^2}=&\frac{4}{H^3}\Big(\frac{1}{v}|\nabla H|^2-h^j_i\nabla^iH\nabla_jH\Big)+2\frac{n+|A|^2}{H^2}|A|^2+\frac{2}{Hv}(|A|^2-n)-\frac{4}{H}h^j_ih^k_jh^i_k\\&-\frac{2}{H^2}\nabla h^j_i\nabla h^i_j+\frac{2}{H}\Big(-\frac{\triangle\omega}{v^3}+\frac{3}{v^5}|\nabla\omega|^2\Big)+\frac{2}{H^2}\nabla\Big(\frac{2}{v}\Big)\cdot\nabla H\\&-\frac{n}{H^2}\Big(-\frac{2\triangle\omega}{v^4}+\frac{8}{v^6}|\nabla\omega|^2\Big)+\frac{2}{v^2}(Hv-n)\frac{1}{H^2\sinh^2 r}\Big[\frac{\widetilde{\sigma}^{ij}}{v^2}\omega_{i,j}-a^i\omega_i\\&-a^{ij}\sigma_{ij}|D\varphi|^2+a^{ij}\varphi_i\varphi_j-\frac{n\sinh^2 r}{v^2}|D\varphi|^2-\frac{\widetilde{\sigma}^{ij}}{v^2}\sigma^{kl}\varphi_{ik}\varphi_{jl}\Big]\\\le&-\frac{4n}{H^2}G+C\cdot\mathrm{e}^{-(\frac{3}{n}+3\beta)t}+C\cdot\mathrm{e}^{-(\frac{4}{n}+\beta)t}.\end{aligned}$$

Proof If we define $\omega = \frac{1}{2}|D\varphi|^2$ as before, then $v = \sqrt{1+2\omega}$ and

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial t}|A|^2 - \frac{2}{v}\frac{\partial H}{\partial t} + \frac{2}{v^4}(Hv-n)\frac{\partial}{\partial t}\omega.$$

By Lemma 5.1(ii), (vii) and (4.2), we have

$$\begin{aligned}
\frac{\partial G}{\partial t} - \frac{\triangle_M G}{H^2} &= \frac{\partial}{\partial t}|A|^2 - \frac{1}{H^2}\triangle|A|^2 - \frac{2}{v}\Big(\frac{\partial H}{\partial t} - \frac{\triangle H}{H^2}\Big) + \frac{1}{H}\triangle\Big(\frac{2}{v}\Big) + \frac{2}{H^2}\nabla\Big(\frac{2}{v}\Big)\cdot\nabla H \\
&\quad + \frac{2}{H^2 v}\triangle H + \frac{2}{v^4}(Hv-n)\dot{\omega} - \frac{1}{H^2}\triangle\Big(\frac{n}{v^2}\Big) \\
&= -\frac{4}{H^3}h_i^j\nabla^i H\nabla_j H - \frac{2}{H^2}\nabla h_i^j\cdot\nabla h_j^i + 2\frac{n+|A|^2}{H^2}|A|^2 - \frac{4}{H}h_i^j h_j^k h_k^i \\
&\quad - \frac{2}{v}\Big(-\frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H} + \frac{n}{H}\Big) + \frac{1}{H}\triangle\Big(\frac{2}{v}\Big) + \frac{2}{H^2}\nabla\Big(\frac{2}{v}\Big)\cdot\nabla H \\
&\quad - \frac{1}{H^2}\triangle\Big(\frac{n}{v^2}\Big) + \frac{2}{v^4}(Hv-n)\frac{1}{F^2}\Big[\frac{\widetilde{\sigma}^{ij}}{v^2}\omega_{i,j} - a^i\omega_i - a^{ij}\sigma_{ij}|D\varphi|^2 \\
&\quad + a^{ij}\varphi_i\varphi_j - \frac{n\sinh^2 r}{v^2}|D\varphi|^2 - \frac{\widetilde{\sigma}^{ij}}{v^2}\sigma^{kl}\varphi_{ik}\varphi_{jl}\Big].
\end{aligned} \tag{5.12}$$

Since $v = \sqrt{1+2\omega}$ and $F = H\frac{\sinh r}{v}$, we have

$$\begin{aligned}
\frac{\partial G}{\partial t} - \frac{\triangle_M G}{H^2} &= \frac{4}{H^3}\Big(\frac{1}{v}|\nabla H|^2 - h_i^j\nabla^i H\nabla_j H\Big) + 2\frac{n+|A|^2}{H^2}|A|^2 + \frac{2}{Hv}(|A|^2-n) - \frac{4}{H}h_i^j h_j^k h_k^i \\
&\quad - \frac{2}{H^2}\nabla h_i^j\nabla h_j^i + \frac{2}{H}\Big(-\frac{\triangle\omega}{v^3} + \frac{3}{v^5}|\nabla\omega|^2\Big) + \frac{2}{H^2}\nabla\Big(\frac{2}{v}\Big)\cdot\nabla H \\
&\quad - \frac{n}{H^2}\Big(-\frac{2\triangle\omega}{v^4} + \frac{8}{v^6}|\nabla\omega|^2\Big) + \frac{2}{v^2}(Hv-n)\frac{1}{H^2\sinh^2 r}\Big[\frac{\widetilde{\sigma}^{ij}}{v^2}\omega_{i,j} - a^i\omega_i \\
&\quad - a^{ij}\sigma_{ij}|D\varphi|^2 + a^{ij}\varphi_i\varphi_j - \frac{n\sinh^2 r}{v^2}|D\varphi|^2 - \frac{\widetilde{\sigma}^{ij}}{v^2}\sigma^{kl}\varphi_{ik}\varphi_{jl}\Big].
\end{aligned}$$

By (2.8), we have

$$g^{ij} = \frac{1}{\sinh^2 r}\widetilde{\sigma}^{ij}.$$

Combining $a^{ij} = \frac{1}{v^2}\widetilde{\sigma}^{ij}$, (5.8) and (5.10), we conclude

$$\begin{aligned}
\frac{\partial G}{\partial t} - \frac{\triangle G}{H^2} &= \frac{4}{H^3}\Big(\frac{1}{v}|\nabla H|^2 - h_i^j\nabla^i H\nabla_j H\Big) + 2\frac{n+|A|^2}{H^2}|A|^2 + \frac{2}{Hv}(|A|^2-n) - \frac{4}{H}h_i^j h_j^k h_k^i \\
&\quad - \frac{2}{H^2}\nabla h_i^j\nabla h_j^i + \frac{2}{H}\Big(-\frac{\triangle\omega}{v^3} + \frac{3}{v^5}|\nabla\omega|^2\Big) + \frac{2}{H^2}\nabla\Big(\frac{2}{v}\Big)\cdot\nabla H \\
&\quad - \frac{n}{H^2}\Big(-\frac{2\triangle\omega}{v^4} + \frac{8}{v^6}|\nabla\omega|^2\Big) + \frac{2}{v^4}(Hv-n)\frac{\triangle\omega}{H^2} - \frac{2(Hv-n)}{H^2v^2\sinh^2 r}a^i\omega_i \\
&\quad + O(\mathrm{e}^{-\frac{5t}{n}}) + O(\mathrm{e}^{-(\frac{3}{n}+3\beta)t}) - \frac{4n}{v^4}\cdot\frac{Hv-n}{H^2}\omega \\
&= \frac{4}{H^3}\Big(\frac{1}{v}\delta_i^j - h_i^j\Big)\nabla^i H\nabla_j H + 2\frac{n+|A|^2}{H^2}|A|^2 + \frac{2}{Hv}(|A|^2-n) - \frac{4}{H}h_i^j h_j^k h_k^i \\
&\quad - 4n\frac{Hv-n}{H^2v^4}\omega - \frac{2}{H^2}\nabla h_i^j\nabla h_j^i + \frac{4}{H^2}\nabla\Big(\frac{1}{v}\Big)\cdot\nabla H - \frac{2(Hv-n)}{H^2v^2\sinh^2 r}a^i\omega_i \\
&\quad + \frac{6}{Hv^5}|\nabla\omega|^2 - \frac{8n}{H^2v^6}|\nabla\omega|^2 + O(\mathrm{e}^{-\frac{5t}{n}}) + O(\mathrm{e}^{-(\frac{3}{n}+3\beta)t}).
\end{aligned}$$

Since $\nabla h_i^j \nabla h_j^i$ is independent of the frame, we can choose normal coodinates $\{X_i\}_{i=1}^n$ on M. Then

$$\nabla h_i^j \nabla h_j^i = \sum_{i,j} \nabla h(X_i, X_j) \nabla h(X_i, X_j) \geq \sum_i \nabla h(X_i, X_i) \nabla h(X_i, X_i) \geq \frac{1}{n} |\nabla H|^2.$$

Hence

$$\begin{aligned}
&\frac{4}{H^3}\Big(\frac{1}{v}\delta_i^j - h_i^j\Big)\nabla^i H \nabla_j H - \frac{2}{H^2}\nabla h_i^j \nabla h_j^i + \frac{4}{H^2}\nabla\Big(\frac{1}{v}\Big)\cdot \nabla H + \frac{6}{Hv^5}|\nabla\omega|^2 - \frac{8n}{H^2 v^6}|\nabla\omega|^2 \\
&\leq -C\mathrm{e}^{-(\frac{1}{n}+\beta)t}|\nabla H|^2 - \frac{2}{nH^2}|\nabla H|^2 + \frac{4}{H^2}\nabla\Big(\frac{1}{v}\Big)\cdot\nabla H - \frac{v}{H}\Big(\frac{8n}{Hv} - 6\Big)\Big|\nabla\Big(\frac{1}{v}\Big)\Big|^2 \\
&\leq \frac{4}{H^4}\Big|\nabla\Big(\frac{1}{v}\Big)\Big|^2 \frac{1}{\frac{2}{nH^2} + C\mathrm{e}^{-(\frac{1}{n}+\beta)t}} - \frac{v}{H}\Big(\frac{8n}{Hv} - 6\Big)\Big|\nabla\Big(\frac{1}{v}\Big)\Big|^2 \\
&\leq C\mathrm{e}^{-(\frac{1}{n}+\beta)t}|\nabla\omega|^2 \leq C\mathrm{e}^{-3(\frac{1}{n}+\beta)t},
\end{aligned}$$

where the second inequality above is obtained by using Cauchy-Schwartz inequality, and the third inequality is obtained by (5.8).

From $a^i = \frac{\partial F}{\partial \varphi_i} = \frac{\partial}{\partial \varphi_i}(\frac{n\lambda' - \widetilde{\sigma}^{ij}\varphi_{i,j}}{v^2})$, we obtain

$$\frac{2(Hv-n)}{H^2 v^2 \sinh^2 r} a^i \omega_i \leq C\mathrm{e}^{-(\frac{4}{n}+\beta)t}.$$

Thus we conclude

$$\begin{aligned}
\frac{\partial}{\partial t}G - \frac{\triangle G}{H^2} \leq\ & 2\frac{n+|A|^2}{H^2}|A|^2 + \frac{2}{Hv}(|A|^2 - n) - \frac{4}{H}h_i^j h_j^k h_k^i - 2n\frac{Hv-n}{H^2 v^4}(v^2-1) \\
&+ O(\mathrm{e}^{-(\frac{4}{n}+\beta)t}) + O(\mathrm{e}^{-(\frac{3}{n}+3\beta)t}).
\end{aligned}$$

If $\{\lambda_i\}_1^n$ are the princple curvatures of M in N, and $\mu_i \triangleq \lambda_i - \frac{1}{v}$, then (5.10) implies

$$|\mu_i| \leq C\cdot \mathrm{e}^{-(\frac{1}{n}+\beta)t}.$$

Combining $v^2 - 1 = 2\omega \leq C\cdot\mathrm{e}^{-\frac{2t}{n}}$, we deduce

$$\begin{aligned}
&2H\sum h_i^j h_j^k h_k^i - |A|^2(|A|^2+n) - \frac{H}{v}(|A|^2 - n) \\
&= 2\sum_i \lambda_i \sum_j \lambda_j^3 - \sum_i (1+\lambda_i^2)\sum_j \lambda_j^2 - \frac{1}{v}\sum_i \lambda_i \sum_j (\lambda_j^2 - 1) \\
&= 2\sum_i\left(\mu_i + \frac{1}{v}\right)\sum_j\left(\mu_j + \frac{1}{v}\right)^3 - \sum_i\left(1 + \left(\mu_i + \frac{1}{v}\right)^2\right)\sum_j\left(\mu_j + \frac{1}{v}\right)^2 \\
&\quad - \frac{1}{v}\sum_i\left(\mu_i + \frac{1}{v}\right)\sum_j\left(\left(\mu_j + \frac{1}{v}\right)^2 - 1\right) \\
&= \sum_{i,j}\left(2\mu_i\mu_j^3 - \mu_i^2\mu_j^2 + \frac{1}{v}\mu_i\mu_j^2 + \frac{2}{v}\mu_j^3 - \mu_j^2 + \frac{3}{v^2}\mu_j^2 - \frac{1}{v}\mu_j + \frac{1}{v^3}\mu_j\right) \\
&= \sum_{i,j}\left(2\mu_i\mu_j^3 - \mu_i^2\mu_j^2 + \mu_i\mu_j^2 + \mu_j^3 + 2\mu_j^2 - \frac{1}{v}\mu_j + \frac{1}{v^3}\mu_j\right) + O(\mathrm{e}^{-(\frac{4}{n}+\beta)t}) \\
&= n\sum_j\left(2\mu_j^2 + \frac{\mu_j}{v^3}(1-v^2)\right) + O(\mathrm{e}^{-(\frac{4}{n}+\beta)t}) + O(\mathrm{e}^{-(\frac{3}{n}+3\beta)t}).
\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t}G-\frac{\triangle G}{H^2}\le&-\frac{2n}{H^2}\sum_j\Big(2\mu_j^2+\frac{\mu_j}{v^3}(1-v^2)+\frac{1}{v^4}(\lambda_j v-1)(v^2-1)\Big)\\&+O(\mathrm{e}^{-(\frac{4}{n}+\beta)t})+O(\mathrm{e}^{-(\frac{3}{n}+3\beta)t})\\=&-\frac{2n}{H^2}\sum_j 2\mu_j^2+O(\mathrm{e}^{-(\frac{4}{n}+\beta)t})+O(\mathrm{e}^{-(\frac{3}{n}+3\beta)t})\\=&-\frac{4n}{H^2}G+O(\mathrm{e}^{-(\frac{4}{n}+\beta)t})+O(\mathrm{e}^{-(\frac{3}{n}+3\beta)t}).\end{aligned}\tag{5.13}$$

Let $f(t)=\sup\limits_{x\in\mathbb{S}^n}G(x,t)$. From Theorem 2.1 and (5.7), we deduce

$$\frac{\mathrm{d}f}{\mathrm{d}t}\le-\frac{4f}{n+C\mathrm{e}^{-\frac{t}{n}}}+C\mathrm{e}^{-(\frac{4}{n}+\beta)t}+C\mathrm{e}^{-(\frac{3}{n}+3\beta)t},$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}[(n\mathrm{e}^{\frac{t}{n}}+C)^4f]\le C(n\mathrm{e}^{\frac{t}{n}}+C)^4(\mathrm{e}^{-(\frac{4}{n}+\beta)t}+\mathrm{e}^{-(\frac{3}{n}+3\beta)t}).$$

From the solution to above differential inequality, we obtain

$$G(x,t)=\sum_{i,j}(h^i_j-\delta^i_j)(h^j_i-\delta^j_i)\le C\cdot\mathrm{e}^{-\frac{3t}{n}}(\mathrm{e}^{-\frac{t}{n}}+\mathrm{e}^{-\frac{5}{2}\beta t}).$$

If we define matrix $B=(B_{ij})$, where $B_{ij}=h^i_j-\delta^i_j$, then matrix B can be diagonalized at a given point, i.e., there exist a reversible matrix P and a real diagonal matrix Λ, such that $B=P\Lambda P^{-1}$, then

$$G=\mathrm{tr}(B^2)=\mathrm{tr}(P\Lambda P^{-1}P\Lambda P^{-1})=\mathrm{tr}(\Lambda^2).$$

We deduce $|B_{ij}|=|h^i_j-\delta^i_j|\le\sqrt{G}$. If $\frac{5}{2}\beta<\frac{1}{n}$, then by (5.1), we have $|D^2\varphi|\le C\cdot\mathrm{e}^{-\frac{t}{2n}-\frac{5}{4}\beta t}$. If we use Theorem 5.1 to iterative the order of $|D^2\varphi|$, then we deduce $G(x,t)\le C\cdot\mathrm{e}^{-\frac{4t}{n}}$, that is $|D^2\varphi|\le C\cdot\mathrm{e}^{-\frac{t}{n}}$. Since

$$h^i_j=\frac{\cosh r}{v\sinh r}\delta^i_j+\frac{\cosh r}{v^3\sinh^3 r}D^irD_jr-\frac{\widetilde{\sigma}^{ik}\cdot r_{k,j}}{v\sinh^2 r},$$

and $\widetilde{\sigma}^{ij}\to\sigma^{ij}$, we have $|D^2r|\le C$, where C is a uniform constant.

Finally,

$$\widetilde{h}^i_j=\frac{\cosh(\frac{n}{t}r)}{\widetilde{v}\sinh(\frac{n}{t}r)}\delta^i_j+\frac{\cosh(\frac{n}{t}r)}{\widetilde{v}^3\sinh^3(\frac{n}{t}r)}\Big(\frac{n}{t}\Big)^2D^irD_jr-\frac{\widetilde{\widetilde{\sigma}}^{ik}\cdot\frac{n}{t}\cdot r_{k,j}}{\widetilde{v}\sinh^2(\frac{n}{t}r)}\to\frac{\cosh 1}{\sinh 1}\delta^i_j,\quad t\to\infty,$$

where $\widetilde{\widetilde{\sigma}}^{ik}$ is $\widetilde{\sigma}^{ik}$ in the version of $\widetilde{X}$, so the rescaled surface converges to a sphere of radius 1.

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