

On Reed-Solomon Codes*

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Abstract The complexity of decoding the standard Reed-Solomon code is a well-known open problem in coding theory. The main problem is to compute the error distance of a received word. Using the Weil bound for character sum estimate, Li and Wan showed that the error distance can be determined when the degree of the received word as a polynomial is small. In the first part, the result of Li and Wan is improved. On the other hand, one of the important parameters of an error-correcting code is the dimension. In most cases, one can only get bounds for the dimension. In the second part, a formula for the dimension of the generalized trace Reed-Solomon codes in some cases is obtained.

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Trace Reed-Solomon code, Trace map

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1 On Improved Bounds for Error Distance of Standard Reed-Solomon Codes

Let \mathbb{F}_q be the finite field of q elements with characteristic p . Fix a subset $\mathcal{D} = \{x_1, \dots, x_n\} \subseteq \mathbb{F}_q$, which is called the evaluation set. The generalized Reed-Solomon code $\mathcal{C}_q(\mathcal{D}, k)$ of length n and dimension k over \mathbb{F}_q is

$$\mathcal{C}_q(\mathcal{D}, k) = \{(f(x_1), \dots, f(x_n)) \in \mathbb{F}_q^n \mid f(x) \in \mathbb{F}_q[x], \deg f(x) \leq k-1\}.$$

Its elements are called codewords. The most widely used cases are $\mathcal{D} = \mathbb{F}_q$ or \mathbb{F}_q^* . These two cases are essentially equivalent. We call the case $\mathcal{D} = \mathbb{F}_q$ the standard Reed-Solomon code. Note that in other literature, the case $\mathcal{D} = \mathbb{F}_q^*$ is called standard.

For a linear code \mathcal{C} of length n over \mathbb{F}_q and a word $u \in \mathbb{F}_q^n$, we define the error distance of u to the code \mathcal{C} to be

$$d(u, \mathcal{C}) = \min_{v \in \mathcal{C}} d(u, v),$$

where $d(\cdot, \cdot)$ denote the Hamming distance. It is clear that $d(u, \mathcal{C}) = 0$ if and only if u is a codeword. The covering radius of \mathcal{C} is defined to be

$$\rho(\mathcal{C}) = \max_{u \in \mathbb{F}_q^n} d(u, \mathcal{C}).$$

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The minimal distance of \mathcal{C} is defined to be

$$d(\mathcal{C}) = \min_{u \neq v \in \mathcal{C}} d(u, v) = \min_{0 \neq v \in \mathcal{C}} d(0, v).$$

It is easy to see that the minimal distance of the generalized Reed-Solomon code $\mathcal{C}_q(\mathcal{D}, k)$ is $n - k + 1$, and its covering radius ρ is $n - k$. The most important algorithmic problem in coding theory is the maximal likelihood decoding (MLD): given a word $u \in \mathbb{F}_q^n$, find a codeword $v \in \mathcal{C}$ such that $d(u, v) = d(u, \mathcal{C})$. The decision version of this problem is essentially computing the error distance $d(u, \mathcal{C})$ for a received word u . This is well-known to be NP-complete.

Given a received word $u \in \mathbb{F}_q^n$, if the error distance is small, say, $d(u, \mathcal{C}) \leq n - \sqrt{nk}$, then the list decoding algorithm of Sudan [11] and Guruswami-Sudan [5] provides a polynomial time algorithm for the decoding of u . When the error distance increases, the decoding becomes more complicated, in fact, NP-complete for generalized Reed-Solomon codes (see [7]).

For $u = (u_1, \dots, u_n) \in \mathbb{F}_q^n$, let

$$u(x) = \sum_{i=1}^n u_i \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} \in \mathbb{F}_q[x],$$

that is, $u(x)$ is the unique polynomial of degree at most $n - 1$ such that $u(x_i) = u_i$ for $1 \leq i \leq n$. For $u \in \mathbb{F}_q^n$, we define $\deg u := \deg u(x)$, called the degree of u . As mentioned above, the fundamental decoding problem is to compute the error distance $d(u, \mathcal{C})$ for received word $u \in \mathbb{F}_q^n$. It is clear that $d(u, \mathcal{C}) = 0$ if and only if $\deg u \leq k - 1$. Without loss of generality, we can assume that $k \leq \deg u \leq n - 1$.

Lemma 1.1 (see [8]) *For $k \leq \deg u \leq n - 1$, we have the inequality*

$$n - \deg u \leq d(u, \mathcal{C}) \leq n - k = \rho.$$

In particular, the word u is called a deep hole if the above upper bound is an equality, i.e., if $d(u, \mathcal{C}) = n - k$. The word u is called ordinary if the above lower bound is an equality, i.e., if $d(u, \mathcal{C}) = n - \deg u$.

By definition, we have the following result.

Lemma 1.2 (see [8]) *Let $u \in \mathbb{F}_q^n$ be a word with $\deg u = k + d$, where $k + 1 \leq k + d \leq n - 1$. Then the error distance $d(u, \mathcal{C}) \leq n - k - r$ ($1 \leq r \leq d$) if and only if there exists a subset $\{x_1, \dots, x_{k+r}\} \subseteq \mathcal{D}$ and a monic polynomial $w(x) \in \mathbb{F}_q[x]$ of degree $d - r$ such that*

$$u(x) - v(x) = (x - x_1) \cdots (x - x_{k+r}) w(x)$$

for some $v(x) \in \mathbb{F}_q[x]$ with $\deg v(x) \leq k - 1$.

From now on, we assume that \mathcal{C} is the standard Reed-Solomon code $\mathcal{C}_q(\mathbb{F}_q, k)$. For the standard Reed-Solomon code \mathcal{C} , the complexity of decoding is unknown and much more subtle. It was shown in [2, 4] to be at least as hard as the discrete logarithm in a large extension \mathbb{F}_{q^h} , where h can be as large as \sqrt{q} . If $\deg u(x) = k$, then u is a deep hole. Based on numerical calculations, Cheng and Murray [1] conjectured that there are no other deep holes for standard

Reed-Solomon codes. As a theoretical evidence, they proved that their conjecture is true if $d := \deg u - k$ is small and q is sufficiently large compared to $d + k$. More precisely, they showed

Proposition 1.1 *Let $u \in \mathbb{F}_q^q$ such that $1 \leq d := \deg u(x) - k \leq q - k - 1$. Assume that*

$$q \geq \max(k^{7+\epsilon}, d^{\frac{13}{3}+\epsilon})$$

for some constant $\epsilon > 0$. Then $d(u, \mathcal{C}) < q - k$, that is, u is not a deep hole.

For the words with small degree represented by a polynomial in $\mathbb{F}_q[x]$, Li and Wan [9] applied the method of Cheng and Wan [2] to study the error distance $d(u, \mathcal{C})$ for the standard Reed-Solomon code. They proved the following two results.

Proposition 1.2 (see [9, Theorem 1.4]) *Let $u \in \mathbb{F}_q^q$ be such that $1 \leq d := \deg u(x) - k \leq q - k - 1$. Assume that*

$$q > \max((k+1)^2, d^{2+\epsilon}), \quad k > \left(\frac{2}{\epsilon} + 1\right)d + \frac{8}{\epsilon} + 2$$

for some constant $\epsilon > 0$. Then $d(u, \mathcal{C}) < q - k$, that is, u is not a deep hole.

More precisely, the error distance can be determined with a similar hypothesis.

Proposition 1.3 (see [9, Theorem 1.5]) *Let $u \in \mathbb{F}_q^q$ be such that $1 \leq d := \deg u(x) - k \leq q - k - 1$. Assume that*

$$q > \max((k+1)^2, d^{2+\epsilon}), \quad k > \left(\frac{4}{\epsilon} + 1\right)d + \frac{4}{\epsilon} + 2$$

for some constant $\epsilon > 0$. Then $d(u, \mathcal{C}) = q - (k + d)$, that is, u is ordinary.

This result can be used to determine the error distance $d(u, \mathcal{C})$ only for the received word $u \in \mathbb{F}_q^q$ with small degree.

In this part, for the standard Reed-Solomon code $\mathcal{C} = \mathcal{C}_q(\mathbb{F}_q, k)$, we generalize the above results. In fact, we prove the following result.

Theorem 1.1 *Let $r \geq 1$ be an integer. For any received word $u \in \mathbb{F}_q^q$ with $u(x)$ as its interpolation polynomial of degree m , if $m \geq k + r$,*

$$q > \max \left\{ 2 \binom{k+r}{2} + (m-k), (m-k)^{2+\epsilon} \right\}$$

and

$$k > \frac{1}{1+\epsilon} \left(r + (2+\epsilon) \left(\frac{m}{2} + 1 \right) \right)$$

for some constant $\epsilon > 0$, then we have $d(u, \mathcal{C}) \leq q - k - r$. So u is not a deep hole.

Taking $h = 0$ in the following proof and $r = 1$ or $m - k$ in Theorem 1.1, we get Propositions 1.2 and 1.3 respectively.

Proof Let $h(x)$ be a fixed monic polynomial in $\mathbb{F}_q[x]$ of degree $0 \leq h \leq \min\{m-k+1, k-1\}$ with no zero in \mathbb{F}_q . Let $\bar{h}(x) = x^{m-k+1}h(\frac{1}{x}) \in \mathbb{F}_q[x]$, $A = (\mathbb{F}_q[x]/(\bar{h}(x)))^*$ and \hat{A} denotes the set of all characters of A . Then $|\hat{A}| = \Phi(\bar{h}(x)) \leq q^{\deg \bar{h}(x)} - 1 = q^{m-k+1} - 1$, where $\Phi(\bar{h}(x))$ is

the Euler function of the polynomial $\bar{h}(x)$, i.e., $\Phi(\bar{h}(x))$ is equal to the number of units in A . Then $\hat{B} = \{\chi \in \hat{A} \mid \chi(\mathbb{F}_q^*) = 1\}$ is an abelian group of order $\leq q^{m-k}$.

For $g(x) \in \mathbb{F}_q[x]$, define

$$\chi(g(x)) = \begin{cases} \chi(g(x) \pmod{\bar{h}(x)}), & \text{if } \gcd(g(x), \bar{h}(x)) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This defines a multiplicative function of the polynomial ring $\mathbb{F}_q[x]$. By the Weil bound as given in [14], if $\chi \neq 1$ and $\chi(\mathbb{F}_q^*) = 1$, we have

$$\left| \sum_{\substack{g(0)=1 \\ \deg g(x)=m-(k+r)}} \Lambda(g(x)) \chi(g(x)) \right| \leq (m-k) q^{\frac{m-(k+r)}{2}}, \quad (1.1)$$

where $\Lambda(g(x))$ is the Von-Mangoldt function on $\mathbb{F}_q[x]$, i.e., $\Lambda(g(x))$ is equal to $\deg P$ if g is a power of an irreducible polynomial P and is otherwise equal to zero.

For a polynomial $f(x) \in \mathbb{F}_q[x]$ of degree at most $k-1-\deg h(x)$ (thus $f(x)$ represents a codeword), the sum

$$\frac{u(x)}{h(x)} + f(x) = \frac{u(x) + f(x)h(x)}{h(x)}$$

has at most $\deg u(x) = m$ roots in \mathbb{F}_q since

$$\deg u(x) = m \geq k + h - 1 \geq \deg f(x)h(x).$$

Then by Lemma 1.2, we know that

$$d(u, \mathcal{C}) \leq q - k - r$$

if there exists a subset $\{x_1, \dots, x_{k+r}\} \subseteq \mathbb{F}_q$ and a monic polynomial $v(x) \in \mathbb{F}_q[x]$ of degree $m - (k+r)$ such that $v(0) \neq 0$ and

$$(x - x_1) \cdots (x - x_{k+r})v(x) = u(x) + f(x)h(x)$$

for some $f(x) \in \mathbb{F}_q[x]$, $\deg f(x) \leq k-1-\deg h(x)$. This is equivalent to the equation

$$(1 - xx_1) \cdots (1 - xx_{k+r})x^{m-(k+r)}v\left(\frac{1}{x}\right) = x^m u\left(\frac{1}{x}\right) + x^m f\left(\frac{1}{x}\right)h\left(\frac{1}{x}\right).$$

If we denote $\tilde{u}(x) = x^m u\left(\frac{1}{x}\right)$, $\tilde{h}(x) = x^h h\left(\frac{1}{x}\right)$, $\tilde{f}(x) = x^{k-1-\deg h(x)} f\left(\frac{1}{x}\right)$, $\tilde{v}(x) = x^{m-(k+r)} v\left(\frac{1}{x}\right)$, then

$$\begin{aligned} \deg \tilde{u}(x) &= \deg u(x) = m, & \deg \tilde{h}(x) &= \deg h(x) = h, \\ \deg \tilde{v}(x) &= m - (k+r), & \deg \tilde{f}(x) &\leq k-1-\deg h(x). \end{aligned}$$

That is

$$(1 - xx_1) \cdots (1 - xx_{k+r})\tilde{v}(x) = \tilde{u}(x) + x^{m-(k+h-1)}\tilde{f}(x)\tilde{h}(x). \quad (1.2)$$

By definition

$$\bar{h}(x) = x^{m-k+1}h\left(\frac{1}{x}\right) = x^{m-k-h+1}\tilde{h}(x),$$

we have

$$x^{m-(k+h-1)}\tilde{f}(x)\tilde{h}(x) \equiv 0 \pmod{\bar{h}(x)}.$$

Thus the equation (1.2) is equivalent to the congruence

$$(1 - xx_1) \cdots (1 - xx_{k+r})\tilde{v}(x) \equiv \tilde{u}(x) \pmod{\bar{h}(x)}. \quad (1.3)$$

Since $(u(x), h(x)) = 1$ and $h(x)$ has no zero, $\deg u(x) = m$ and $\tilde{u}(0) \neq 0$, we know that $(\bar{h}(x), \tilde{u}(x)) = 1$. Thus (1.3) is equivalent to the congruence

$$(1 - xx_1) \cdots (1 - xx_{k+r}) \frac{\tilde{v}(x)}{\tilde{u}(x)} \equiv 1 \pmod{\bar{h}(x)}.$$

The number of solutions of this congruence in x_i 's is

$$N_u = \# \left\{ (x_1, \dots, x_{k+r}, \tilde{v}(x)) \left| \begin{array}{l} (1 - xx_1) \cdots (1 - xx_{k+r}) \frac{\tilde{v}(x)}{\tilde{u}(x)} \equiv 1 \pmod{\bar{h}(x)}, x_i \in \mathbb{F}_q, \\ \text{distinct, } 1 \leq i \leq k+r, \tilde{v}(0) = 1, \deg \tilde{v}(x) = m - (k+r) \end{array} \right. \right\}.$$

Denote

$$N = \frac{1}{|\hat{B}|} \sum_{\substack{x_i \in \mathbb{F}_q, \text{distinct} \\ 1 \leq i \leq k+r}} \sum_{\substack{\tilde{v}(x) \in \mathbb{F}_q[x], \tilde{v}(0)=1 \\ \deg \tilde{v}(x)=m-(k+r)}} \Lambda(\tilde{v}(x)) \sum_{\chi \in \hat{B}} \chi \left(\frac{(1 - xx_1) \cdots (1 - xx_{k+r}) \tilde{v}(x)}{\tilde{u}(x)} \right).$$

One can easily check that if $N > 0$, then $N_u > 0$. So, in order to show that $N_u > 0$, it is enough to show that $N > 0$. Since the second summand of the above formula is always non-negative, applying inclusion and exclusion principle, we deduce

$$\begin{aligned} N &\geq \frac{1}{|\hat{B}|} \left\{ \sum_{\substack{x_i \in \mathbb{F}_q \\ 1 \leq i \leq k+r}} \sum_{\substack{\tilde{v}(x) \in \mathbb{F}_q[x], \tilde{v}(0)=1 \\ \deg \tilde{v}(x)=m-(k+r)}} \Lambda(\tilde{v}(x)) \sum_{\chi \in \hat{B}} \chi \left(\frac{(1 - xx_1) \cdots (1 - xx_{k+r}) \tilde{v}(x)}{\tilde{u}(x)} \right) \right. \\ &\quad \left. - \sum_{1 \leq i < j \leq k+r} \sum_{x_i = x_j \in \mathbb{F}_q} \sum_{\substack{\tilde{v}(x) \in \mathbb{F}_q[x], \tilde{v}(0)=1 \\ \deg \tilde{v}(x)=m-(k+r)}} \Lambda(\tilde{v}(x)) \sum_{\chi \in \hat{B}} \chi \left(\frac{(1 - xx_1) \cdots (1 - xx_{k+r}) \tilde{v}(x)}{\tilde{u}(x)} \right) \right\}. \end{aligned}$$

Separating the trivial character, we obtain

$$\begin{aligned} N &\geq \frac{1}{|\hat{B}|} \left\{ q^m - \binom{k+r}{2} q^{m-1} \right. \\ &\quad + \sum_{\substack{\chi \in \hat{B} \\ \chi \neq 1}} \sum_{\substack{x_i \in \mathbb{F}_q \\ 1 \leq i \leq k+r}} \sum_{\substack{\tilde{v}(x) \in \mathbb{F}_q[x], \tilde{v}(0)=1 \\ \deg \tilde{v}(x)=m-(k+r)}} \Lambda(\tilde{v}(x)) \chi \left(\frac{(1 - xx_1) \cdots (1 - xx_{k+r}) \tilde{v}(x)}{\tilde{u}(x)} \right) \\ &\quad \left. - \sum_{\substack{\chi \in \hat{B} \\ \chi \neq 1}} \sum_{1 \leq i < j \leq k+r} \sum_{x_i = x_j \in \mathbb{F}_q} \sum_{\substack{\tilde{v}(x) \in \mathbb{F}_q[x], \tilde{v}(0)=1 \\ \deg \tilde{v}(x)=m-(k+r)}} \Lambda(\tilde{v}(x)) \chi \left(\frac{(1 - xx_1) \cdots (1 - xx_{k+r}) \tilde{v}(x)}{\tilde{u}(x)} \right) \right\}. \end{aligned}$$

If χ is non-trivial and $\chi(\mathbb{F}_q^*) = 1$, Weil's estimate (see [14]) gives

$$\left| \sum_{x_i \in \mathbb{F}_q} \chi(1 - xx_i) \right| = \left| 1 + \sum_{a \in \mathbb{F}_q} \chi(x - a) \right| \leq (m - k) q^{\frac{1}{2}}.$$

Thus

$$\left| \prod_{i=1}^{k+r} \sum_{x_i \in \mathbb{F}_q} \chi(1 - x_i x) \right| \leq (m - k)^{k+r} q^{\frac{k+r}{2}}.$$

If $\chi^2 \neq 1$,

$$\left| \sum_{1 \leq i < j \leq k+r} \sum_{x_i = x_j \in \mathbb{F}_q} \chi((1 - xx_1) \cdots (1 - xx_{k+r})) \right| \leq (m-k)^{k+r-1} q^{\frac{k+r-1}{2}} \binom{k+r}{2}.$$

If $\chi \neq 1$ but $\chi^2 = 1$,

$$\left| \sum_{1 \leq i < j \leq k+r} \sum_{x_i = x_j \in \mathbb{F}_q} \chi((1 - xx_1) \cdots (1 - xx_{k+r})) \right| \leq (m-k)^{k+r-2} q^{\frac{k+r}{2}} \binom{k+r}{2}.$$

By (1.1), we know that for $\chi \neq 1$ and $\chi(\mathbb{F}_q^*) = 1$,

$$\left| \sum_{\substack{\tilde{v}(x) \in \mathbb{F}_q[x], \tilde{v}(0)=1 \\ \deg \tilde{v}(x)=m-(k+r)}} \Lambda(\tilde{v}(x)) \chi(\tilde{v}(x)) \right| \leq (m-k) q^{\frac{m-(k+r)}{2}}.$$

Thus for $\chi \neq 1$ and $\chi(\mathbb{F}_q^*) = 1$,

$$\begin{aligned} & \left| \sum_{\substack{x_i \in \mathbb{F}_q \\ 1 \leq i \leq k+r}} \sum_{\substack{\tilde{v}(x) \in \mathbb{F}_q[x], \tilde{v}(0)=1 \\ \deg \tilde{v}(x)=m-(k+r)}} \Lambda(\tilde{v}(x)) \chi\left(\frac{(1-xx_1) \cdots (1-xx_{k+r}) \tilde{v}(x)}{\tilde{u}(x)}\right) \right. \\ & \quad \left. - \sum_{1 \leq i < j \leq k+r} \sum_{x_i = x_j \in \mathbb{F}_q} \sum_{\substack{\tilde{v}(x) \in \mathbb{F}_q[x], \tilde{v}(0)=1 \\ \deg \tilde{v}(x)=m-(k+r)}} \Lambda(\tilde{v}(x)) \chi\left(\frac{(1-xx_1) \cdots (1-xx_{k+r}) \tilde{v}(x)}{\tilde{u}(x)}\right) \right| \\ & \leq (m-k)^{k+r} q^{\frac{m}{2}} \left(\binom{k+r}{2} + (m-k) \right). \end{aligned}$$

Since $|\hat{B}| \leq q^{m-k}$, we have

$$N \geq \frac{1}{|\hat{B}|} \left(\left(q - \binom{k+r}{2} \right) q^{m-1} - q^{m-k} (m-k)^{k+r} q^{\frac{m}{2}} \left(\binom{k+r}{2} + (m-k) \right) \right).$$

By our assumption, $q > 2 \binom{k+r}{2} + (m-k)$. To prove $N > 0$, it suffices to show that

$$q^{m-1} > q^{m-k} (m-k)^{k+r} q^{\frac{m}{2}}.$$

Now since $q > (m-k)^{2+\varepsilon}$ and $k > \frac{1}{1+\varepsilon} (r + (2+\varepsilon)(\frac{m}{2} + 1))$ for some constant $\varepsilon > 0$, we deduce that $N > 0$. Thus $N_u > 0$. The proof is completed.

2 A Formula for the Dimension of Trace Reed-Solomon Codes

One of the important parameters of an error-correcting code is the dimension. In most cases, we only have bounds for the dimension (see [10, Chapter 7, 12]). It is interesting to try to improve these bounds, or better, to determine the true dimension. This part is a contribution to the solution of the dimension problem for generalized Reed-Solomon codes in some cases.

For $1 \leq k \leq n \leq q^m$, and the generalized Reed-Solomon code $\mathcal{C}_{q^m}[\mathcal{D}, k]$, the following code over \mathbb{F}_q

$$\text{Tr}_{q^m}[\mathcal{D}, k] = \{(\text{Tr}(f(x_1)), \dots, \text{Tr}(f(x_n))) \in \mathbb{F}_q^n \mid f(x) \in \mathbb{F}_{q^m}[x], \deg(f(x)) \leq k-1\}$$

is called the trace Reed-Solomon code of $\mathcal{C}_{q^m}[\mathcal{D}, k]$, where Tr is the Trace map from \mathbb{F}_{q^m} to \mathbb{F}_q .

By the general bound for the dimension of the trace code given in [6], we have the following result.

Proposition 2.1 (Trivial Bound) (see [6, Lemma VIII.1.3]) *For $1 \leq k \leq n \leq q^m - 1$, we have*

$$k \leq \dim_{\mathbb{F}_q} (\text{Tr}_{q^m}[\mathcal{D}, k]) \leq mk.$$

In 1991, Marcel van der Vluge [12, 13] improved the above upper bound for the dimension of Reed-Solomon codes in some special cases. In this section, we obtain an explicit formula for the dimension of trace Reed-Solomon codes in some cases.

Theorem 2.1 *Let $n - 1$ be a positive divisor of $q^m - 1$, $\mathcal{D} = \{x_1, \dots, x_{n-1}\} \cup \{0\}$ and $\{x_1, \dots, x_{n-1}\}$ be the subgroup of order $n - 1$ of the multiplicative group $\mathbb{F}_{q^m}^*$. Suppose that $S = \{1, \dots, n - 1\} \cup \{0\}$ and q acts on S as follows:*

$$q : S \rightarrow S, \quad 0 \mapsto 0, \quad u \mapsto \langle qu \rangle, \quad \forall u \in S \setminus \{0\},$$

where $\langle qu \rangle$ denotes the least nonnegative residue of qu modulo $n - 1$. For any $u \in S$, denote the q -orbit of u by

$$\Omega_u = \{u, \langle qu \rangle, \dots, \langle q^{h_u-1}u \rangle\},$$

where $h_u \in \mathbb{Z}^+$ is the least positive integer such that $q^{h_u} \cdot u \equiv u \pmod{n - 1}$ and u is the smallest integer of Ω_u . Then

$$\dim_{\mathbb{F}_q} \text{Tr}_{q^m}[\mathcal{D}, k] = mk - (m - 1) \cdot \left(1 + \sum_{\substack{u \in S/\sim \\ u \in [1, k-1]}} h_u\right),$$

where “ \sim ” is the equivalence relation on $S \times S$ given by the q -action on S .

Proof For the generalized Reed-solomon code

$$\mathcal{C}_{q^m}[\mathcal{D}, k] = \{(f(x_1), \dots, f(x_n)) \mid f(x) \in \mathbb{F}_{q^m}[x], \deg f(x) \leq k - 1\},$$

the trace code is defined to be

$$\text{Tr}_{q^m}[\mathcal{D}, k] = \{(\text{Tr}(f(x_1)), \dots, \text{Tr}(f(x_n))) \mid f(x) \in \mathbb{F}_{q^m}[x], \deg f(x) \leq k - 1\},$$

where $\text{Tr}(\alpha)$ is the trace map of $\alpha \in \mathbb{F}_{q^m}$ over \mathbb{F}_q . Suppose that K_k is the kernel of the trace map TR which is defined to be

$$\begin{aligned} \text{TR} : \mathbb{F}_{q^m}[x]_{\leq k-1} &\rightarrow \text{Tr}_q[\mathcal{D}, k], \\ f(x) &\mapsto (\text{Tr}(f(x_1)), \dots, \text{Tr}(f(x_n))), \end{aligned}$$

where $\mathbb{F}_{q^m}[x]_{\leq k-1}$ is the set of polynomials in $\mathbb{F}_{q^m}[x]$ with degree $\leq k - 1$. Then

$$\dim_{\mathbb{F}_q} \text{Tr}_{q^m}[\mathcal{D}, k] = \dim_{\mathbb{F}_q} \mathcal{C}_{q^m}[\mathcal{D}, k] - \dim_{\mathbb{F}_q} K_k = mk - \dim_{\mathbb{F}_q} K_k. \quad (2.1)$$

For any $u \in S$, set $f_u(x) = cx^u - c^q x^{\langle qu \rangle} \in \mathbb{F}_{q^m}[x]$. Then $\deg f_u(x) \leq q^m - 1$. Note that $qu \equiv \langle qu \rangle \pmod{n - 1}$, thus $\forall \alpha \in \mathcal{D}$, $\alpha^{\langle qu \rangle} = \alpha^{qu}$. This means that

$$\text{Tr}(f_u(\alpha)) = \text{Tr}(c\alpha^u) - \text{Tr}(c^q \alpha^{\langle qu \rangle}) = \text{Tr}(c\alpha^u) - \text{Tr}(c^q \alpha^{qu}) = 0.$$

And so

$$\{cx^u - c^q x^{\langle qu \rangle} \mid u \in S, c \in \mathbb{F}_{q^m}\} \cap \{f(x) \in \mathbb{F}_{q^m}[x] \mid \deg f(x) \leq k-1\} \subseteq K_k.$$

For a fixed $i = 1, \dots, h_u$, set $A_i = \{cx^u - c^{q^i} x^{\langle q^i u \rangle} \mid c \in \mathbb{F}_{q^m}\}$. Take $\alpha \in \mathbb{F}_q$ and $f(x) = cx^u - c^{q^i} x^{\langle q^i u \rangle} \in A_i$, $g(x) = dx^u - d^{q^i} x^{\langle q^i u \rangle} \in A_i$. Then

$$f(x) - g(x) = (c - d)x^u - (c^{q^i} - d^{q^i})x^{\langle q^i u \rangle} = (c - d)x^u - (c - d)^{q^i} x^{\langle q^i u \rangle} \in A_i$$

and

$$\alpha f(x) = \alpha cx^u - \alpha c^{q^i} x^{\langle q^i u \rangle} = (\alpha c)x^u - (\alpha c)^{q^i} x^{\langle q^i u \rangle} \in A_i.$$

Hence, A_i is an \mathbb{F}_q -vector space. Furthermore, for any $f(x) \in A_i \cap A_j$, $1 \leq i \neq j \leq h_u$, from

$$f(x) = cx^u - c^{q^i} x^{\langle q^i u \rangle} = dx^u - d^{q^j} x^{\langle q^j u \rangle},$$

we can get $g(x) = (c - d)x^u - c^{q^i} x^{\langle q^i u \rangle} + d^{q^j} x^{\langle q^j u \rangle} \equiv 0$. While $\deg g(x) \leq q^m - 1$, thus $c = d$ and $c^{q^i} = d^{q^j} = 0$ or $c^{q^i} - d^{q^j} = 0$ according as $\langle q^i u \rangle = \langle q^j u \rangle$ or not. Fix a $u \in \{1, \dots, n-1\}$ and $1 \leq i \neq j \leq h_u - 1$. By the definition of h_u , we have $\langle q^i u \rangle \neq \langle q^j u \rangle$. This means that $f(x) \equiv 0$, i.e., $A_i \cap A_j = \{0\}$, $1 \leq i \neq j \leq h_u - 1$. Therefore

$$V_u = \bigoplus_{i=1}^{h_u} \{cx^u - c^{q^i} x^{\langle q^i u \rangle} \mid c \in \mathbb{F}_{q^m}\}$$

is an \mathbb{F}_q -vector space of $\mathbb{F}_{q^m}[x]$, and $\dim_{\mathbb{F}_q} V_u = \sum_{i=1}^{h_u-1} \dim_{\mathbb{F}_q} A_i + \dim_{\mathbb{F}_q} A_{h_u}$.

Now from

$$A_{h_u} = \{cx^u - c^{q^{h_u}} x^{\langle q^{h_u} u \rangle} = cx^u - c^{q^{h_u}} x^u \mid c \in \mathbb{F}_{q^m}\}$$

and

$$A_i = \{cx^u - c^{q^i} x^{\langle q^i u \rangle} \mid c \in \mathbb{F}_{q^m}\}, \quad \forall i = 1, \dots, h_u - 1,$$

we have

$$\dim_{\mathbb{F}_q} A_{h_u} = m - h_u, \quad \dim_{\mathbb{F}_q} A_i = m, \quad \forall i = 1, \dots, h_u - 1.$$

Namely,

$$\dim_{\mathbb{F}_q} V_u = \sum_{i=1}^{h_u-1} m + (m - h_u) = h_u \cdot m - h_u.$$

And so

$$\dim_{\mathbb{F}_q} V = \sum_{u \in S/\sim} h_u(m-1) = n(m-1),$$

where $V = \bigoplus_{u \in S/\sim} V_u$, “ \sim ” is the equivalence relation on $S \times S$ given by the q -action on S .

Taking $f(x) \in V$ and $\alpha \in \mathcal{D}$, we have $\text{Tr}(f(\alpha)) = 0$, which means that $V \subseteq \text{Ker } T$, where the map T is defined to be

$$\begin{aligned} T : \mathbb{F}_{q^m}[x]_{\leq n-1} &\rightarrow \mathbb{F}_q^n, \\ f(x) &\mapsto (\text{Tr}(f(x_1)), \dots, \text{Tr}(f(x_n))). \end{aligned}$$

On the other hand, it is well-known that the trace map is surjective, hence $\dim_{\mathbb{F}_q} \text{Ker } T = m \cdot n - n = n(m-1) = \dim_{\mathbb{F}_q} V$, and then $V = \text{Ker } T$. Therefore

$$\begin{aligned} K_k &= \text{Ker } T \cap \mathbb{F}_{q^m}[x]_{\leq k-1} = V \cap \mathbb{F}_{q^m}[x]_{\leq k-1} \\ &= \bigoplus_{\substack{u \in S/\sim \\ u \in [0, k-1]}} \bigoplus_{\substack{i=1 \\ \langle q^i u \rangle \leq k-1}}^{h_u} \{c_i x^u - c_i^{q^i} x^{\langle q^i u \rangle} \mid c_i \in \mathbb{F}_{q^m}\}. \end{aligned}$$

Thus

$$\dim_{\mathbb{F}_q} K_k = m-1 + (m-1) \cdot \sum_{\substack{u \in S/\sim \\ u \in [1, k-1]}} h_u.$$

Now from (2.1) we immediately have

$$\dim_{\mathbb{F}_q} \text{Tr}_{q^m}[\mathcal{D}, k] = mk - (m-1) \cdot \left(1 + \sum_{\substack{u \in S/\sim \\ u \in [1, k-1]}} h_u\right).$$

Corollary 2.1 *With the assumptions as in Theorem 2.1, we have*

$$\begin{aligned} &mk - (m-1) \cdot \left(1 + \sum_{u \in [1, k-1]/\sim} m\right) \\ &\leq \dim_{\mathbb{F}_q} \text{Tr}_{q^m}[\mathcal{D}, k] \leq mk - (m-1) \cdot \left(1 + \sum_{u \in [1, k-1]/\sim} 1\right) < mk, \end{aligned}$$

where \sim is the equivalent relation given by the q -action on S .

Proof Since $n-1$ is a positive divisor of q^m-1 , $q^m \equiv 1 \pmod{n-1}$. Note that $qu \equiv \langle qu \rangle \pmod{n-1}$ and $\Omega_u = \{u, \langle qu \rangle, \dots, \langle q^{h_u-1}u \rangle\}$ is the q -orbit of u . Therefore $h_u \mid m$, and so

$$\begin{aligned} &mk - (m-1) \cdot \left(1 + \sum_{u \in [1, k-1]/\sim} m\right) \\ &\leq \dim_{\mathbb{F}_q} \text{Tr}_{q^m}[\mathcal{D}, k] \leq mk - (m-1) \cdot \left(1 + \sum_{u \in [1, k-1]/\sim} 1\right) < mk, \end{aligned}$$

Taking $\mathcal{D} = \mathbb{F}_{q^m}$ in Theorem 2.1, we get a formula for the dimension of the standard trace Reed-Solomon code.

Corollary 2.2 *Let $S = \{1, \dots, q^m-1\} \cup \{0\}$. Suppose that q acts on the set S as follows:*

$$\begin{aligned} q : S &\rightarrow S \\ 0 &\mapsto 0, \\ u &\mapsto \langle qu \rangle, \quad \forall u \in S \setminus \{0\}, \end{aligned}$$

where $\langle qu \rangle$ denotes the residue of qu modulo q^m-1 . $\forall u \in S$, denote the q -orbit of u to be

$$\Omega_u = \{u, \langle qu \rangle, \dots, \langle q^{h_u-1}u \rangle\},$$

where $h_u = |\Omega_u|$ and u is the smallest integer in Ω_u . Then

$$\dim_{\mathbb{F}_q} \text{Tr}_{q^m}[\mathbb{F}_{q^m}, k] = mk - (m-1) \cdot \left(1 + \sum_{\substack{u \in S/\sim \\ u \in [1, k-1]}} h_u\right).$$

And so

$$\begin{aligned}
& mk - (m-1) \cdot \left(1 + \sum_{u \in [1, k-1]/\sim} m\right) \\
& \leq \dim_{\mathbb{F}_q} \text{Tr}_{q^m}[\mathbb{F}_{q^m}, k] \leq mk - (m-1) \cdot \left(1 + \sum_{u \in [1, k-1]/\sim} 1\right) < mk,
\end{aligned}$$

where \sim is the equivalent relation given by the q -action on S .

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