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Dynamics about Neural Array with Simple Lateral Inhibitory Connections

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Abstract Lateral inhibitory effect is a well-known feature of information processing in neural systems. This paper presents a neural array model with simple lateral inhibitory connections. After detailed examining into the dynamics of this kind of neural array, the author gives the sufficient conditions under which the outputs of the network will tend to a special stable pattern called spatial sparse pattern in which if the output of a neuron is 1, then the outputs of the neurons in its neighborhood are 0. This ability called spatial sparse coding plays an important role in self-coding, self-organization and associative memory for patterns and pattern sequences. The main conclusions about the dynamics of this kind of neural array which is related to spatial sparse coding are introduced.

Keywords Simple lateral inhibitory connections, Spatial sparse coding, Spatial sparse pattern, Neural array
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1 Introduction

Lateral inhibitory effect is an important feature of information processing in neural systems. Lots of early researches were done. Many researchers presented homogeneous neural field models and examined their dynamics of pattern formation and propagation. For example, Amari [1] studied the formation and the interaction of patterns in a homogeneous neural field by mathematical analysis and provided a good example of non-homogeneous pattern formation in a homogeneous neural field (see also [2–7]). Lots of attentions were paid on the dynamical pattern progressing and propagation by various computer stimulations. Inspired by the early researchers, we study the static pattern formation of a discrete neural array called simple lateral inhibitory neural array. For each neuron in this array, we define a neighborhood. Each neuron has an excitatory feedback from itself and has inhibitory connections with the neurons in its neighborhood. We call this kind of structure the simple lateral inhibitory connection. The dynamics of this kind network is very complex. However, the research interests of this paper is to find sufficient conditions under which the outputs of the neural array will tend to a stable pattern called spatial sparse pattern in which if the output of a neuron is 1, then the outputs of the neurons in its neighborhood are 0. The spatial sparse patterns play an important role in the self-organization, the self-coding and associative memory for patterns and pattern sequences

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that will be introduced in other papers (see [8–11]) and neural networks presented in these papers can also be used for secrete communication and information security. So this research is very important from the mathematical point of view. Here we give the sufficient conditions that make the neural array tend to spatial sparse patterns and introduce the main conclusions that characterize this dynamical process. It should be mentioned that the network presented in this paper is very similar to the cellular neural network (see [12, 13]) which is deeply researched and widely used in image processing in the last twenty years. However, the dynamics of the two networks that we expect are quite different. The former is a competitive network that if a neuron excites, the neurons in its neighborhood are inhibited and the later plays a role of high pass filters. In addition, the neural array model with simple lateral inhibitory connections has also the ability of short term memory which will be introduced in [10] and plays an important role in the associative memory for pattern sequences in [10].

2 Simple Lateral Inhibitory Neural Array

We first introduce the concept of a neural array. A neural array is a set of neurons with 2-dimensional index, such as

$$\mathrm{AN} = \{n_{ij}; (i,j) \in \mathrm{ID}\}, \quad \mathrm{ID} \subset \{1,2,\cdots,N_a\} \times \{1,2,\cdots,N_a\}.$$

For simplicity, a neural array can be denoted in the form

$$\mathrm{AN} = \{(i,j) \in \mathrm{ID}\}.$$

For each neuron (i,j), we define a neighborhood

$$N(i,j) \subset \mathrm{AN}, \quad (i,j) \in \mathrm{AN}.$$

There are many methods to define a neighborhood. The simplest way is using distance. For example, we define

$$\begin{aligned} &d((s,t),(i,j)) = \max\{|s-i|, |t-j|\}, \\ &\qquad N(i,j) = \{(s,t);\ d((s,t),(i,j)) \le d_n,\ (s,t) \in \mathrm{AN}\} \subset \mathrm{AN}, \end{aligned}$$

where d_n is a constant. Obviously, this kind of neighborhoods have the property that if $(s,t) \in N(i,j)$ then $(i,j) \in N(s,t)$.

For each neuron (i,j), we define three variables. They are the state variables $u_{ij}(t)$, the input signals $s_{ij}(t)$ and the output $z_{ij}(t)$. The following difference equations give the relationship of these variables

$$\begin{cases} u_{ij}(t+1) = (1-a)u_{ij}(t) + aw_e z_{ij}(t) - aw_i \displaystyle\sum_{\substack{(s,t)\neq(i,j)\\(s,t)\in N(i,j)}} z_{st}(t) + as_{ij}(t), \\ z_{ij}(t) = g(u_{ij}(t)), \qquad\qquad\qquad\qquad\qquad (i,j) \in \mathrm{AN}, \\ u_{ij}(t_0) = u_{ij}^0, \end{cases} \tag{2.1}$$

where $g(\,\cdot\,)$ is a sigmoid function. Thus we defined a neural network called simple lateral inhibitory network.

Proposition 2.1 *Let the input signals in difference equations* (2.1) *be constants* $\mathbf{s} = \{s_{ij}, (i,j) \in \mathrm{AN}\}$. *Denote all the equilibrium points corresponding to* $\mathbf{s}$ *by* $U_e(\mathbf{s})$. *Then* $U_e(\mathbf{s}) \neq \emptyset$, *and*

$$U_e(\mathbf{s}) \subset [-K_f, K_f]^{|\mathrm{AN}|},$$

where

$$K_f = w_e + w_i(N_n - 1) + s_m, \quad N_n = \max\{|N(i,j)|, (i,j) \in \mathrm{AN}\},$$
$$s_m = \max\{|s_{ij}|, (i,j) \in \mathrm{AN}\}.$$

We first divide the neural array into three parts

$$\mathrm{AN} = \mathrm{AN}_0 \cup \mathrm{AN}_1 \cup \mathrm{AN}_2, \quad \mathrm{AN}_i \cap \mathrm{AN}_j = \emptyset, \quad i \neq j.$$

The set of all the divisions is denoted by

$$D(\mathrm{AN}) = \{(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2);\ \mathrm{AN} = \mathrm{AN}_0 \cup \mathrm{AN}_1 \cup \mathrm{AN}_2,\ \mathrm{AN}_i \cap \mathrm{AN}_j = \emptyset,\ i \neq j\}.$$

For each division, the set of all the corresponding equilibrium points under the input $\mathbf{s}$ is denoted by

$$U_e(\mathbf{s})(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2) = \{\overline{\mathbf{u}};\ \overline{u}_{ij} \leq 0,\ (i,j) \in \mathrm{AN}_0;\ 0 < \overline{u}_{ij} < h,\ (i,j) \in \mathrm{AN}_1;$$
$$\overline{u}_{ij} \geq h, (i,j) \in \mathrm{AN}_2\}.$$

It can be shown that if $-(w_e - h) < s_{ij} < h < 1,\ (i,j) \in \mathrm{AN}$, then

$$\bigcup_{\substack{(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2) \in D(\mathrm{AN}) \\ \mathrm{AN}_1 \neq \emptyset}} U_e(s)(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2) \neq \emptyset.$$

But we hope that the network will tend to the following equilibrium points:

$$\bigcup_{\substack{(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2) \in D(\mathrm{AN}) \\ \mathrm{AN}_1 = \emptyset}} U_e(s)(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2).$$

For examining into the dynamics of the neural array, we define vectors and matrixes whose indexes are pairs of integers. For example, a matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = \{a_{ijst};\ (i,j) \in \mathrm{AN}_1, (s,t) \in \mathrm{AN}_2\},$$

and a vector $\mathbf{x}$ is defined as

$$\mathbf{x} = \{x_{ij};\ (i,j) \in \mathrm{AN}\}.$$

Obviously, the theories about matrixes and vectors are suitable to our case.

At time t, the neural array can be derived into three parts as follows:

$$\mathrm{AN}_0(t) = \{(i,j);\ u_{ij}(t) \leq 0,\ (i,j) \in \mathrm{AN}\},$$
$$\mathrm{AN}_1(t) = \{(i,j);\ 0 < u_{ij}(t) < h,\ (i,j) \in \mathrm{AN}\},$$
$$\mathrm{AN}_2(t) = \{(i,j);\ u_{ij}(t) \geq h,\ (i,j) \in \mathrm{AN}\}.$$

So the state vector $\mathbf{u}(t) = \{u_{ij}(t);\, (i,j) \in \mathrm{AN}\}$ can be derived into three parts

$$\mathbf{u}_0(t) = \{u_{ij}(t);\ (i,j) \in \mathrm{AN}_0(t)\},$$
$$\mathbf{u}_1(t) = \{u_{ij}(t);\ (i,j) \in \mathrm{AN}_1(t)\},$$
$$\mathbf{u}_2(t) = \{u_{ij}(t);\ (i,j) \in \mathrm{AN}_2(t)\}.$$

The input vector $\mathbf{s} = \{s_{ij};\ (i,j) \in \mathrm{AN}\}$ can also be divided into three parts

$$\mathbf{s}_{0t} = \{s_{ij};\ (i,j) \in \mathrm{AN}_0(t)\},\quad \mathbf{s}_{1t} = \{s_{ij};\ (i,j) \in \mathrm{AN}_1(t)\},\quad \mathbf{s}_{2t} = \{s_{ij};\ (i,j) \in \mathrm{AN}_2(t)\}.$$

Let the output function of a simple lateral inhibitory neural array be

$$g(u) = \begin{cases} 0, & u \le 0, \\ \dfrac{1}{h}u, & 0 < u < h, \\ 1, & u \ge h. \end{cases}$$

Then difference equations (2.1) can be described by the matrixes in the following way:

(1) $$\mathbf{u}(t+1) = \mathbf{A}(t)\mathbf{x}(t) + a\mathbf{b}(t) + a\mathbf{s},$$

where

(a) $$\mathbf{A}(t) = \{a_{ijst};\ ((i,j),(s,t)) \in \mathrm{AN} \times \mathrm{AN}\}$$

for $(i,j) \in \mathrm{AN}_0(t) \cup \mathrm{AN}_2(t)$. We have

$$a_{ijst} = \begin{cases} 1 - a, & (i,j) = (s,t), \\ -\dfrac{aw_i}{h}, & (s,t) \in \mathrm{AN}_1(t) \cap N(i,j) - \{(i,j)\}, \\ 0, & \text{otherwise} \end{cases}$$

for $(i,j) \in \mathrm{AN}_1(t)$. We have

$$a_{ijst} = \begin{cases} 1 + a\Big(\dfrac{w_e}{h} - 1\Big), & (i,j) = (s,t), \\ -\dfrac{aw_i}{h}, & (s,t) \in \mathrm{AN}_1(t) \cap N(i,j) - \{(i,j)\}, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $$\mathbf{b}(t) = \{b_{ij}; (i,j) \in \mathrm{AN}(i,j)\},$$

where

$$b_{ij} = -w_i|\mathrm{AN}_2(t) \cap N(i,j)|,\quad (i,j) \in \mathrm{AN}_0(t) \cap \mathrm{AN}_1(t),$$
$$b_{ij} = -w_i(|\mathrm{AN}_2(t) \cap N(i,j)| - 1) + w_e,\quad (i,j) \in \mathrm{AN}_2(t).$$

(2) $$\{u_{ij}(t+1); (i,j) \in \mathrm{AN}_1(t)\} = \mathbf{A}_1(t)\mathbf{u}_1(t) + a\mathbf{b}_1(t) + a\mathbf{s}_{1t},$$

where

(a) $$\mathbf{A}_1(t) = \{a^1_{ijst}(t);\ ((i,j),(s,t)) \in \mathrm{AN}_1(t) \times \mathrm{AN}_1(t)\},$$

where

$$a_{ijst} = \begin{cases} 1 + a\Big(\dfrac{w_e}{h} - 1\Big), & (i,j) = (s,t), \\ -\dfrac{aw_i}{h}, & (s,t) \in \mathrm{AN}_1(t) \cap N(i,j) - \{(i,j)\}, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $$\mathbf{b}_1(t) = \{b^1_{ij}(t);\ (i,j) \in \mathrm{AN}_1(t)\},$$

where

$$b_{ij} = -w_i|\mathrm{AN}_2(t) \cap N(i,j)|, \quad (i,j) \in \mathrm{AN}_0(t) \cup \mathrm{AN}_1(t),$$
$$b_{ij} = -w_i(|\mathrm{AN}_2(t) \cap N(i,j)| - 1) + w_e, \quad (i,j) \in \mathrm{AN}_2(t).$$

If $\mathrm{AN}_1(t) = \mathrm{AN}_1(t_0)$ $(t_0 \le t \le t_1)$, then

$$\mathbf{u}_1(t+1) = \mathbf{A}_1(t_0)\mathbf{u}_1(t) + a\mathbf{b}_1(t_0) + a\mathbf{s}_{1t}, \quad t_0 \le t \le t_1 - 1. \tag{2.2}$$

(3) $$\mathbf{u}(t+1) = \mathbf{A}(0 \le t' \le t)\mathbf{u}(0) + a\mathbf{B}(0 \le t' \le t)\mathbf{s} + a\mathbf{d}(0 \le t' \le t), \quad t > 0, \tag{2.3}$$

where

$$\mathbf{A}(0 \le t' \le t) = \prod_{t'=0}^{t} \mathbf{A}(t'), \tag{2.4}$$

$$\mathbf{B}(0 \le t' \le t) = \mathbf{I} + \sum_{i=1}^{t} \prod_{t'=i}^{t} \mathbf{A}(t'), \quad \mathbf{B}(t'=0) = \mathbf{I}, \tag{2.5}$$

$$\mathbf{d}(0 \le t' \le t) = \mathbf{b}(t) + \sum_{i=1}^{t} \mathbf{b}(i-1) \prod_{t'=i}^{t} \mathbf{A}(t'), \quad \mathbf{d}(t'=0) = \mathbf{I}. \tag{2.6}$$

Proposition 2.2 *Suppose that the neighborhoods of a simple lateral inhibitory neural array satisfy the condition* $(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t)$. *Then we have*

(1) $\mathbf{A}(t)$ *and* $\mathbf{A}_1(t)$ *are symmetry.*

(2) *Let the maximum value of the eigenvalues of* $\mathbf{A_1}(t)$ *be* $\lambda^1_{\max}$. *Then* $\lambda^1_{\max} > 1 + a(\frac{w_e}{h} - 1)$.

(3) *If* $a < \frac{h}{[h + w_i(N_n - 1)]}$, *then* $|\mathbf{A}(0 \le t' \le t)| > 0$.

Note 2.1 Suppose that the neighborhoods of a simple lateral inhibitory neural array satisfy the condition $(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t)$. When $t_0 \le t \le t_1$, we suppose

$$\mathrm{AN}_0(t) = \mathrm{AN}_0(t_0), \quad \mathrm{AN}_1(t) = \mathrm{AN}_1(t_0), \quad \mathrm{AN}_2(t) = \mathrm{AN}_2(t_0).$$

Then $\forall(k,l)\in \mathrm{AN}_1(t_0)$, when $t_0\le t\le t_1+1$, we have

$$u_{kl}(t)=\sum_{(i,j)\in \mathrm{AN}_1(t_0)} p_{ijkl}(t_0)\lambda_{ij}^{t-t_0}\cdot\Big(\sum_{(s,t)\in \mathrm{AN}} a^*_{ijst}(t_0)u_{st}(0)+\sum_{(s,t)\in \mathrm{AN}} b^*_{ijst}(t_0)s_{st}+ac^*_{ij}(t_0)\Big)$$
$$+\sum_{(s,t)\in \mathrm{AN}} d^*_{klst}(t_0)s_{st}+e^*_{kl}(t_0).$$

Here

(a) $\lambda_{ij},\ (i,j)\in \mathrm{AN}_1(t_0)$ are eigenvalues of $\mathbf{A}_1(t_0)$.

(b) $\{p_{ijkl}(t_0);\{(i,j),(k,l)\}\in \mathrm{AN}_1(t_0)\times \mathrm{AN}_1(t_0)\}\equiv \mathbf{P}_1(t_0)$, which satisfies

$$\mathbf{P}_1(t_0)\mathbf{A}_1(t_0)\mathbf{P}_1^{\mathrm{T}}(t_0)=\Lambda_1=\{\lambda_{ijst};\ ((i,j),(s,t))\in \mathrm{AN}_1(t_0)\times \mathrm{AN}_1(t_0)\},$$

where

$$\lambda_{ijst}=\begin{cases}\lambda_{ij}, & (s,t)=(i,j),\\ 0, & (s,t)\neq(i,j).\end{cases}$$

(c) $\{a^*_{ijst}(t_0);\ ((i,j),(s,t))\in \mathrm{AN}_1(t_0)\times \mathrm{AN}(t_0)\}$

$\equiv \mathbf{A}^*(t_0)=\mathbf{P}_1(t_0)\mathbf{T}_1(t_0)\mathbf{A}(0\le t'\le t_0-1),$

$\{b^*_{ijst}(t_0);\ ((i,j),(s,t))\in \mathrm{AN}_1(t_0)\times \mathrm{AN}_1(t_0)\}$

$\equiv \mathbf{B}^*(t_0)=\mathbf{P}_1(t_0)\mathbf{T}_1(t_0)\mathbf{B}(0\le t'\le t_0-1)-(\mathbf{I}-\Lambda_1)^{-1}\mathbf{P}_1(t_0)\mathbf{T}_1(t_0),$

$\{d^*_{ijst}(t_0);\ ((i,j),(s,t))\in \mathrm{AN}_1(t_0)\times \mathrm{AN}\}\equiv \mathbf{D}^*(t_0)=\mathbf{P}_1^{\mathrm{T}}(t_0)(\mathbf{I}-\Lambda_1)^{-1}\mathbf{P}_1(t_0)\mathbf{T}_1(t_0),$

$\{c^*_{ij}(t_0);\ (i,j)\in \mathrm{AN}_1(t_0)\}$

$\equiv \mathbf{c}^*(t_0)=\mathbf{P}_1(t_0)\mathbf{T}_1(t_0)\mathbf{d}(0\le t'\le t_0-1)-(\mathbf{I}-\Lambda_1)^{-1}\mathbf{P}_1(t_0)\mathbf{b}_1(t_0)$

$\{e^*_{kl}(t_0);\ (k,l)\in \mathrm{AN}_1(t_0)\}\equiv \mathbf{e}^*(t_0)=\mathbf{P}_1^{\mathrm{T}}(t_0)(\mathbf{I}-\Lambda_1)^{-1}\mathbf{P}_1(t_0)\mathbf{b}_1(t_0),$

where

$$\mathbf{T}_1(t_0)=\{t_{ijst}(t_0);\ ((i,j),(s,t))\in \mathrm{AN}_1(t_0)\times \mathrm{AN}\},$$
$$t_{ijst}=\begin{cases}1, & (s,t)=(i,j),\\ 0, & (s,t)\neq(i,j).\end{cases}$$

Denote $N_n=\max\{|N(i,j)|;\ (i,j)\in \mathrm{AN}_1(t)\}$. If $a<\frac{h}{[h+w_i(N_n-1)]}$, then $|\mathbf{A}(0\le t\le t_0-1)|\neq 0$. Since $|\mathbf{P}_1|\neq 0$, $\mathrm{rank}(\mathbf{T}_1)=|\mathrm{AN}_1(t_0)|$. So $\mathrm{rank}(\mathbf{A}^*)=|\mathrm{AN}_1(t_0)|$. Therefore, $\forall(i,j)\in \mathrm{AN}_1(t_0),\ \exists(k_1,l_1)\in \mathrm{AN}_1(t_0)$, s.t. $p_{ijk_1l_1}(t_0)\neq 0$, and $\exists(s_1,t_1)\in \mathrm{AN}$, s.t. $a^*_{ijs_1t_1}(t_0)\neq 0$.

Note 2.2 From Note 2.1, we know that as long as

$$\sum_{(s,t)\in \mathrm{AN}} a^*_{ijst}(t_0)u_{st}(0)+\sum_{(s,t)\in \mathrm{AN}} b^*_{ijst}(t_0)s_{st}+ac^*_{ij}(t_0)\neq 0,$$

the component of the state $u_{kl}(t)$ corresponding to eigenvalue λ_{ij}

$$u_{kl}(t)(\lambda_{ij})=p_{ijkl}(t_0)\lambda_{ij}^{t-t_0}\Big(\sum_{(s,t)\in \mathrm{AN}} a^*_{ijst}(t_0)u_{st}(0)+\sum_{(s,t)\in \mathrm{AN}} b^*_{ijst}(t_0)s_{st}+ac^*_{ij}(t_0)\Big)$$

will not be zero.

Let $(\Omega, \mathfrak{F}, \mathcal{P})$ be a probability space. The initial value is a random vector $\mathbf{u}(0)(\omega)$. Therefore, the state vector is also random and is denoted by $\mathbf{u}(t)(\omega)$. Then the division of a neural array is also random and can be denoted by

$$\begin{aligned}\mathrm{AN}_0(t)(\omega) &= \{(i,j);\ u_{ij}(t)(\omega) \le 0,\ (i,j) \in \mathrm{AN}\},\\ \mathrm{AN}_1(t)(\omega) &= \{(i,j);\ 0 < u_{ij}(t)(\omega) < h,\ (i,j) \in \mathrm{AN}\},\\ \mathrm{AN}_2(t)(\omega) &= \{(i,j);\ u_{ij}(t)(\omega) \ge h,\ (i,j) \in \mathrm{AN}\}.\end{aligned}$$

The set of all the division of AN is denoted by $D(\mathrm{AN})$. For the given

$$(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2) \in D(\mathrm{AN})$$

and the time t, the following subset of Ω can be obtained:

$$\begin{aligned}&W(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2)(t)\\ &= \{\omega; u_{ij}(t)(\omega) \le 0, (i,j) \in \mathrm{AN}_0; 0 < u_{ij}(t)(\omega) < h, (i,j) \in \mathrm{AN}_1, u_{ij}(t)(\omega) \ge h, (i,j) \in \mathrm{AN}_2\}\end{aligned}$$

which satisfies the following conditions:

(a) $W(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2)(t) \in \Im$,

(b) If $(\mathrm{AN}_0, , \mathrm{AN}_1, \mathrm{AN}_2) \neq (\mathrm{AN}'_0, , \mathrm{AN}'_1, \mathrm{AN}'_2)$, then

$$W(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2)(t) \cap W(\mathrm{AN}'_0, \mathrm{AN}'_1, \mathrm{AN}'_2)(t) = \emptyset,$$

(c) $\displaystyle\bigcup_{(\mathrm{AN}_0,\mathrm{AN}_1,\mathrm{AN}_2)\in D(\mathrm{AN})} W(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2)(t) = \Omega.$

For simplicity, $W(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2)(t)$ is denoted by $W(t)$. The set of all the subset $W(t)$ is denoted by

$$\overline{W}(t) = \{W(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2)(t); (\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2) \in D(\mathrm{AN})\}.$$

Furthermore, we define

$$W(0 \le t \le t_0) = \bigcap_{0 \le t \le t_0} W(\mathrm{AN}_{0t}, \mathrm{AN}_{1t}, \mathrm{AN}_{2t})(t)$$

and denote

$$\overline{W}(0 \le t \le t_0) = \{W(0 \le t \le t_0); W(0 \le t \le t_0) = \bigcap_{t=0}^{t_0} W(t), W(t) \in \overline{W}(t)\}.$$

Obviously, $\overline{W}(0 \le t \le t_0)$ is also a division of Ω.

Let $\omega_0 \in W(0 \le t \le t_0) \neq \emptyset$. $\forall\, w \in W(0 \le t \le t_0)$, since

$$\omega, \omega_0 \in W(\mathrm{AN}_{0t}, \mathrm{AN}_{1t}, \mathrm{AN}_{2t})(t), \quad 0 \le t \le t_0,$$

we know

$$\begin{aligned}&\mathrm{AN}_0(t)(\omega) = \mathrm{AN}_0(t)(\omega_0) \equiv \mathrm{AN}_{0t}, \quad \mathrm{AN}_1(t)(\omega) = \mathrm{AN}_1(t)(\omega_0) \equiv \mathrm{AN}_{1t},\\ &\mathrm{AN}_2(t)(\omega) = \mathrm{AN}_2(t)(\omega_0) \equiv \mathrm{AN}_{2t}, \quad 0 \le t \le t_0.\end{aligned}$$

Therefore, $\forall\,\omega \in W(0 \le t \le t_0),\ \forall\, 0 \le t \le t_0$, we have

$$\begin{aligned}
\mathbf{A}_1(t)(\omega) &= \mathbf{A}_1(t)(\omega_0) \equiv \mathbf{A}_1(t)_W, \quad & \mathbf{A}(t)(\omega) &= \mathbf{A}(t)(\omega_0) \equiv \mathbf{A}(t)_W,\\
\mathbf{b}_1(t)(\omega) &= \mathbf{b}_1(t)(\omega_0) \equiv \mathbf{b}_1(t)_W, & \mathbf{b}(t)(\omega) &= \mathbf{b}(t)(\omega_0) \equiv \mathbf{b}(t)_W,\\
\mathbf{c}(t)(\omega) &= \mathbf{c}(t)(\omega_0) \equiv \mathbf{c}(t)_W, & \mathbf{d}(t)(\omega) &= \mathbf{d}(t)(\omega_0) \equiv \mathbf{d}(t)_W,\\
\mathbf{e}(t)(\omega) &= \mathbf{e}(t)(\omega_0) \equiv \mathbf{e}(t)_W.
\end{aligned}$$

Here

$$\mathbf{A}_1(t)_W, \quad \mathbf{A}(t)_W, \quad \mathbf{b}_1(t)_W, \quad \mathbf{b}(t)_W, \quad \mathbf{c}(t)_W, \quad \mathbf{d}(t)_W, \quad \mathbf{e}(t)_W$$

can be obtained based on Note 2.1. Thus we get

$$\begin{aligned}
a^*_{ijst}(t,\omega) &= a^*_{ijst}(t,\omega_0) \equiv a^*_{ijst}(t)_W, \quad & b^*_{ijst}(t,\omega) &= b^*_{ijst}(t,\omega_0) \equiv b^*_{ijst}(t)_W,\\
c^*_{ij}(t,\omega) &= c^*_{ij}(t,\omega_0) \equiv c^*_{ij}(t)_W, & d^*_{klst}(t,\omega) &= d^*_{klst}(t,\omega_0) \equiv d^*_{klst}(t)_W,\\
e^*_{kl}(t,\omega) &= e^*_{kl}(t,\omega_0) \equiv e^*_{kl}(t)_W.
\end{aligned}$$

If

$$\mathrm{AN}_0(t) = \mathrm{AN}_0(t_0), \quad \mathrm{AN}_1(t) = \mathrm{AN}_1(t_0), \quad \mathrm{AN}_2(t) = \mathrm{AN}_2(t_0), \quad t_0 \le t \le t_1,$$

then according to Note 2.1, $\forall (k,l) \in \mathrm{AN}_1(t_0)$, if $t_0 \le t \le t_1 + 1$, we have

$$\begin{aligned}
u_{kl}(t)(\omega) = &\sum_{(i,j)\in \mathrm{AN}_1(t_0)} p_{ijkl}(t_0)_W \lambda_{ij}^{t-t_0} \cdot \Big(\sum_{(s,t)\in \mathrm{AN}} a^*_{ijst}(t_0)_W u_{st}(0)(\omega)\\
&+ \sum_{(s,t)\in \mathrm{AN}} b^*_{ijst}(t_0)_W s_{st} + a c^*_{ij}(t_0)_W \Big)\\
&+ \sum_{(s,t)\in \mathrm{AN}} d^*_{klst}(t_0)_W s_{st} + e^*_{kl}(t_0)_W.
\end{aligned}$$

This means that on

$$W(0 \le t \le t_0) = \bigcap_{0 \le t \le t_0} W(\mathrm{AN}_{0t}, \mathrm{AN}_{1t}, \mathrm{AN}_{2t})(t),$$

though $\mathbf{u}(0)(\omega)$ is random, the coefficients are constants.

Lemma 2.1 *The simple lateral inhibitory network is given by difference equations* (2.1). *The following conditions are satisfied:*

(C1) *The neighborhoods satisfy* $(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t)$.

(C2) $g(u) = \begin{cases} 0, & u \le 0,\\ \dfrac{1}{h}u, & 0 < u < h,\\ 1, & u \ge h. \end{cases}$

(C3) $w_e > h,\ 0 < a < \frac{h}{[h + w_i(N_n - 1)]}$.

(C4) *Initial value* $\mathbf{u}(0)(\omega)$ *is a continuous random vector on a complete probability space* $(\Omega, \Im, P)$.

(C5) *Input is a constant vector* $\mathbf{s} = \{s_{ij}, (i,j) \in \mathrm{AN}\}$ *satisfying* $s_{ij} \le h,\ (i,j) \in \mathrm{AN}$.

Suppose $\mathrm{AN}_1(t_0) \neq \emptyset$*. Then*

$$P(\mathrm{AN}_1(t) = \mathrm{AN}_1(t_0), t \geq t_0) = 0.$$

Furthermore, $\exists (k_1, l_1) \in \mathrm{AN}_1(t_0)$*, if we suppose*

$$\mathrm{AN}_1(t) = \mathrm{AN}_1(t_0) \neq \emptyset, \quad t \geq t_0,$$

then it will lead $\lim\limits_{t\to\infty} |u_{k_1 l_1}(t)(\omega)| = \infty$.

Note 2.3 From Lemma 2.1, we know that as long as $\mathrm{AN}_1(t_0) \neq \emptyset$, the neurons in $\mathrm{AN}_1(t_0)$ will leave $\mathrm{AN}_1(t_0)$ with the time. That is $\exists (k_1, l_1) \in \mathrm{AN}_1(t_0)$ and $\exists\, t_1 > t_0$, such that $u_{k_1 l_1}(t_1 - 1) < h,\ u_{k_1 l_1}(t_1) > h$ or $u_{k_1 l_1}(t_1 - 1) > 0,\ u_{k_1 l_1}(t_1) < 0$.

Lemma 2.2 *The simple lateral inhibitory neural array is given by difference equations* (2.1)*, and conditions* (C1)*,* (C2)*,* (C4) *and* (C5) *are satisfied. Furthermore, the following condition is satisfied too:*

(C6) $w_i > (1+\delta) w_e,\ w_i > (1+\delta) h,\ w_e > h,\ h < 1,\ a < \frac{h}{[h + w_i(N_n - 1)]}$.

Then if $u_{ij}(t_0) - u_{ij}(t_0 - 1) > 0,\ u_{ij}(t_0 - 1) > \frac{h}{(1+\delta)},\ \exists\, t_1 > t_0,$ *when* $t \geq t_1,\ u_{ij}(t) > h,$ $u_{st}(t) < -a\delta h,\ (s,t) \in N(i,j) - \{(i,j)\},\ \lim\limits_{t\to\infty} u_{ij}(t) = w_e + w_{ij} > h.$

Note 2.4 From Lemmas 2.1 and 2.2, we have a perceived understanding about dynamics of the simple lateral inhibitory neural array. At first, some of the neurons in $\mathrm{AN}_1(0)$ enter into $\mathrm{AN}_2(t_0)$. Then the neurons in the neighborhoods of these neurons enter into $\mathrm{AN}_0(t_1)$ due to the inhibitory reaction. In this way, at last $\mathrm{AN}_1(t_n) = \emptyset$, and

$$\mathbf{z}(t)(\omega) = \overline{\mathbf{z}}(\omega), \quad \mathbf{u}(t)(\omega) \to \overline{\mathbf{u}}(\omega), \quad t \to \infty.$$

Theorem 2.1 *The simple lateral inhibitory neural array is given by difference equations* (2.1)*, and conditions* (C1), (C2), (C4)–$(C6)$ *are satisfied. Then* $\exists\, N \in \Im,\ P(N) = 0,$ *for arbitrary input* $\mathbf{s} = \{s_{ij}, (i,j) \in \mathrm{AN}\}$*, as long as* $\omega \in \Omega - N$*, there exists* $T(\mathbf{s}, \omega) > 0$*, such that*

$$\mathbf{z}(t)(\omega) = \overline{\mathbf{z}}(\omega) \in \overline{Z}, \quad t > T(\mathbf{s}, \omega), \quad \mathbf{u}(t)(\omega) \to \overline{\mathbf{u}}(\omega) \in \overline{U}_e, \quad t \to \infty.$$

Note 2.5 The output $\overline{\mathbf{z}}(\omega) = \{\overline{z}_{ij};\ (i,j) \in \mathrm{AN}\}$ has the following properties:

(1) $\overline{z}_{ij} = 1 \Rightarrow \overline{z}_{st} = 0,\quad (s,t) \in N(i,j) - \{(i,j)\},$

(2) $\overline{z}_{ij} = 0 \Rightarrow s_{ij} \leq 0$ or $\exists (s,t) \in N(i,j) - \{(i,j)\},\ \overline{z}_{st} = 1.$

The output $\overline{\mathbf{z}}(\omega)$ is called spatial sparse pattern if it has these two properties. The set of all the spatial sparse patterns is denoted by $\overline{Z}$, and the set of the corresponding equilibrium points is denoted by $\overline{U}_e$.

Theorem 2.2 *The simple lateral inhibitory neural array is given by difference equations* (2.1)*. Conditions* (C1)*,* (C2)*,* (C5) *and* (C6) *are satisfied. Then* $\exists\, B \in \mathfrak{B}(\mathbf{R}^{|\mathrm{AN}|}), \lambda(B) = 0$ $(\lambda(\cdot)$ *is Lebesgue measure*$)$*,* $\forall \mathbf{s}$*, as long as* $\mathbf{u}(0) \notin B$*, there exists* $T(\mathbf{u}(0), \mathbf{s}) > 0$*, such that*

$$\mathbf{z}(t) = \overline{\mathbf{z}} \in \overline{Z}, \quad t > T(\mathbf{u}(0), \mathbf{s}), \quad \mathbf{u}(t) \to \overline{\mathbf{u}} \in \overline{U}_e, \quad t \to \infty.$$

Corollary 2.1 *The simple lateral inhibitory neural array is given by difference equations* (2.1)*. Conditions* (C1)*,* (C2)*,* (C5) *and* (C6) *are satisfied. Then for arbitrary input* $\mathbf{s}$*, the equilibrium point*

$$\mathbf{u}_e \in \bigcup_{\substack{(\mathrm{AN}_0,\mathrm{AN}_1,\mathrm{AN}_2)\in D(\mathrm{AN})\\ \mathrm{AN}_1\neq\emptyset}} U_e(\mathbf{s})(\mathrm{AN}_0,\mathrm{AN}_1,\mathrm{AN}_2)$$

is unstable, and the Lebesgue measure of the set of the initial points that make the neural array tend to these unstable equilibrium points is zero.

Proposition 2.3 *The simple lateral inhibitory neural array is given by difference equations* (2.1)*. Conditions* (C1)*,* (C2)*,* (C5) *and* (C6) *are satisfied. Further more, the following condition is satisfied too.*

(C7) *Input is a constant vector* $\mathbf{s}$ *satisfying*

$$-(1-\alpha)(w_e-h) < s_{ij} < h, \quad 0<\alpha\le 1.$$

Denote

$$\begin{aligned}
&N_n = \max\{|N(i,j)|;\ (i,j)\in \mathrm{AN}\},\\
&\varepsilon_1 \equiv \frac{(w_e-h)h}{(N_n-1)w_i}\alpha, \quad \varepsilon_2 \equiv \frac{w_i-(1+\delta)h}{w_e-h}h, \quad \varepsilon_3 \equiv \frac{w_i-h}{\frac{(w_e-h)}{h^2}+\frac{w_i}{h}},\\
&\varepsilon_4 \equiv \frac{w_e-(1+\delta)h}{w_e-h+(N_n-1)w_i}\alpha h, \quad \varepsilon \equiv \min\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}.
\end{aligned}$$

For $\overline{\mathbf{u}} = \{\overline{u}_{ij}\} \in \overline{U}_e$, let

$$[\overline{\mathbf{u}}]^2 \equiv \{(i,j);\ \overline{u}_{ij} > h, (i,j)\in\mathrm{AN}\}, \quad [\overline{\mathbf{u}}]^0 \equiv \{(i,j);\ N(i,j)\cap[\overline{\mathbf{u}}]^2=\emptyset,\ (i,j)\in\mathrm{AN}\}.$$

We define a neighborhood of $\overline{\mathbf{u}}$ as follows

$$N(\overline{\mathbf{u}},\varepsilon) = \{\mathbf{u};\ u_{ij} > h-\varepsilon,\ u_{st}<\varepsilon, (i,j)\in[\overline{\mathbf{u}}]^2,\ (s,t)\in N(i,j)-\{(i,j)\},\ u_{ij}<0, (i,j)\in[\overline{\mathbf{u}}]^0\}.$$

Then $\forall\, \mathbf{u}(0)\in N(\overline{\mathbf{u}},\varepsilon),\ \exists T(\mathbf{u}(0),\mathbf{s})$, when $t > T(\mathbf{u}(0),\mathbf{s})$, we have

$$\mathbf{z}(t) = \overline{\mathbf{z}}, \quad \lim_{t\to\infty}\mathbf{u}(t) = \overline{\mathbf{u}}.$$

Note 2.6 From Proposition 2.3, we know that the spatial sparse patterns are local stable.

3 Conclusion

In this paper, we have given the sufficient conditions that ensure the neural array with simple lateral inhibitory connections tending to spatial sparse patterns. We have also given the conclusions that characterize this dynamical process under these conditions. The Lebesgue measure of the set of the initial states which lead the array to other equilibrium points is zero and these points are unstable. We also prove that the spatial sparse patterns are locally stable.

Appendix Proofs of Main Results

Proof of Proposition 2.1 Denote

$$\mathbf{u} = \{u_{ij}, (i,j) \in \text{AN}\}, \quad F_{ij}(\mathbf{u}) = w_e g(u_{ij}) - w_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} g(u_{st}) + s_{ij},$$
$$\mathbf{F} = \{F_{ij}(\mathbf{u});\ (i,j) \in \text{AN}\}, \quad \mathbf{f}(\mathbf{u}) = (1-a)\mathbf{u} + a\mathbf{F}(\mathbf{u}).$$

Let $\mathbf{u}_f$ be a fixed point of $\mathbf{f}(\bullet)$. Then

$$\mathbf{u}_f = (1-a)\mathbf{u}_f + a\mathbf{F}(\mathbf{u}_f), \quad \mathbf{u}_f = \mathbf{F}(\mathbf{u}_f).$$

So the fixed point of $\mathbf{f}(\bullet)$ is also the fixed point of $\mathbf{F}(\bullet)$. If $\mathbf{u}_f$ is the fixed point of $\mathbf{F}(\bullet)$, then

$$\mathbf{u}_f = \mathbf{F}(\mathbf{u}_f), \quad \mathbf{u}_f = (1-a)\mathbf{u}_f + a\mathbf{F}(\mathbf{u}_f).$$

So the fixed point of $\mathbf{F}(\bullet)$ is the fixed point of $\mathbf{f}(\bullet)$ too. Let

$$\mathbf{F}(\mathbf{u}) = \mathbf{y} = \{y_{ij};\ (i,j) \in \text{AN}\}, \quad y_{ij} = w_e g(u_{ij}) - w_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} g(u_{st}) + s_i.$$

Since

$$|y_{ij}| \le w_e + w_i(|N(i,j)| - 1) + s_m = K_f,$$

we know $\forall\, \mathbf{u} \in [-K_f, K_f]^{|\text{AN}|}$, $\mathbf{F}(\mathbf{u}) \in [-K_f, K_f]^{|\text{AN}|}$. Since $\mathbf{F}(\bullet)$ is a continuous map from $[-K_f, K_f]^{|\text{AN}|}$ to $[-K_f, K_f]^{|\text{AN}|}$, $\mathbf{F}(\bullet)$ and $\mathbf{f}(\bullet)$ have fixed points on $[-K_f, K_f]^{|\text{AN}|}$. Hence $U_e(\mathbf{s}) \neq \emptyset$.

Now we prove $U_e(\mathbf{s}) \subset [-K_f, K_f]^{|\text{AN}|}$. Since $|u_{ij}(t+1)| \le (1-a)|u_{ij}(t)| + aK_f$, we consider the difference equation

$$\begin{cases} x_{ij}(t+1) = (1-a)x_{ij}(t) + aK_f, \\ x_{ij}(t_0) = |u_{ij}(t_0)|. \end{cases}$$

Thus we know

$$|u_{ij}(t)| \le x_{ij}(t) \to K_f, \quad t \to \infty.$$

Therefore $\overline{\mathbf{u}} \in [-K_f, K_f]^{|\text{AN}|}$.

Proof of Proposition 2.2 First, we prove (1). Consider $\mathbf{A}_1(t)$. Suppose $(s,t) \neq (i,j)$. Then

$$a^1_{ijst} = \begin{cases} -\dfrac{aw_i}{h}, & (s,t) \in \text{AN}_1(t) \cap N(i,j) - \{(i,j)\}, \\ 0, & (s,t) \notin \text{AN}_1(t) \cap N(i,j) - \{(i,j)\}, \end{cases}$$

$\forall (i,j), (s,t) \in \text{AN}_1(t)$,

$$(i,j) \in \text{AN}_1(t) \cap N(s,t) - \{(s,t)\} \Leftrightarrow (s,t) \in \text{AN}_1(t) \cap N(i,j) - \{(i,j)\},$$
$$(i,j) \notin \text{AN}_1(t) \cap N(s,t) - \{(s,t)\} \Leftrightarrow (s,t) \notin \text{AN}_1(t) \cap N(i,j) - \{(i,j)\}.$$

Thus we know

$$a^1_{ijst} = -\frac{aw_i}{h} \Leftrightarrow a^1_{stij} = -\frac{aw_i}{h}, \quad a^1_{ijst} = 0 \Leftrightarrow a^1_{stij} = 0.$$

So $\mathbf{A}_1(t)$ is symmetry. In the same way, we can prove that $\mathbf{A}(t)$ is symmetry too.

Next, let us prove (2). Take $(i_1, j_1) \in \mathrm{AN}_1(t)$ and let the vector $\mathbf{x}_1$ be

$$\mathbf{x}_1 = \{x^1_{ij};\ (i,j) \in \mathrm{AN}_1(t)\},$$

where

$$x^1_{ij} = \begin{cases} 1, & (i,j) = (i_1, j_1), \\ 0, & (i,j) \neq (i_1, j_1). \end{cases}$$

Since for $\langle \mathbf{x}_1, \mathbf{x}_1 \rangle = 1$,

$$\langle \mathbf{A}_1(t)\mathbf{x}_1, \mathbf{x}_1 \rangle = \Big(\sum_{(i,j)\in \mathrm{AN}_1(t)} a^1_{iji_1j_1} x^1_{ij};\ (i,j) \in \mathrm{AN}_1(t) \Big) = a^1_{i_1j_1i_1j_1} = 1 + a\Big(\frac{1}{h} w_e - 1\Big),$$

we know

$$\lambda^1_{\max} = \sup_{|x|\neq 0} \frac{\langle \mathbf{A}_1(t)\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} \geq \frac{\langle \mathbf{A}_1(t)\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle} = 1 + a\Big(\frac{1}{h} w_e - 1\Big).$$

On the other hand,

$$\lambda_{\min} \in \bigcup_{(i,j)\in \mathrm{AN}_1(t)} \Big\{ \lambda;\ |\lambda - a_{ijij}| \leq \sum_{(s,t)\in \mathrm{AN}_1(t) - \{(i,j)\}} |a_{ijst}| \Big\}.$$

Since $a_{ijij} = 1 + a(\frac{1}{h} w_e - 1)$ or $a_{ijij} = 1 - a$, we have

$$\sum_{(s,t)\in \mathrm{AN}_1(t) - \{(i,j)\}} |a_{ijst}| \leq a \frac{w_i}{h} (N_n - 1).$$

We know

$$\lambda_{\min} > 1 - a - aw_i \frac{(N_n - 1)}{h}.$$

Finally, we prove (3). Since $a < \frac{h}{[h + w_i(N_n - 1)]}$, we have

$$\lambda_{\min} > 1 - a - aw_i \frac{(N_n - 1)}{h} > 0, \quad |\mathbf{A}(t)| > 0.$$

Hence $|\mathbf{A}(0 \leq t' \leq t)| = \Big| \prod_{t'=0}^{t} \mathbf{A}(t') \Big| > 0$.

Proof of Lemma 2.1 We take

$$(\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2) \in D(\mathrm{AN}), \quad \mathrm{AN}_1 \neq \emptyset,$$
$$W(0 \leq t \leq t_0) = \bigcap_{0 \leq t \leq t_0} W(\mathrm{AN}_{0t}, \mathrm{AN}_{1t}, \mathrm{AN}_{2t})(t) \in \overline{W}(0 \leq t \leq t_0).$$

Here $(\mathrm{AN}_{0t_0}, \mathrm{AN}_{1t_0}, \mathrm{AN}_{2t_0}) = (\mathrm{AN}_0, \mathrm{AN}_1, \mathrm{AN}_2)$.

Furthermore, suppose $\mathrm{AN}_1(t) = \mathrm{AN}_1$ $(t \geq t_0)$. According to Note 2.1, $\forall\,\omega \in W(0 \leq t \leq t_0)$,

$$\mathbf{u}_1(t) = \mathbf{P}_1^{\mathrm{T}}\mathbf{\Lambda}_1^{t-t_0}[\mathbf{A}^*(t_0)\mathbf{u}(0)(\omega) + a\mathbf{B}^*(t_0)\mathbf{s} + a\mathbf{c}^*(t_0)] + a\mathbf{D}^*(t_0)\mathbf{s} + a\mathbf{e}^*(t_0).$$

Since $\mathbf{u}(0)(\omega)$ is continuous, $|\mathbf{A}^*(t_0)| \neq 0$, we know that

$$\mathbf{A}^*(t_0)\mathbf{u}(0)(\omega) + a\mathbf{B}^*(t_0)\mathbf{s} + a\mathbf{c}^*(t_0)$$

is also continuous. Therefore

$$P\Big(\sum_{(s,t)\in\mathrm{AN}} a^*_{ijst}(t_0)u_{st}(0)(\omega) + \sum_{(s,t)\in\mathrm{AN}} b^*_{ijst}(t_0)s_{st} + ac^*_{ij}(t_0) = 0 \,\Big|\, W(0 \leq t \leq t_0)\Big)$$
$$= 0, \quad (i,j) \in \mathrm{AN}_1.$$

Let

$$N_{W(0\leq t\leq t_0)} = \bigcup_{(i,j)\in\mathrm{AN}_{1t_0}} \Big(\omega;\ \sum_{(s,t)\in\mathrm{AN}} a^*_{ijst}(t_0)u_{st}(0)(\omega) + \sum_{(s,t)\in\mathrm{AN}} b^*_{ijst}(t_0)s_{st} + ac^*_{ij}(t_0) = 0 \wedge W(0 \leq t \leq t_0)\Big).$$

Then

$$P(N_{W(0\leq t\leq t_0)}) = \sum_{(i,j)\in\mathrm{AN}_{1t_0}} P\Big(\omega;\ \sum_{(s,t)\in\mathrm{AN}} a^*_{ijst}(t_0)u_{st}(0)(\omega) + \sum_{(s,t)\in\mathrm{AN}} b^*_{ijst}(t_0)s_{st} + ac^*_{ij}(t_0) = 0 | W(0 \leq t \leq t_0)\Big) \times P(W(0 \leq t \leq t_0)) = 0.$$

Define

$$\Omega_{W(0\leq t\leq t_0)} = \{\omega;\ \mathrm{AN}_{kt_0}(t)(\omega) = \mathrm{AN}_k,\ k = 0,1,2,\ t \geq t_0,\ \omega \in W(0 \leq t \leq t_0)\}.$$

Suppose $\Omega_{W(0\leq t\leq t_0)} - N_{W(0\leq t\leq t_0)} \neq \emptyset$. Then, $\forall\,\omega \in \Omega_{W(0\leq t\leq t_0)} - N_{W(0\leq t\leq t_0)}$,

$$\mathrm{AN}_1(t)(\omega) = \mathrm{AN}_{1t_0}, \quad t \geq t_0,$$
$$\sum_{(s,t)\in\mathrm{AN}} a^*_{ijst}(t_0)u_{st}(0)(\omega) + \sum_{(s,t)\in\mathrm{AN}} b^*_{ijst}(t_0)s_{st} + ac^*_{ij}(t_0) \neq 0, \quad (i,j) \in \mathrm{AN}_1.$$

Denote the maximum eigenvalue of $\mathbf{A}_1(t_0)$ by $\lambda_{i_1j_1}$. Since $w_e > h$, according to Proposition 2.2, $\lambda_{i_1j_1} > 1$. For this (i_1, j_1), we know from Note 2.1,

$$\exists(k_1,l_1) \in \mathrm{AN}_1(t_0),\ \text{s.t. } p^1_{i_1j_1k_1l_1}(t_0) \neq 0, \quad \exists(s_1,t_1) \in \mathrm{AN},\ \text{s.t. } a^*_{i_1j_1s_1t_1}(t_0) \neq 0,$$

the component of the state variable $u_{k_1l_1}(t)(\omega)$ corresponding to $\lambda_{i_1j_1}$ is

$$u_{k_1l_1}(t)(\omega)(\lambda_{i_1j_1}) = p_{i_1j_1k_1l_1}(t_0)\lambda^{t-t_0}_{i_1j_1}\Big(\sum_{(s,t)\in\mathrm{AN}} a^*_{i_1j_1st}(t_0)u_{st}(0)(\omega) + \sum_{(s,t)\in\mathrm{AN}} b^*_{i_1j_1st}(t_0)s_{st} + ac^*_{i_1j_1}(t_0)\Big).$$

Obviously,

$$\lim_{t\to\infty} |u_{k_1l_1}(t)(\omega)(\lambda_{i_1j_1})| = \infty, \quad \lim_{t\to\infty} |u_{k_1l_1}(t)(\omega)| = \infty.$$

This contracts the definition of $\Omega_{W(0\le t\le t_0)}$. So

$$\Omega_{W(0\le t\le t_0)} - N_{W(0\le t\le t_0)} = \emptyset, \quad \Omega_{W(0\le t\le t_0)} \subset N_{W(0\le t\le t_0)}.$$

Let

$$\Omega_1 = \{\omega;\ \mathrm{AN}_1(t)(\omega) = \mathrm{AN}_{1t_0},\ t \ge t_0,\ \omega \in \Omega\}.$$

Then

$$\Omega_1 = \bigcup_{W(0\le t\le t_0)\in \overline{W}(0\le t\le t_0)} \Omega_{W(0\le t\le t_0)} \subset \bigcup_{W(0\le t\le t_0)\in \overline{W}(0\le t\le t_0)} N_{W(0\le t\le t_0)}, \quad P(\Omega_1) = 0.$$

Now, we give some lemmas which will be used in the proof of Lemma 2.2.

Lemma A.1 *The simple lateral inhibitory neural array is given by difference equations* (2.1). *The following conditions are satisfied:*

$$w_i > w_e > h, \quad s_{ij} < h, \quad (i,j) \in \mathrm{AN}.$$

If

$$\Delta u_{ij}(t) = u_{ij}(t) - u_{ij}(t-1) > 0, \quad u_{ij}(t-1) \ge 0,$$

then

(a) $\displaystyle\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\ne(i,j)}} z_{st}(t-1) < \frac{w_e}{w_i} < 1,$

(b) $u_{st}(t-1) < h,\ (s,t) \in N(i,j) - \{(i,j)\}.$

Proof We first consider the case $0 \le u_{ij}(t-1) \le h$. Since

$$u_{ij}(t) = \Big[1 + a\Big(\frac{w_e}{h} - 1\Big)\Big]u_{ij}(t-1) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\ne(i,j)}} z_{st}(t-1) + as_{ij}.$$

We have

$$\Delta u_{ij}(t) = u_{ij}(t) - u_{ij}(t-1) = a\Big(\frac{w_e}{h} - 1\Big)u_{ij}(t-1) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\ne(i,j)}} z_{st}(t-1) + as_{ij} > 0,$$

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\ne(i,j)}} z_{st}(t-1) < \frac{1}{w_i}\Big(\Big(\frac{1}{h}w_e - 1\Big)u_{ij}(t-1) + s_{ij}\Big) < \frac{1}{w_i}\Big(\Big(\frac{1}{h}w_e - 1\Big)h + h\Big) = \frac{w_e}{w_i}.$$

Now we consider the case $u_{ij}(t-1) > h$. Since

$$u_{ij}(t) = (1-a)u_{ij}(t-1) + aw_e - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\ne(i,j)}} z_{st}(t-1) + as_{ij},$$

we have

$$\Delta u_{ij}(t) = u_{ij}(t) - u_{ij}(t-1) = -au_{ij}(t-1) + aw_e - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t-1) + as_{ij} > 0,$$

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t-1) < \frac{w_e}{w_i} < 1,$$

$$z_{st}(t-1) < 1, \quad u_{st}(t-1) < h, \quad (s,t) \in N(i,j) - \{(i,j)\}.$$

Lemma A.2 *The simple lateral inhibitory neural array is given by difference equations* (2.1)*. Its neighborhoods satisfy* $(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t)$*. And the following conditions are satisfied:*

$$w_i > (1+\delta)h, \quad s_{ij} \le h, \quad (i,j) \in \mathrm{AN}.$$

If $u_{ij}(t) > \frac{h}{(1+\delta)}$*, then* $\mathrm{AN}_0(t+1) \cap N(i,j) \supset \mathrm{AN}_0(t) \cap N(i,j)$.

Proof Let $(s,t) \in \mathrm{AN}_0(t) \cap N(i,j)$. Since

$$(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t),$$

and $u_{ij}(t) > \frac{h}{(1+\delta)}$, we have $\frac{1}{(1+\delta)} < z_{ij}(t) \le 1$,

$$\begin{aligned} u_{st}(t+1) &= (1-a)u_{st}(t) - aw_i \sum_{\substack{(k,l)\in N(s,t)\\(k,l)\neq(s,t)}} z_{kl}(t) + as_{st} \\ &\le -aw_i z_{ij}(t) + as_{st} < -a(1+\delta)\frac{h}{1+\delta} + ah = 0. \end{aligned}$$

Therefore

$$(s,t) \in \mathrm{AN}_0(t+1) \cap N(i,j), \quad \mathrm{AN}_0(t+1) \cap N(i,j) \supset \mathrm{AN}_0 \cap N(i,j).$$

Lemma A.3 *The simple lateral inhibitory neural array is given by difference equations* (2.1)*. Its neighborhoods satisfy* $(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t)$*. And the following conditions are satisfied:*

$$w_i > (1+\delta)w_e, \quad w_i > (1+\delta)h, \quad w_e > h, \quad s_{ij} < h, \quad (i,j) \in \mathrm{AN}.$$

If $u_{ij}(t) > \frac{h}{1+\delta}$, $u_{st}(t) < h$, $(s,t) \in N(i,j) - \{(i,j)\}$*, then* $u_{st}(t+1) < h$.

Proof The case that $u_{st}(t) < 0$ has been discussed in the proof of Lemma A.2. So we just consider the case $0 \le u_{st} \le h$. Since

$$z_{ij}(t) = \begin{cases} \dfrac{u_{ij}(t)}{h} > \dfrac{1}{(1+\delta)}, & \dfrac{h}{(1+\delta)} < u_{ij}(t) \le h, \\ 1, & u_{ij}(t) > h, \end{cases}$$

$$(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t),$$

we have

$$\begin{aligned}u_{st}(t+1) &= \Big[1 + a\Big(\frac{w_e}{h-1}\Big)\Big]u_{st}(t) - aw_i \sum_{\substack{(p,q)\in N(s,t)\\(p,q)\neq(s,t)}} z_{pq}(t) + as_{st}\\ &\leq \Big[1 + a\Big(\frac{w_e}{h-1}\Big)\Big]u_{st}(t) - aw_i z_{ij}(t) + as_{st}\\ &< \Big[1 + a\Big(\frac{w_e}{h-1}\Big)\Big]h - a(1+\delta)\frac{w_e}{(1+\delta)} + ah = h.\end{aligned}$$

Lemma A.4 *The simple lateral inhibitory neural array is given by difference equations* (2.1). *Its neighborhoods satisfy*

$$(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t).$$

And the following conditions are satisfied:

$$w_i > (1+\delta)w_e,\ w_i > (1+\delta)h,\ w_e > h, \quad s_{ij} < h < 1,\ (i,j) \in \text{AN}.$$

If $\Delta u_{ij}(t) = u_{ij}(t) - u_{ij}(t-1) > 0,\ u_{ij}(t-1) > \frac{h}{(1+\delta)}$, *then*

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t) \begin{cases} < -a\Big(\dfrac{1}{h-1}\Big), & N_1(i,j)(t-1) \neq \emptyset,\\ = 0, & N_1(i,j)(t-1) = \emptyset,\end{cases}$$

where

$$N_1(i,j)(t-1) = \{(s,t);\ (s,t) \in N(i,j),\ (s,t) \neq (i,j),\ 0 < u_{st}(t-1) < h\}.$$

Proof Denote

$$\begin{aligned}N_0(i,j)(t) &= \{(s,t);\ (s,t) \in N(i,j),\ (s,t) \neq (i,j),\ u_{st}(t) \leq 0\},\\ N_1(i,j)(t) &= \{(s,t);\ (s,t) \in N(i,j),\ (s,t) \neq (i,j),\ 0 < u_{st}(t) < h\},\\ N_2(i,j)(t) &= \{(s,t);\ (s,t) \in N(i,j),\ (s,t) \neq (i,j),\ u_{st}(t) \geq h\}.\end{aligned}$$

Obviously,

$$N(i,j) = N_0(i,j)(t) \cup N_1(i,j)(t) \cup N_2(i,j)(t) \cup \{(i,j)\}.$$

For $(s,t) \in N(i,j) - \{(i,j)\}$,

$$z_{st}(t) = \begin{cases} 0, & (s,t) \in N_0(i,j)(t),\\ \dfrac{u_{st}(t)}{h}, & (s,t) \in N_1(i,j)(t),\\ 1, & (s,t) \in N_2(i,j)(t).\end{cases}$$

Therefore,

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t) = \frac{1}{h} \sum_{\substack{(s,t)\in N_1(i,j)(t)\\(s,t)\neq(i,j)}} u_{st}(t) + |N_2(i,j)(t)| - 1.$$

Since

$$w_i > w_e > h, \quad s_{pq} < h, \quad (p,q) \in \mathrm{AN}, \quad \Delta u_{ij}(t) > 0, \quad u_{ij}(t-1) > 0,$$

according to Lemma A.1, we have

$$u_{st}(t-1) < h, \quad (s,t) \in N(i,j) - \{(i,j)\}, \quad N_2(i,j)(t-1) = \emptyset.$$

Again since

$$\begin{aligned} &w_i > (1+\delta)w_e, \quad w_i > (1+\delta)h, \quad w_e > h, \quad s_{pq} < h, \ (p,q) \in \mathrm{AN}, \\ &u_{ij}(t-1) > \frac{h}{(1+\delta)}, \quad u_{st}(t-1) < h, \quad (s,t) \in N(i,j) - \{(i,j)\}, \end{aligned}$$

according to Lemma A.3, we have

$$u_{st}(t) < h, \quad (s,t) \in N(i,j) - \{(i,j)\}, \quad N_2(i,j)(t) = \emptyset.$$

Since

$$w_i > (1+\delta)h, \quad s_{pq} < h, \quad (p,q) \in \mathrm{AN}, \quad u_{ij}(t-1) > \frac{h}{(1+\delta)},$$

from Lemma A.2, we have $N_0(i,j)(t) \supset N_0(i,j)(t-1)$, so $N_1(i,j)(t) \subset N_1(i,j)(t-1)$. Hence, we know

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t) = \frac{1}{h} \sum_{\substack{(s,t)\in N_1(i,j)(t)\\(s,t)\neq(i,j)}} u_{st}(t), \tag{A.1}$$

$$\begin{aligned} \sum_{\substack{(s,t)\in N_1(i,j)(t)\\(s,t)\neq(i,j)}} u_{st}(t-1) &= \sum_{\substack{(s,t)\in N_1(i,j)(t-1)\\(s,t)\neq(i,j)}} u_{st}(t-1) - \sum_{\substack{(s,t)\in N_1(i,j)(t-1)\\(s,t)\notin N_1(i,j)(t)\\(s,t)\neq(i,j)}} u_{st}(t-1) \\ &\leq \sum_{\substack{(s,t)\in N_1(i,j)(t-1)\\(s,t)\neq(i,j)}} u_{st}(t-1). \end{aligned} \tag{A.2}$$

Suppose $N_1(i,j)(t-1) \neq \emptyset$. If $(s,t) \in N_1(i,j)(t)$, then $(i,j) \in N(s,t)$. So we have

$$\begin{aligned} u_{st}(t) &= \left[1 + a\left(\frac{w_e}{h} - 1\right)\right] u_{st}(t-1) - a w_i \sum_{\substack{(p,q)\in N(s,t)\\(p,q)\neq(s,t)}} z_{pq}(t-1) + a s_{st} \\ &\leq \left[1 + a\left(\frac{w_e}{h} - 1\right)\right] u_{st}(t-1) - a w_i z_{ij}(t-1) + ah. \end{aligned} \tag{A.3}$$

From (A.2) and (A.3), we get

$$\begin{aligned} \sum_{(s,t)\in N_1(i,j)(t)} u_{st}(t) \leq \left[1 + a\left(\frac{w_e}{h} - 1\right)\right] &\sum_{(s,t)\in N_1(i,j)(t)} u_{st}(t-1) \\ &- [a w_i z_{ij}(t-1) - ah]|N_1(i,j)(t)|. \end{aligned}$$

From equation (A.1) and (A.3), we get

$$h\Big(\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t) - \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t-1)\Big)$$
$$\le a\Big(\frac{w_e}{h}-1\Big)h \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t-1)) - [aw_i z_{ij}(t-1) - ah],$$
$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t) \le a\Big(\frac{w_e}{h}-1\Big) \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t-1) - \frac{1}{h}[aw_i z_{ij}(t-1) - ah].$$

Since

$$w_i > w_e > h, \quad s_{pq} < h, \quad (p,q)\in \mathrm{AN}, \quad \Delta u_{ij}(t) > 0, \quad u_{ij}(t-1) > 0,$$

according to Lemma A.1, we have

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t-1) < \frac{w_e}{w_i} < 1.$$

Since

$$w_i > (1+\delta)w_e, \quad h < 1, \quad z_{ij}(t-1) > \frac{1}{h}\cdot\frac{h}{(1+\delta)} = \frac{1}{(1+\delta)},$$

we get

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t) \le a\Big(\frac{w_e}{h}-1\Big) - \frac{1}{h}\Big[a(1+\delta)w_e\frac{1}{1+\delta} - ah\Big] < -\frac{a}{h} + a \le -a\Big(\frac{1}{h}-1\Big) < 0.$$

In the case of $N_1(i,j)(t-1) = \emptyset$, we have $N_1(i,j)(t) = \emptyset$, so $\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t) = 0$.

Proof of Lemma 2.2 If $u_{ij}(t_0) \ge h$, then $\exists\, t_1 > t_0$, $u_{ij}(t_1) < h$, or $u_{ij}(t) \ge h$, $t > 0$. If $u_{ij}(t_0) \le 0$, then $\exists\, t_1 > t_0$, $0 < u_{ij}(t_1) < h$, or $u_{ij}(t) \le 0$, $t > t_0$. So we just consider the case that $0 < u_{ij}(t_0) < h$.

Step 1 We proof that $\exists\, t_1 > t_0$, $u_{ij}(t_1) \ge h$. In fact, if

$$0 < u_{ij}(t) < h, \quad 0 < u_{ij}(t-1) < h,$$

then

$$u_{ij}(t+1) = \Big[1 + a\Big(\frac{w_e}{h}-1\Big)\Big]u_{ij}(t) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t) + as_{ij},$$
$$u_{ij}(t) = \Big[1 + a\Big(\frac{w_e}{h}-1\Big)\Big]u_{ij}(t-1) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(t-1) + as_{ij},$$
$$\Delta u_{ij}(t+1) = \Big[1 + a\Big(\frac{w_e}{h}-1\Big)\Big]\Delta u_{ij}(t) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t). \tag{A.4}$$

Since

$$\Delta u_{ij}(t_0) > 0, \quad \frac{h}{(1+\delta)} < u_{ij}(t_0 - 1) < h,$$

from Lemma A.4, we have

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t_0) < 0.$$

So $\Delta u_{ij}(t_0+1) > 0$, therefore $u_{ij}(t_0+1) > \frac{h}{(1+\delta)}$. If $u_{ij}(t_0+1) < h$, according to Lemma A.4, we have

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t_0+2) < 0.$$

By mathematical induction, we know that as long as $u_{ij}(t) < h$, it holds that

$$\Delta u_{ij}(t) > 0, \quad \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t) < 0.$$

From (A.4), we know

$$\Delta u_{ij}(t+1) > \left[1 + a\left(\frac{w_e}{h} - 1\right)\right]\Delta u_{ij}(t).$$

Since $1 + a(\frac{w_e}{h} - 1) > 1$, we know that $\Delta u_{ij}(t)\uparrow$. Therefore $\exists\, t_1 > t_0$,

$$u_{ij}(t_1) \geq h, \quad \frac{h}{(1+\delta)} < u_{ij}(t_1 - 1) < h.$$

Step 2 We prove that when $t > t_1$, it holds that

$$u_{ij}(t) > h, \quad \Delta u_{ij}(t) > 0, \quad u_{st}(t) < h, \quad (s,t) \in N(i,j) - \{(i,j)\}.$$

In fact, $u_{ij}(t_1) \geq h$, $\frac{h}{(1+\delta)} < u_{ij}(t_1 - 1) < h$. According to Lemma A.4, we have

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t_1) < 0.$$

So

$$\sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t_1+1) \leq 0,$$

$$\Delta u_{ij}(t_1+1) > (1-a)\Delta u_{ij}(t_1) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} \Delta z_{st}(t_1) > 0, \quad u_{ij}(t_1+1) > h.$$

By mathematical induction, we know that when $t > t_1$,

$$\Delta u_{ij}(t) > 0, \quad u_{ij}(t) > h.$$

Since $\Delta u_{ij}(t_1) > 0$, $u_{ij}(t_1 - 1) > \frac{h}{(1+\delta)}$, according to Lemma A.1, we know $u_{st}(t_1 - 1) < h$, $(s,t) \in N(i,j) - \{(i,j)\}$; according to Lemma A.3, we know $u_{st}(t_1) < h$, $(s,t) \in N(i,j) -$

$\{(i,j)\}$. By mathematical induction, we know that when $t > t_1$, $u_{st}(t) < h$, $(s,t) \in N(i,j) - \{(i,j)\}$.

Step 3 We show that $\exists\, t_2 > t_1,\ t > t_2$,

$$u_{st}(t) < -a\delta h, \quad (s,t) \in N(i,j) - \{(i,j)\}, \quad \lim_{t\to\infty} u_{ij}(t) = w_e + s_{ij} \geq h.$$

For simplicity, we only consider the case of $0 < u_{st}(t_1) < h$. By mathematical induction, we can show that as long as $0 < u_{st}(t) < h$,

$$\begin{aligned} u_{st}(t+1) &= \Big(1 + a\Big(\frac{w_e}{h} - 1\Big)\Big)u_{st}(t) - aw_i \sum_{\substack{(p,q)\in N(s,t)\\(p,q)\neq(s,t)}} z_{pq}(t) + as_{st} \\ &\leq \Big[1 + a\Big(\frac{w_e}{h} - 1\Big)\Big]u_{st}(t) - aw_i z_{ij}(t) + as_{st} \\ &= \Big[1 + a\Big(\frac{w_e}{h} - 1\Big)\Big]u_{st}(t) - aw_i + ah, \\ \Delta u_{st}(t+1) &\leq a\Big(\frac{w_e}{h} - 1\Big)u_{st}(t) - aw_i + ah \\ &< a\Big(\frac{w_e}{h} - 1\Big)h - a(1+\delta)h + ah = -a\delta h. \end{aligned}$$

Hence $\exists\, t_2(s,t) > t_1$, when $t = t_2(s,t)$, $u_{st} < 0$, we have $u_{st}(t-1) > 0$; when $t = t_2(s,t) + 1$, we have

$$\begin{aligned} u_{st}(t+1) &= (1-a)u_{st}(t) - aw_i \sum_{\substack{(p,q)\in N(s,t)\\(p,q)\neq(s,t)}} z_{pq}(t) + as_{st} \\ &< -aw_i + as_{st} < -a(1+\delta)h + ah < -a\delta h. \end{aligned}$$

According to mathematical induction, we have

$$u_{st}(t) < -a\delta h, \quad (s,t) \in N(i,j) - \{(i,j)\}, \quad t > t_2 = \max\{t_2(s,t); (s,t) \in N(i,j) - \{(i,j)\}\}.$$

Therefore,

$$u_{ij}(t+1) = (1-a)u_{ij}(t) + aw_e + as_{ij},\ t \geq t_2, \quad \lim_{t\to\infty} u_{ij}(t) = w_e + s_{ij} \geq h.$$

Next, we give some lemmas which will be used in the proof of Theorem 2.1.

Lemma A.5 *The simple lateral inhibitory neural array is given by difference equations* (2.1). *The conditions* (C1), (C2), (C4)–(C6) *are satisfied. Then for input* $\mathbf{s}, \exists\, N_s \in \Im,\ P(N_s) = 0$, *when* $\omega \in \Omega - N_s,\ \exists\, T(\mathbf{s},\omega) > 0$, *s.t.*

$$\mathbf{z}(t)(\omega) = \overline{\mathbf{z}}(\omega) \in \overline{Z}, \quad t > T(\mathbf{s},\omega), \quad \mathbf{u}(t)(\omega) \to \overline{\mathbf{u}}(\omega) \in \overline{U}_e, \quad t \to \infty.$$

Proof If $u_{ij}(0)(\omega) > h$, then $u_{ij}(t)(\omega) \geq h$, $t > 0$, or $\exists\, t_1 > 0$, $u_{ij}(t_1 - 1)(\omega) > h$, $u_{ij}(t_1)(\omega) < h$. If $u_{ij}(0)(\omega) < 0$, then $u_{ij}(t)(\omega) \leq 0$, $t > 0$, or $\exists\, t_1 > 0$, $u_{ij}(t_1 - 1)(\omega) \leq 0$, $u_{ij}(t_1)(\omega) > 0$. For simplicity, suppose

$$0 < u_{ij}(0)(\omega) < h, \quad (i,j) \in \text{AN}.$$

Therefore $\mathrm{AN}_1(0) = \mathrm{AN}$. According to Lemma 2.1, $\exists (k_1, l_1) \in \mathrm{AN}$, $\exists N_1$, $P(N_1) = 0$, if $\omega \in \Omega - N_1$, then when $t > 0$, $u_{k_1 l_1}(t)(\omega) \uparrow h$, or $u_{k_1 l_1}(t)(\omega) \downarrow 0$. For simplicity, suppose $u_{k_1 l_1}(t)(\omega) \uparrow h$. Then $\exists\, t_1 > 0$, such that

$$u_{ij}(t_1)(\omega) - u_{ij}(t_1 - 1)(\omega) > 0, \quad u_{ij}(t_1 - 1)(\omega) > \frac{h}{(1+\delta)}.$$

According to Lemma 2.2, $\exists\, t_2 > t_1$, when $t > t_2$,

$$u_{k_1 j_1}(t)(\omega) > h, \quad u_{st}(t)(\omega) < -a\delta h, \quad (s,t) \in N(k_1, l_1) - \{(k_1, l_1)\},$$
$$\lim_{t\to\infty} u_{k_1 j_1}(t)(\omega) = w_e + w_i > h.$$

If $\mathrm{AN}_1(t_2) \neq \emptyset$, then $\exists\,(k_2, l_2) \in \mathrm{AN}_1(t_2)$, $\exists N_2, P(N_2) = 0$, when $\omega \in \Omega - (N_1 \cup N_2)$, $u_{k_2 l_2}(t)(\omega) \uparrow h$ or $u_{k_2 l_2}(t)(\omega) \downarrow h$. For simplicity, suppose $u_{k_2 l_2}(t)(\omega) \uparrow h$. Then $\exists\, t_3 > t_2$, when $t > t_3$,

$$u_{k_2 j_2}(t)(\omega) > h, \quad u_{st}(t)(\omega) < -a\delta h, \quad (s,t) \in N(k_2, l_2) - \{(k_2, l_2)\},$$
$$\lim_{t\to\infty} u_{k_2 j_2}(t)(\omega) = w_e + w_i > h.$$

At last $\exists\, N_n,\ P(N_n) = 0,\ \exists\, t_{n+1}$, as long as

$$\omega \in \Omega - (N_1 \cup N_2 \cup \cdots \cup N_n),$$

$\mathrm{AN}_1(t) = \emptyset,\ t \geq t_{n+1}$. It is easy to show $AN_2(t) = AN_2(t_n + 1),\ t \geq t_{n+1}$. Therefore

$$\lim_{t\to\infty} u_{ij}(t)(\omega) = w_e - w_i(|\mathrm{AN}_2(t_{n+1}) \cap N(i,j)| - 1) + s_{ij} = \overline{u}_{ij}(\omega) > h, \quad (i,j) \in \mathrm{AN}_2(t_n + 1),$$
$$\lim_{t\to\infty} u_{st}(t)(\omega) = -w_i|\mathrm{AN}_2(t_{n+1}) \cap N(s,t)| + s_{st} = \overline{u}_{st}(\omega) < 0, \quad (s,t) \in \mathrm{AN}_0(t_n + 1),$$
$$\mathbf{z}(t)(\omega) = \overline{\mathbf{z}}(\omega) \in \overline{Z}, \quad t \geq t_n + 1.$$

Lemma A.6 *The simple lateral inhibitory neural array is given by difference equations* (2.1)*. The neighborhoods satisfy*

$$(s,t) \in N(i,j) \Rightarrow (i,j) \in N(s,t).$$

And it also satisfies

$$0 < a < \frac{h}{[h + w_i(N_n - 1)]}.$$

Then $\mathbf{u}(t)(\omega)$ $(t > 0)$ *is continuous as long as* $\mathbf{u}(0)(\omega)$ *is continuous.*

Proof According to Note 2.2 and Note 2.5, on

$$W(0 \leq t' \leq t) = \bigcap_{0 \leq t' \leq t} W(\mathrm{AN}_{0t}, \mathrm{AN}_{1t}, \mathrm{AN}_{2t})(t'),$$

we have

$$\mathbf{u}(t+1)(\omega) = A(0 \leq t' \leq t)_W \mathbf{u}(0)(\omega) + a\mathbf{B}(0 \leq t' \leq t)_W \mathbf{s} + a\mathbf{d}(0 \leq t' \leq t)_W,$$

where $A(0 \leq t' \leq t)_W$, $\mathbf{B}(0 \leq t' \leq t)_W$ and $\mathbf{d}(0 \leq t' \leq t)_W$ are defined by equations (2.4), (2.5) and (2.6) respectively. According to Proposition 2.2, $|\mathbf{A}(0 \leq t' \leq t)_W| > 0$, hence because

$\mathbf{u}(0)(\omega)$ is continuous, $\mathbf{u}(t+1)(\omega)$ is continuous too. So $\mathbf{u}(t)(\omega)$, $t > 0$ is continuous on $W(0 \le t' \le t) = \bigcap_{0 \le t' \le t} W(\mathrm{AN}_{0t}, \mathrm{AN}_{1t}, \mathrm{AN}_{2t})(t')$. Therefore, $\mathbf{u}(t)(\omega)$ $(t > 0)$ is continuous on Ω.

Lemma A.7 *The simple lateral inhibitory neural array is given by difference equations* (2.1). *Conditions* (C1), (C2), (C4)–(C6) *are satisfied. Then for constant input* $\mathbf{s}$, $\exists N_s \in \Im$, *s.t.* $P(N_s) = 0$ *and*

$$\exists\, O(\mathbf{s}, \delta(\mathbf{s}, \mathbf{u}(0)(\omega))) = \{\mathbf{s}';\ |\mathbf{s}' - \mathbf{s}| < \delta(\mathbf{s}, \mathbf{u}(0)(\omega))\},$$

if $\mathbf{s}' \in O(\mathbf{s}, \delta(\mathbf{s}, \mathbf{u}(0)(\omega)))$ *and* $\omega \in \Omega - N_s$, *then*

$$\mathbf{z}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)} = \overline{\mathbf{z}}(\omega), \quad t > T(\mathbf{s}, \omega), \quad \mathbf{u}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)} \to \overline{\mathbf{u}}'(\omega) \in \overline{U}_e, \quad t \to \infty,$$

where $\mathbf{z}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)}$ *and* $\mathbf{u}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)}$ *are the output and the state vector when the input is* $\mathbf{s}'$ *and the initial state is* $\mathbf{u}(0)(\omega)$ *respectively.*

Proof From Lemma A.5, we know that for the input $\mathbf{s}$, $\exists N_s \in \Im$, $P(N_s) = 0$, when $\omega \in \Omega - N_{\mathbf{s}}$, $\exists T'(\mathbf{s}, \omega) > 0$, s.t.

$$\mathbf{z}(t)(\omega) = \overline{\mathbf{z}}(\omega) \in \overline{Z}, \quad t > T'(\mathbf{s}, \omega), \quad \mathbf{u}(t)(\omega) \to \overline{\mathbf{u}}(\omega) \in \overline{U}_e, \quad t \to \infty.$$

According to Lemma A.6, $\mathbf{u}(t)(\omega)$ $(t > 0)$ is continuous. Thus $\exists N_{\mathbf{s}}^t$, s.t. $P(N_{\mathbf{s}}^t) = 0$, if $\omega \in \Omega - N_{\mathbf{s}}^t$, then $u_{ij}(t)(\omega) \ne 0, u_{ij}(t)(\omega) \ne h$. Let $N''_{\mathbf{s}} = \bigcup_{t>0} N_{\mathbf{s}}^t$, $N_{\mathbf{s}} = N'_{\mathbf{s}} \cup N''_{\mathbf{s}}$. For $\omega \in \Omega - N_{\mathbf{s}}$, denote

$$[\overline{\mathbf{z}}(\omega)]_1 = \{(i,j);\ \overline{z}_{ij}(\omega) = 1,\ (i,j) \in \mathrm{AN}\},$$
$$[\overline{\mathbf{z}}(\omega)]_0 = \{(i,j);\ \overline{z}_{ij}(\omega) = 0 \wedge \overline{z}_{st}(\omega) = 0,\ (s,t) \in N(i,j),\ (i,j) \in \mathrm{AN}\}.$$

$\exists T''(\mathbf{s}, \omega) > 0$, when $t > T''(\mathbf{s}, \omega)$,

$$u_{ij}(t)(\omega) > h, \quad u_{st}(t)(\omega) < 0, \quad (s,t) \in N(i,j) - (i,j), \quad (i,j) \in [\overline{\mathbf{z}}(\omega)]_1,$$
$$u_{ij}(t)(\omega) < 0, \quad (i,j) \in [\overline{\mathbf{z}}(\omega)]_0.$$

Let

$$\varepsilon_1 = \min\{u_{ij}(T''(\mathbf{s},\omega))(\omega) - h,\ -u_{st}(T''(\mathbf{s},\omega))(\omega);\ (s,t) \in N(i,j) - \{(i,j)\}, (i,j) \in [\overline{\mathbf{z}}(\omega)]_1\},$$
$$\varepsilon_2 = \min\{-u_{ij}(T''(\mathbf{s},\omega))(\omega);\ (i,j) \in [\overline{\mathbf{z}}(\omega)]_0\}, \quad \varepsilon = \min\left\{\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}\right\}.$$

Since difference equations (2.1) satisfy the Lipschitz condition, $\mathbf{u}(t)(\omega)$ is dependent continuously on $\mathbf{s}$. Corresponding to this ε, $\exists \delta(\mathbf{s}, \mathbf{u}(0)(\omega)) > 0$, if $\mathbf{s}' \in O(\mathbf{s}, \delta(\mathbf{s}, \mathbf{u}(0)(\omega)))$, then

$$u_{ij}(T''(\mathbf{s},\omega))(\omega) > h, \quad u_{st}(T''(\mathbf{s},\omega))(\omega) < 0, \quad (s,t) \in N(i,j) - \{(i,j)\},\ (i,j) \in [\overline{\mathbf{z}}(\omega)]_1,$$
$$u_{ij}(T''(\mathbf{s},\omega))(\omega) < 0, \quad (i,j) \in [\overline{\mathbf{z}}(\omega)]_0.$$

For $(i,j) \in [\overline{\mathbf{z}}(\omega)]_1$, it is easy to show that when $t > T''(\mathbf{s}, \omega)$,

$$u_{ij}(t)(\omega) > h, \quad u_{st}(t)(\omega) < 0, \quad (s,t) \in N(i,j) - \{(i,j)\}, \quad (i,j) \in [\overline{\mathbf{z}}(\omega)]_1,$$
$$u_{ij}(t)(\omega) < 0, \quad (i,j) \in [\overline{\mathbf{z}}(\omega)]_0.$$

Let $T(\mathbf{s},\omega)=\max\{T'(\mathbf{s},\omega),T''(\mathbf{s},\omega)\}$. Then

$$\mathbf{z}(t)(\omega)=\overline{\mathbf{z}}(\omega),\quad t>T(\mathbf{s},\omega),\quad \mathbf{u}(t)(\omega)\to\overline{\mathbf{u}}'(\omega)\in\overline{U}_\varepsilon,\quad t\to\infty.$$

Proof of Theorem 2.1 Let

$$S_Q=\{\mathbf{s};\ s_{ij}\text{ is rational},\ (i,j)\in\text{AN}\},\quad N=\bigcup_{\mathbf{s}\in S_Q}N_{\mathbf{s}}.$$

Here $N_{\mathbf{s}}$ is defined by Lemma A.5, so $P(N_s)=0$. From Lemma A.7, we know that for $\mathbf{s}\in S_Q$, as long as $\omega\in\Omega-N_s,\ \exists\, T(\mathbf{s},\omega)>0$, s.t.

$$\mathbf{z}(t)|_{\mathbf{s},\mathbf{u}(0)(\omega)}=\overline{\mathbf{z}}(\omega)\in\overline{Z},\quad t>T(\mathbf{s},\omega),\quad \mathbf{u}(t)|_{\mathbf{s},\mathbf{u}(0)(\omega)}\to\overline{\mathbf{u}}(\omega)\in\overline{U}_\varepsilon,\quad t\to\infty.$$

Furthermore, for this $\mathbf{s},\ \exists\, O(\mathbf{s},\delta(\mathbf{s},\mathbf{u}(0)(\omega)))=\{\mathbf{s}';\ |\mathbf{s}'-\mathbf{s}|<\delta(\mathbf{s},\mathbf{u}(0)(\omega))\}$, if $\mathbf{s}'\in O(\mathbf{s},\delta(\mathbf{s},\mathbf{u}(0)(\omega)))$, then

$$\mathbf{z}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)}=\overline{\mathbf{z}}(\omega)\in\overline{Z},\quad t>T(\mathbf{s},\omega),\quad \mathbf{u}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)}\to\overline{\mathbf{u}}\in\overline{U}_\varepsilon,\quad t\to\infty.$$

Since

$$\bigcup_{s\in S_Q}O(\mathbf{s},\delta(\mathbf{s},\ \mathbf{u}(0)))\supset\{s,s_{ij}\le h,\ (i,j)\in\text{AN}\},$$

$\forall\,\mathbf{s}'\in\{\mathbf{s},s_{ij}\le h,(i,j)\in\text{AN}\}$ and $\forall\,\omega\in\Omega-N_s,\ \exists\,\mathbf{s}\in S_Q$, s.t. $\mathbf{s}'\in O(\mathbf{s},\delta(\mathbf{s},\mathbf{u}(0)(\omega)))$, such that

$$\begin{aligned}&\mathbf{z}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)}=\overline{\mathbf{z}}(\omega)\in\overline{Z},\qquad t>T(\mathbf{s},\omega),\\&\mathbf{u}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)}\to\overline{\mathbf{u}}(\omega)\in\overline{U}_e,\quad t\to\infty.\end{aligned}$$

Let $N=\bigcup_{\mathbf{s}\in S_Q}N_{\mathbf{s}}$. Then $P(N)=0$. $\forall\,\mathbf{s}$, if $\omega\notin N^c$, we have

$$\mathbf{z}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)}=\overline{\mathbf{z}}(\omega)\in\overline{Z},\quad t>T(\mathbf{s},\omega),\quad \mathbf{u}(t)|_{\mathbf{s}',\mathbf{u}(0)(\omega)}\to\overline{\mathbf{u}}(\omega)\in\overline{U}_e,\quad t\to\infty.$$

Proof of Theorem 2.2 Denote

$$\Omega_0=\Big\{\omega;\ \exists\,\overline{\mathbf{u}}\in\overline{U}_e,\ \exists\,\mathbf{s},\ \lim_{t\to\infty}\mathbf{u}(t)|_{\mathbf{s},\mathbf{u}(\omega)}=\overline{\mathbf{u}},\ \omega\in\Omega\Big\}.$$

Corresponding to this set, there exists

$$\overline{U}_0=\Big\{\mathbf{u}(0);\ \exists\,\overline{\mathbf{u}}\in\overline{U}_e,\ \exists\,\mathbf{s},\ \lim_{t\to\infty}\mathbf{u}(t)|_{\mathbf{s},\mathbf{u}(0)}=\overline{\mathbf{u}}\Big\}.$$

Denote $N=\Omega-\Omega_0,\ U=\mathbf{R}^{|\text{AN}|}-\overline{U}_0$. According to Theorem 2.1, $P(N)=0$.
Let

$$O_k=\{\mathbf{x};\|\mathbf{x}\|\le k,\ \mathbf{x}\in\mathbf{R}^{|\text{AN}|}\},\quad k=1,2,3,\cdots.$$

Then

$$U=\lim_{k\to\infty}U\cap O_k.$$

Let $\mathbf{u}(0)(\omega)$ be random variable with uniform probability distribution on O_k $(k \geq 1)$. It is easy to prove

$$P(N) = 0 \Rightarrow \lambda(U \cap O_k) = 0.$$

Therefore $\lambda(U) = \lim_{k\to\infty} \lambda(U \cap O_k) = 0$. Here $\lambda(\cdot)$ is Lebesgue measure.

Proof of Proposition 2.3 Let $\mathbf{u}(0) \in N(\overline{\mathbf{u}}, \varepsilon)$. We first consider the neuron $(i,j) \in [\overline{\mathbf{u}}]^2$. From the definition of $N(\overline{\mathbf{u}}, \varepsilon)$, we know

$$u_{st}(0) < \varepsilon, \quad (s,t) \in N(i,j) - \{(i,j)\}.$$

There are two cases.

Case 1 In this case, $u_{ij}(0) > h$, so

$$\begin{aligned} u_{ij}(1) &= (1-a)u_{ij}(0) + aw_e - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(0) + as_{ij} \\ &> (1-a)h + aw_e - aw_i(N_n - 1)\frac{1}{h}\varepsilon - a(1-\alpha)(w_e - h) \\ &= h + a\Big[\alpha(w_e - h) - \varepsilon w_i \frac{(N_n - 1)}{h}\Big]. \end{aligned}$$

Since $\varepsilon < \varepsilon_1 \equiv \alpha\frac{((w_e-h)h)}{[(N_n-1)w_i]}$, we know $u_{ij}(1) > h$.

If $u_{st}(0) < 0,\ \forall (s,t) \in N(i,j) - \{(i,j)\}$, then

$$u_{st}(1) = (1-a)u_{st}(0) - aw_i \sum_{\substack{(p,q)\in N(s,t)\\(p,q)\neq(s,t)}} z_{pq}(0) + as_{st} < -aw_i + ah < 0.$$

If $u_{st}(0) < \varepsilon,\ \forall (s,t) \in N(i,j) - \{(i,j)\}$, then

$$\begin{aligned} u_{st}(1) &= \Big[1 + a\Big(\frac{w_e}{h} - 1\Big)\Big]u_{st}(0) - aw_i \sum_{\substack{(p,q)\in N(s,t)\\(p,q)\neq(s,t)}} z_{pq}(0) + as_{st}, \\ \Delta u_{st}(1) &= a\Big(\frac{w_e}{h} - 1\Big)u_{st}(0) - aw_i \sum_{\substack{(p,q)\in N(s,t)\\(p,q)\neq(s,t)}} z_{pq}(0) + as_{st} \\ &< a\Big(\frac{w_e}{h} - 1\Big)\varepsilon - aw_i + ah = a(w_e - h)\frac{\varepsilon}{h} - a(w_i - h). \end{aligned}$$

Since $\varepsilon < \varepsilon_2 \equiv \frac{w_i h - (1+\delta)h^2}{w_e - h}$, we have $\Delta u_{st}(1) < -a\delta h$. Using mathematical induction, we know that when $t \geq 1$, we have

(a) $u_{ij}(t) > h$,

(b) $u_{st}(t-1) < 0 \Rightarrow u_{st}(t) < 0$,

(c) $0 < u_{st}(t-1) < \varepsilon \Rightarrow \Delta u_{st}(t) < -a\delta h$.

Hence $\exists\, t_{ij} > 0$, when $t > t_{ij}$, $u_{ij}(t) > h,\ u_{st}(0) < 0,\ (s,t) \in N(i,j) - \{(i,j)\}$.

Case 2 In this case, $h - \varepsilon < u_{ij}(0) < h$, so

$$u_{ij}(1) = \Big[1 + a\Big(\frac{w_e}{h} - 1\Big)\Big]u_{ij}(0) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(0) + as_{ij},$$

$$\begin{aligned}\Delta u_{ij}(1) &= a\Big(\frac{w_e}{h} - 1\Big)u_{ij}(0) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(0) + as_{ij}\\ &> a\Big(\frac{w_e}{h} - 1\Big)(h-\varepsilon) - aw_i(N_n - 1)\frac{\varepsilon}{h} - a\alpha(w_e - h)\\ &= a(a-\alpha)(w_e - h) + \frac{a}{h}[w_e - h - w_i(N_n - 1)]\varepsilon.\end{aligned}$$

Since

$$\varepsilon < \varepsilon_4 \equiv \frac{w_e - (1+\delta)h}{w_e - h + (N_n - 1)w_i}h,$$

we have $\Delta u_{ij}(1) > a\alpha\sigma h$. $\forall (s,t) \in N(i,j) - \{(i,j)\}$, if $u_{st}(0) < 0$, then

$$\begin{aligned}u_{st}(1) &= (1-a)u_{st}(0) - aw_i \sum_{\substack{(p,q)\in N(s,t)\\(p,q)\neq(s,t)}} z_{pq}(0) + as_{st}\\ &< -aw_i\frac{h-\varepsilon}{h} + ah = -a(w_i - h) + a\frac{w_i}{h}\varepsilon.\end{aligned}$$

Since

$$\varepsilon < \varepsilon_3 \equiv \frac{w_i - h}{\frac{(w_e - h)}{h^2} + \frac{w_i}{h}} < \frac{w_i - h}{\frac{w_i}{h}},$$

we have $u_{st}(1) < 0$. If $0 < u_{st}(0) < \varepsilon$, then

$$u_{ij}(1) = \Big[1 + a\Big(\frac{w_e}{h} - 1\Big)\Big]u_{ij}(0) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(0) + as_{ij},$$

$$\begin{aligned}\Delta u_{ij}(1) &= a\Big(\frac{w_e}{h} - 1\Big)u_{ij}(0) - aw_i \sum_{\substack{(s,t)\in N(i,j)\\(s,t)\neq(i,j)}} z_{st}(0) + as_{ij}\\ &< a\Big(\frac{w_e}{h} - 1\Big)\frac{1}{h}\varepsilon - aw_i\frac{h-\varepsilon}{h} + ah = a\Big(\frac{w_e - h}{h^2} + \frac{w_i}{h}\Big)\varepsilon - a(w_i - h).\end{aligned}$$

Since

$$\varepsilon < \varepsilon_3 \equiv \frac{w_i - h}{\frac{w_e - h}{h^2} + \frac{w_i}{h}} < \frac{w_i - h}{\frac{w_i}{h}}, \quad \Delta u_{st}(1) < 0, \quad u_{st}(1) < \varepsilon,$$

using mathematical induction, we know that when $t \geq 1$,

(a) $u_{ij}(t-1) < h \Rightarrow \Delta u_{ij}(t) > a\alpha\sigma h$,

(b) $u_{st}(t) < \varepsilon,\ (s,t) \in N(i,j) - \{(i,j)\}$.

Hence $\exists\, t'_{ij} > 0$, such that $u_{ij}(t'_{ij} - 1) < h,\ u_{ij}(t'_{ij}) > h$. This is just the same as Case 1. Therefore, $\exists\, t_{ij} > 0$, when $t > t_{ij}$, $u_{ij}(t) > h,\ u_{st}(0) < 0,\ (s,t) \in N(i,j) - \{(i,j)\}$.

Then we consider the neuron $(i,j) \in [\overline{\mathbf{u}}]^0$. Since $s_{ij} \leq 0,\ u_{ij}(0) < 0$,

$$u_{ij}(1) \leq (1-a)u_{ij}(0) + as_{ij} < 0.$$

By mathematical induction, we know that when $t \geq 1$,

$$u_{ij}(t) \leq (1-a)u_{ij}(t-1) + as_{ij} < 0.$$

So when $t > T \equiv \max\{t_{ij}; (i,j) \in [\overline{\mathbf{u}}]^2\}$,

$$u_{ij}(t+1) = (1-a)u_{ij}(t) + aw_e + as_{ij} > h, \quad (i,j) \in [\overline{\mathbf{u}}]^2,$$
$$u_{st}(t+1) = (1-a)u_{st}(t) - aw_i|[\overline{\mathbf{u}}]^2 \cap N(i,j)| + as_{st} < 0, \quad (s,t) \in N(i,j),\ (i,j) \in [\overline{\mathbf{u}}]^2.$$

Therefore when $t > T$,

$$\mathbf{z}(t) = \overline{\mathbf{z}} \in \overline{Z}(\mathbf{s}), \quad \lim_{t\to\infty} \mathbf{u}(t) = \overline{\mathbf{u}} \in \overline{U}(\mathbf{s}).$$

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