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Essential Norms of Composition Operators Between Hardy Spaces of the Unit Disc*

Luo LUO[1] Kai LI[1]

Abstract The authors express the essential norms of composition operators between Hardy spaces of the unit disc in terms of the natural Nevanlinna counting function.

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1 Introduction

Let D be the open unit disk in the complex plane and $H(D)$ denote the space of all holomorphic functions in D. For each p $(0 < p < \infty)$, the Hardy space $H^p(D)$ is defined by

$$H^p(D) = \Big\{f \in H(D) : \sup_{0<r<1} \int_{\partial D} |f(r\xi)|^p \mathrm{d}\sigma(\xi) < \infty\Big\}, \quad \|f\|_p = \Big[\int_{\partial D} |f^*(\xi)|^p \mathrm{d}\sigma(\xi)\Big]^{\frac{1}{p}},$$

where f^* denotes the radial limit of f and $\mathrm{d}\sigma$ is the normalized Lebesgue measure on the boundary ∂D of D. For $1 < p < \infty$, the Hardy space $H^p(D)$ is a Banach space.

Let $\varphi : D \to D$ be a holomorphic self-map of D. For a holomorphic function f on D, denote the composition $f \circ \varphi$ by $C_\varphi f$ and call C_φ the composition operator induced by φ.

Let X and Y be Banach spaces. For a bounded linear operator $T : X \to Y$, the essential norm $\|T\|_{e,X\to Y}$ is defined to be the distance from T to the set of the compact operators $K : X \to Y$, namely,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\| : K \text{ is compact from } X \text{ into } Y\},$$

where $\|\cdot\|$ denotes the usual operator norm.

J. H. Shapiro [3] expressed the essential norm of the composition operator $C_\varphi : H^2(D) \to H^2(D)$ in terms of natural Nevanlinna counting function of the inducing map φ.

The natural Nevanlinna counting function for φ, N_φ, provides such a measure. It is defined by

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}\{w\}} \log\Big(\frac{1}{|z|}\Big), \quad w \in D\backslash\{\varphi(0)\}.$$

As usual, $z \in \varphi^{-1}\{w\}$ is repeated according to the multiplicity of the zero of $\varphi - w$ at z.

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[1]Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.
E-mail: lluo@ustc.edu.cn awaypa@mail.ustc.edu.cn
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The main goal of this paper is to compute the essential norm of $C_\varphi : H^p(D) \to H^q(D)$ for $1 < p \le q < \infty$ in terms of the natural Nevanlinna counting function of the inducing map φ.

In this paper, we get the following theorem.

Theorem 1.1 *Let φ be a holomorphic self-map of D, $1 < p \le q < \infty$. If $C_\varphi : H^p(D) \to H^q(D)$ is bounded, then there exist constants C_1 and C_2, such that*

$$C_1 \limsup_{|a|\to 1^-} \frac{N_\varphi(a)}{[\log(\frac{1}{|a|})]^{\frac{q}{p}}} \le \|C_\varphi\|^q_{e,H^p\to H^q} \le C_2 \limsup_{|a|\to 1^-} \frac{N_\varphi(a)}{[\log(\frac{1}{|a|})]^{\frac{q}{p}}}.$$

Particularly, we get the corollary.

Corollary 1.1 For $1 < p \le q < \infty$, $C_\varphi : H^p(D) \to H^q(D)$ is compact if and only if

$$\limsup_{|a|\to 1^-} \frac{N_\varphi(a)}{[\log(\frac{1}{|a|})]^{\frac{q}{p}}} = 0.$$

In the case $p = q = 2$, Theorem 1.1 and Corollary 1.1 were given by J. H. Shapiro [3].

Throughout the paper, C denotes a positive constant, whose value may change from one occurrence to the next one, but it is independent of f and φ.

2 Proof of Theorem 1.1

Recall that a holomorphic function f in D has the Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

For the Taylor expansion of f and any integer $n \ge 1$, let

$$R_n f(z) = \sum_{k=n}^{\infty} a_k z^k$$

and $K_n = I - R_n$ where $If = f$ is the identity operator.

The operator K_n has a connection with the following natural question: When does the partial sums of the Taylor expansion of f converge to f in the norm topology of the function space? K. Zhu [5] considered the question for various analytic function spaces on the unit disc. In order to prove our main result, we need some of his results.

Lemma 2.1 *Suppose that X is a Banach space of holomorphic functions in D with the property that the polynomials are dense in X. Then $\|K_n f - f\|_X \to 0$ as $n \to \infty$ if and only if $\sup\{\|K_n\| : n \ge 1\} < \infty$.*

Lemma 2.2 *If $1 < p < \infty$, then $\|K_n f - f\|_p \to 0$ as $n \to \infty$ for each $f \in H^p(D)$. Moreover, $\sup\{\|R_n\| : n \ge 1\} < \infty$ and $\sup\{\|K_n\| : n \ge 1\} < \infty$.*

Lemmas 2.1 and 2.2 are Proposition 1 in [5].

To prove Theorem 1.1, we also need the following lemmas.

Lemma 2.3 *For $0 < p < \infty$, $f \in H(D)$ and φ is a holomorphic self-map of D. Then*

$$\|f \circ \varphi\|_p^p = |f(\varphi(0))|^p + \frac{p^2}{2} \int_D |f(w)|^{p-2} |f'(w)|^2 N_\varphi(w) \mathrm{d}A(w),$$

where $\mathrm{d}A$ *is the normalized Lebesgue measure on* D.

Lemma 2.3 is the special case of Lemma 2.2 (see the Change of Variable Formula and (2.1) in [4]).

Lemma 2.4 *Let ψ be a holomorphic self-map of D. If $\psi(0) \neq 0$ and $0 < r < |\psi(0)|$, then*

$$N_\psi(0) \leq \frac{1}{r^2} \int_{rD} N_\psi(w) \mathrm{d}A(w).$$

Lemma 2.4 is the special case of Lemma 4.1 in [4].

Lemma 2.5 *Let ψ be a holomorphic self-map of D. Let $a \in D$ and let*

$$\sigma_a(w) = \frac{a - w}{1 - \overline{a}w}$$

be the Möbius self-map of D that interchanges 0 and a. Then

$$N_\psi \circ \sigma_a = N_{\sigma_a \circ \psi}.$$

Lemma 2.5 is the special case of Lemma 4.2 in [4].

Lemma 2.6 *For $0 < p < \infty$, we have $f \in H^p(D)$ and $w \in D$. Then*

$$|f(w)| \leq \frac{C\|f\|_p}{(1 - |w|)^{\frac{1}{p}}}.$$

Here C is independent of f.

Lemma 2.6 is the special case of Lemma 2.5 in [4].

Proof of Theorem 1.1 At first, we prove

$$\|C_\varphi\|_{e, H^p \to H^q}^q \geq C_1 \limsup_{|a| \to 1^-} \frac{N_\varphi(a)}{[\log(\frac{1}{|a|})]^{\frac{q}{p}}}.$$

For $a \in D$, letting

$$k_a(z) = \Big\{ \frac{1 - |a|^2}{(1 - \overline{a}z)^2} \Big\}^{\frac{1}{p}},$$

we know $\|k_a\|_p = 1$ and, as $|a| \to 1^-$, $k_a \to 0$ uniformly on compact subset of D.

For the moment, fix a compact operator $K : H^p(D) \to H^q(D)$. Since the family $\{k_a\}$ is bounded in $H^p(D)$, and $k_a \to 0$ uniformly on compact subsets of D as $|a| \to 1^-$, we have $\|Kk_a\|_q \to 0$, as $|a| \to 1^-$. Thus

$$\|C_\varphi - K\| \geq \limsup_{|a| \to 1^-} \|(C_\varphi - K)k_a\|_q \geq \limsup_{|a| \to 1^-} (\|C_\varphi k_a\|_q - \|Kk_a\|_q) = \limsup_{|a| \to 1^-} \|C_\varphi k_a\|_q.$$

Upon taking the infimum of both sides of this inequality over all compact operators $K : H^p(D) \to H^q(D)$, we obtain

$$\|C_\varphi\|_{e,H^p\to H^q} \geq \limsup_{|a|\to 1^-} \|C_\varphi k_a\|_q. \tag{2.1}$$

By Lemma 2.3,

$$\|C_\varphi k_a\|_q^q = |k_a(\varphi(0))|^q + \frac{q^2}{2}\int_D |k_a(w)|^{q-2}|k_a'(w)|^2 N_\varphi(w)\mathrm{d}A(w).$$

So, there is a constant C such that

$$\begin{aligned}\|C_\varphi k_a\|_q^q &\geq C\int_D |k_a(w)|^{q-2}|k_a'(w)|^2 N_\varphi(w)\mathrm{d}A(w)\\ &= C\frac{4}{p^2}|a|^2(1-|a|^2)^{\frac{q}{p}}\int_D \frac{N_\varphi(w)}{|1-\overline{a}w|^{2+\frac{2q}{p}}}\mathrm{d}A(w)\\ &= C\frac{4}{p^2}|a|^2(1-|a|^2)^{\frac{q}{p}-2}\int_D \frac{N_\varphi(w)}{|1-\overline{a}w|^{\frac{2q}{p}-2}}|\sigma_a'(w)|\mathrm{d}A(w)\\ &= C\frac{4}{p^2}|a|^2(1-|a|^2)^{\frac{q}{p}-2}\int_D \frac{N_\varphi(\sigma_a(z))}{|1-\overline{a}\sigma_a(z)|^{\frac{2q}{p}-2}}\mathrm{d}A(z).\end{aligned}$$

Here $\sigma_a = \sigma_a^{-1}$ is the Möbius self-map of D as in Lemma 2.5, and the change of variable $z = \sigma_a(w)$ was made. Now,

$$\frac{1}{|1-\overline{a}\sigma_a(z)|} = \frac{|1-\overline{a}z|}{1-|a|^2} \geq \frac{1}{2(1-|a|^2)}, \quad \text{as } |z| \leq \frac{1}{2},$$

so

$$\|C_\varphi k_a\|_q^q \geq \frac{C|a|^2}{(1-|a|^2)^{\frac{q}{p}}}\int_{\frac{1}{2}D} N_\varphi(\sigma_a(z))\mathrm{d}A(z).$$

Since $\sigma_a \circ \varphi(0) > \frac{1}{2}$, if $|a|$ is sufficiently close to 1, applying Lemmas 2.5 and 2.4, we have

$$\int_{\frac{1}{2}D} N_\varphi(\sigma_a(z))\mathrm{d}A(z) = \int_{\frac{1}{2}D} N_{\sigma_a\circ\varphi}(z)\mathrm{d}A(z) \geq 4N_{\sigma_a\circ\varphi}(0) = 4N_\varphi(a).$$

Therefore,

$$\|C_\varphi k_a\|_q^q \geq \frac{C|a|^2 N_\varphi(a)}{(1-|a|^2)^{\frac{q}{p}}}.$$

Since $\log(\frac{1}{|a|})$ is comparable to $(1-|a|^2)$, if $|a|$ is sufficiently close to 1, by (2.1), we get

$$\|C_\varphi\|_{e,H^p\to H^q}^q \geq C_1 \limsup_{|a|\to 1^-} \frac{N_\varphi(a)}{[\log(\frac{1}{|a|})]^{\frac{q}{p}}}.$$

Now, we turn to prove

$$\|C_\varphi\|_{e,H^p\to H^q}^q \leq C_2 \limsup_{|a|\to 1^-} \frac{N_\varphi(a)}{[\log(\frac{1}{|a|})]^{\frac{q}{p}}}.$$

Since, for each n, K_n is compact, we have that $C_\varphi K_n$ is compact and for each n,

$$\|C_\varphi\|_{e,H^p\to H^q} = \|C_\varphi R_n + C_\varphi K_n\|_{e,H^p\to H^q} \leq \|C_\varphi R_n\|. \tag{2.2}$$

Let U denote the closed unit ball in $H^p(D)$, for $f(z) \in U$, by Lemma 2.3,

$$\|C_\varphi R_n f\|_q^q = |R_n f(\varphi(0))|^q + \frac{q^2}{2} \int_D |R_n f(w)|^{q-2} |(R_n f)'(w)|^2 N_\varphi(w) \mathrm{d}A(w). \tag{2.3}$$

For a fixed constant r_0, $\frac{1}{2} < r_0 < 1$, we have

$$\begin{aligned} &\int_D |R_n f(w)|^{q-2} |(R_n f)'(w)|^2 N_\varphi(w) \mathrm{d}A(w) \\ &= \int_{r_0 D} |R_n f(w)|^{q-2} |(R_n f)'(w)|^2 N_\varphi(w) \mathrm{d}A(w) \\ &\quad + \int_{D \setminus r_0 D} |R_n f(w)|^{q-2} |(R_n f)'(w)|^2 N_\varphi(w) \mathrm{d}A(w). \end{aligned} \tag{2.4}$$

Let $M = \sup\limits_{|w| > r_0} \frac{N_\varphi(w)}{[\log(\frac{1}{|w|})]^{\frac{q}{p}}}$. By Lemma 2.6, we have

$$|R_n f(w)|^{q-p} \le \frac{C \|R_n f\|_p^{q-p}}{(1 - |w|)^{\frac{q-p}{p}}}.$$

Then

$$\begin{aligned} &\int_{D \setminus r_0 D} |R_n f(w)|^{q-2} |(R_n f)'(w)|^2 N_\varphi(w) \mathrm{d}A(w) \\ &\le CM \|R_n f\|_p^{q-p} \int_{D \setminus r_0 D} |R_n f(w)|^{p-2} |(R_n f)'(w)|^2 \frac{[\log(\frac{1}{|w|})]^{\frac{q}{p}-1} [\log(\frac{1}{|w|})]}{(1 - |w|)^{\frac{q-p}{p}}} \mathrm{d}A(w). \end{aligned}$$

Since $\log(\frac{1}{|w|}) \le 2(1 - |w|)$ as $|w| \ge \frac{1}{2}$, we have

$$\begin{aligned} &\int_{D \setminus r_0 D} |R_n f(w)|^{q-2} |(R_n f)'(w)|^2 N_\varphi(w) \mathrm{d}A(w) \\ &\le CM \|R_n f\|_p^{q-p} \int_{D \setminus r_0 D} |R_n f(w)|^{p-2} |(R_n f)'(w)|^2 \log\Big(\frac{1}{|w|}\Big) \mathrm{d}A(w). \end{aligned}$$

For $\varphi(z) = z$, we have $N_\varphi(w) = \log(\frac{1}{|w|})$. By Lemma 2.3, we get

$$\int_{D \setminus r_0 D} |R_n f(w)|^{p-2} |(R_n f)'(w)|^2 \log\Big(\frac{1}{|w|}\Big) \mathrm{d}A(w) \le C \|R_n f\|_p^p.$$

By Lemma 2.2 and $f(z) \in U$, we get

$$\int_{D \setminus r_0 D} |R_n f(w)|^{q-2} |(R_n f)'(w)|^2 N_\varphi(w) \mathrm{d}A(w) \le CM \|R_n f\|_p^q \le CM.$$

Using the Cauchy integral formula, for $0 < r < 1$, $w \in rD$, we have

$$(R_n f)'(w) = \frac{1}{2\pi \mathrm{i}} \int_{\partial(rD)} \frac{(R_n f)(\xi)}{(\xi - w)^2} \mathrm{d}\xi.$$

By the Hölder inequality, letting $r \to 1^-$, we get

$$|(R_n f)'(w)| \le \frac{C \|R_n f\|_p}{(1 - |w|)^2}, \quad \text{where } C \text{ is independent of } f.$$

By Lemma 2.6, we get

$$|R_nf(w)|^{q-2}|(R_nf)'(w)|^2 \leq \frac{C\|R_nf\|_p^q}{(1-|w|)^{\frac{q+4p-2}{p}}}.$$

By Lemma 2.2, we get $\|R_nf\|_p \to 0$, as $n \to \infty$. So, as $n \to \infty$,

$$|R_nf(w)|^{q-2}|(R_nf)'(w)|^2 \to 0, \quad \text{uniformly on } r_0D \quad \text{and} \quad |R_nf(\varphi(0))| \to 0. \tag{2.5}$$

By Lemma 2.3, for $f(z) = z$ and $p = 2$, we get

$$\|\varphi\|_2^2 = |\varphi(0))|^2 + 2\int_D N_\varphi(w)\mathrm{d}A(w).$$

So, by Lemma 2.6, we get

$$\int_{r_0D} N_\varphi(w)\mathrm{d}A(w) \leq C, \quad \text{where } C \text{ is independent of } \varphi. \tag{2.6}$$

Combining (2.2)–(2.6) and letting $n \to \infty$, we get

$$\|C_\varphi\|_{e,H^p\to H^q}^q \leq C \sup_{|w|>r_0} \frac{N_\varphi(w)}{[\log(\frac{1}{|w|})]^{\frac{q}{p}}}.$$

Let $r_0 \to 1^-$. Then

$$\|C_\varphi\|_{e,H^p\to H^q}^q \leq C_2 \limsup_{|a|\to 1^-} \frac{N_\varphi(a)}{[\log(\frac{1}{|a|})]^{\frac{q}{p}}}.$$

The proof is completed.

References

[1] Luo, L., The essential norm of a composition operator on the Bergman space of the unit ball, *Complex Variables*, **49**, 2004, 845–850.

[2] Luo, L., The essential norm of a composition operator on Hardy space of the unit ball (in Chinese), *Chin. Ann. Math.*, **28A**(6), 2007, 805–810.

[3] Shapiro, J. H., The Essential Norm of a composition operator, *Ann. of Math.*, **125**, 1987, 375–404.

[4] Smith, W., Composition operators between Bergman and Hardy spaces, *Trans. Amer. Math. Soc.*, **348**, 1996, 2331–2347.

[5] Zhu, K., Duality of Bloch spaces and norm convergence of Taylor series, *Michigan Math. J.*, **38**, 1991, 89–101.

[6] Zhu, K. H., Operator Theory in Function Space, Marcel-Dekker, New York, 1990.