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The Global Existence of Small Solutions to the Oldroyd-B Model

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Abstract The Cauchy problem to the Oldroyd-B model is studied. In particular, it is shown that if the smooth solution (u, τ) to this system blows up at a finite time T^*, then $\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} \mathrm{d}t = \infty$. Furthermore, the global existence of smooth solution to this system is given with small initial data.

Keywords Cauchy problem, Oldroyd-B model, Global existence
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1 Introduction

In this paper, we consider the blow up principle and the global existence of small solutions to the following Oldroyd-B model:

$$\begin{cases} \mathrm{Re}\Big(\dfrac{\partial u}{\partial t} + u \cdot \nabla u\Big) = -\nabla p + (1-\varepsilon)\, \triangle u + \operatorname{div} \tau, \\ \operatorname{div} u = 0, \\ \dfrac{\partial \tau}{\partial t} + (u \cdot \nabla)\tau = (\nabla u)\tau + \tau(\nabla^{\mathrm{T}} u) - \dfrac{1}{\mathrm{We}}\tau + \dfrac{\varepsilon}{\mathrm{We}}(\nabla u + \nabla^{\mathrm{T}} u), \end{cases} \tag{1.1}$$

where $u : (x,t) \in \mathbb{R}^n \times [0,T) \to u(x,t) \in \mathbb{R}^n$ is the fluid velocity, $p : (x,t) \in \mathbb{R}^n \times [0,T) \to p(x,t) \in \mathbb{R}$ represents the hydrodynamic pressure, and $\tau : (x,t) \in \mathbb{R}^n \times [0,T) \to \tau(x,t) \in \mathbb{R}^{n\times n}$ denotes the extra-stress tensor, $n = 2, 3$ is the space dimension. The following parameters are dimensionless: the Reynolds number $\mathrm{Re} \in \mathbb{R}_+$, the Weissenberg number $\mathrm{We} \in \mathbb{R}_+$ and the elastic viscosity fraction $\varepsilon \in (0,1)$. And the initial data satisfy

$$u(x,0) = u_0(x), \quad \tau(x,0) = \tau_0(x). \tag{1.2}$$

The Oldroyd-B model is a classical model for dilute solutions of polymers suspended in a viscous incompressible solvent (see [2]). The system (1.1) describes the motion of the incompressible fluid satisfying the Oldroyd constitutive law (see [17]). Viscoelastic material can be viewed as the intermediate state between the fluid and solid. This kind of material exhibits elastic behavior, such as memory effects as well as fluid properties. Many complicated hydrodynamical and rheological behaviors of complex fluids and the electro-magnetic behavior of the materials (see [2, 5, 6, 10, 14, 18]) can be regarded as consequence of internal elastic properties.

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In 2001, Chemin and Masmoudi [3] established a criterion for the system involving both τ and ∇u. [4, 11, 13, 15] dealt with the inertial Oldroyd-B model for the deformation tensor and established global-in-time existence for small initial data.

In this paper, we will prove a similar result like the well-known Beale-Kato-Majda criterion (see [1]) for the two and three dimensional Oldroyd-B model for the extra-stress tensor with the relaxation term (finite Weissenberg number) and global existence with small initial data. The main results of the paper are presented by the following two theorems:

Theorem 1.1 *Let $s > [\frac{n}{2}] + 1$ for $n = 2, 3$ be an integer, $u_0 \in H^s(\mathbb{R}^n)$, $\tau_0 \in H^s(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$. Then there exists a positive time T, such that Cauchy problem (1.1) with (1.2) has a unique solution on $[0, T]$, $(u, \tau) \in L^\infty([0, T]; H^s(\mathbb{R}^n))$ with $\nabla u \in L^2([0, T]; H^s(\mathbb{R}^n))$. Furthermore, if T^* is the maximal time for existence, then*

$$\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} \mathrm{d}t = \infty. \tag{1.3}$$

Theorem 1.2 *Let $s > [\frac{n}{2}] + 1$ for $n = 2, 3$ be an integer, $u_0 \in H^s(\mathbb{R}^n)$, $\tau_0 \in H^s(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$, and $0 < \varepsilon < \frac{4}{5}$. Then Cauchy problem (1.1) with (1.2) has a unique global solution $(u, \tau) \in L^\infty([0, \infty); H^s(\mathbb{R}^n))$ with $\nabla u \in L^2([0, \infty); H^s(\mathbb{R}^n))$ provided that $\|u_0\|_{H^3}$ and $\|\tau_0\|_{H^3}$ are small enough.*

This paper is organized as follows. Some lemmas are given in Section 2, and Theorem 1.1 and Theorem 1.2 are proved respectively in Section 3 and Section 4.

2 Preliminaries

Before proving the main theorems, we first introduce three useful estimates.

Lemma 2.1 *For $u \in H^s(\mathbb{R}^n)$, $\nabla u \in L^\infty(\mathbb{R}^n)$ where $n = 2$ or 3, and for any positive integer $s > 1$, set $v = \mathrm{D}_x^\alpha u$ where $\alpha \in Z_+^n$ with $|\alpha| = s$. Then we have*

$$\int [\mathrm{D}_x^\alpha (u \cdot \nabla u) - u \cdot \nabla \mathrm{D}_x^\alpha u] v \le C(n) \|\nabla u\|_{L^\infty} \|u\|_{H^s}^2, \tag{2.1}$$

where $C(n)$ is a constant depending only on n.

Proof First

$$\int [\mathrm{D}_x^\alpha (u \cdot \nabla u) - u \cdot \nabla \mathrm{D}_x^\alpha u] v \le \|\mathrm{D}_x^\alpha (u \cdot \nabla u) - u \cdot \nabla \mathrm{D}_x^\alpha u\|_{L^2} \|v\|_{L^2},$$

then we can easily get (2.1) by the Sobolev inequality

$$\|\mathrm{D}_x^\alpha (fg) - f \mathrm{D}_x^\alpha g\|_{L^2} \le C(n)(\|f\|_{H^s} \|g\|_{L^\infty} + \|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}}), \tag{2.2}$$

where $f = u$ and $g = \nabla u$.

Lemma 2.2 *For $(\nabla u, \tau) \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ where $n = 2$ or 3, and for any positive integer $s > 1$, set $v = \mathrm{D}_x^\alpha u$, $\eta = \mathrm{D}_x^\alpha \tau$ where $\alpha \in Z_+^n$ with $|\alpha| = s$. Then we have*

$$\begin{aligned} &\int [\mathrm{D}_x^\alpha (u \cdot \nabla \tau) - u \cdot \nabla \mathrm{D}_x^\alpha \tau] \eta \\ &\le \delta \|\nabla u\|_{H^s}^2 + \frac{C(n)}{\delta} \|\tau\|_{H^s}^2 (\|\nabla u\|_{L^\infty} + \|\tau\|_{L^\infty}^2 + 1) + C(n) \|\nabla u\|_{L^\infty} \|\tau\|_{H^s}^2, \end{aligned} \tag{2.3}$$

where $\delta > 0$ and $C(n)$ is a constant depending only on n.

Proof By the Cauchy inequality, we have

$$\int [\mathrm{D}_x^\alpha(u\cdot\nabla\tau) - u\cdot\nabla\mathrm{D}_x^\alpha\tau]\eta \le \|\mathrm{D}_x^\alpha(u\cdot\nabla\tau) - u\cdot\nabla\mathrm{D}_x^\alpha\tau\|_{L^2}\|\eta\|_{L^2}$$
$$\le C(n)\Big(\sum_{i=1}^{s-1}\|\mathrm{D}_x^{s+1-i}u\|_{L^4}\|\mathrm{D}_x^i\tau\|_{L^4} + \|\nabla u\|_{L^\infty}\|\tau\|_{H^s}\Big)\|\eta\|_{L^2}.$$

Then for any integer $1 \le j < s$, using the Gagliardo-Nirenberg interpolation inequalities (see [7, 9, 16]),

$$\|\mathrm{D}_x^j f\|_{L^4} \le C(n)\|\mathrm{D}_x^s f\|_{L^2}^{\frac{n-4j}{2(n-2s)}}\|f\|_{L^\infty}^{\frac{n-4(s-j)}{2(n-2s)}}, \tag{2.4}$$

we obtain

$$\|\mathrm{D}_x^{s+1-i}u\|_{L^4} \le C(n)\|\nabla v\|_{L^2}^{\frac{n-4(s-i)}{2(n-2s)}}\|\nabla u\|_{L^\infty}^{\frac{n-4i}{2(n-2s)}}, \tag{2.5}$$

$$\|\mathrm{D}_x^i\tau\|_{L^4} \le C(n)\|\eta\|_{L^2}^{\frac{n-4i}{2(n-2s)}}\|\tau\|_{L^\infty}^{\frac{n-4(s-i)}{2(n-2s)}}. \tag{2.6}$$

So

$$\int [\mathrm{D}_x^\alpha(u\cdot\nabla\tau) - u\cdot\nabla\mathrm{D}_x^\alpha\tau]\eta$$
$$\le C(n)\Big(\sum_{i=1}^{s-1}\|\nabla v\|_{L^2}^{\frac{n-4(s-i)}{2(n-2s)}}\|\nabla u\|_{L^\infty}^{\frac{n-4i}{2(n-2s)}}\|\eta\|_{L^2}^{\frac{n-4i}{2(n-2s)}}\|\tau\|_{L^\infty}^{\frac{n-4(s-i)}{2(n-2s)}} + \|\nabla u\|_{L^\infty}\|\eta\|_{L^2}\Big)\|\eta\|_{L^2}$$
$$\le C(n)\sum_{i=1}^{s-1}\|\nabla u\|_{H^s}^{\frac{n-4(s-i)}{2(n-2s)}}\|\tau\|_{H^s}^{\frac{3n-4(s+i)}{2(n-2s)}}\|\nabla u\|_{L^\infty}^{\frac{n-4i}{2(n-2s)}}\|\tau\|_{L^\infty}^{\frac{n-4(s-i)}{2(n-2s)}} + C(n)\|\nabla u\|_{L^\infty}\|\tau\|_{H^s}^2.$$

Finally by means of the Young's inequality we obtain (2.3).

Lemma 2.3 *For $(\nabla u, \tau) \in H^s(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ where $n = 2$ or 3, and for any positive integer $s > 1$, set $v = \mathrm{D}_x^\alpha u$, $\eta = \mathrm{D}_x^\alpha\tau$ where $\alpha \in Z_+^n$ with $|\alpha| = s$. Then we have*

$$\int \mathrm{D}_x^\alpha(\nabla u\tau)\eta + \int \mathrm{D}_x^\alpha(\tau\nabla^{\mathrm{T}}u)\eta \le C(n)\|\nabla u\|_{H^s}\|\tau\|_{L^\infty}\|\tau\|_{H^s} + \delta\|\nabla u\|_{H^s}^2$$
$$+ \frac{C(n)}{\delta}\|\tau\|_{H^s}^2(\|\nabla u\|_{L^\infty} + \|\tau\|_{L^\infty}^2 + 1)$$
$$+ C(n)\|\nabla u\|_{L^\infty}\|\tau\|_{H^s}^2, \tag{2.7}$$

where $\delta > 0$ and $C(n)$ is a constant depending only on n.

Proof By the Cauchy inequality, we get

$$\int \mathrm{D}_x^\alpha(\nabla u\tau)\eta + \int \mathrm{D}_x^\alpha(\tau\nabla^{\mathrm{T}}u)\eta \le (\|\mathrm{D}_x^\alpha(\nabla u\tau)\|_{L^2} + \|\mathrm{D}_x^\alpha(\tau\nabla^{\mathrm{T}}u)\|_{L^2})\|\eta\|_{L^2}$$
$$\le C(n)\Big(\|\nabla v\|_{L^2}\|\tau\|_{L^\infty} + \sum_{i=1}^{s-1}\|\mathrm{D}_x^{s+1-i}u\|_{L^4}\|\mathrm{D}_x^i\tau\|_{L^4}$$
$$+ \|\nabla u\|_{L^\infty}\|\tau\|_{H^s}\Big)\|\eta\|_{L^2}.$$

By using (2.5) and (2.6), we obtain

$$\int \mathrm{D}_x^\alpha(\nabla u\tau)\eta + \int \mathrm{D}_x^\alpha(\tau\nabla^{\mathrm{T}}u)\eta \le C(n)\|\nabla u\|_{H^s}\|\tau\|_{L^\infty}\|\tau\|_{H^s} + C(n)\sum_{i=1}^{s-1}\|\nabla u\|_{H^s}^{\frac{n-4(s-i)}{2(n-2s)}}\|\tau\|_{H^s}^{\frac{3n-4(s+i))}{2(n-2s)}}\|\nabla u\|_{L^\infty}^{\frac{n-4i}{2(n-2s)}}\|\tau\|_{L^\infty}^{\frac{n-4(s-i)}{2(n-2s)}} + C(n)\|\nabla u\|_{L^\infty}\|\tau\|_{H^s}^2.$$

Then by the Young's inequality, we get (2.7).

3 Proof of Theorem 1.1

As the local existence of the Oldroyd-B model is well-known, here we only present the H^s-estimates for the solution of (1.1)–(1.2), $s > [\frac{n}{2}]+1$.

Proof of Theorem 1.1 Actually let (u,τ) be the unique local solution to (1.1)–(1.2). It is sufficient to prove for any integer $s > [\frac{n}{2}]+1$,

$$\|u\|_{H^s} + \|\tau\|_{H^s} \le C_0, \quad t \le T^* \tag{3.1}$$

for some constant C_0 provided that

$$\int_0^{T*} \|\nabla u(t)\|_{L^\infty}\mathrm{d}t \triangleq M_0 < +\infty. \tag{3.2}$$

In what follows, we are going to present the L^2 estimate up to s order derivative of u and τ.

First, a standard energy estimate for system (1.1) gives

$$\begin{aligned}&\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2 + (1-\varepsilon)\int |\nabla u|^2 + \frac{\mathrm{We}}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int |\tau|^2 + \frac{1}{\varepsilon}\int |\tau|^2\\ &= \frac{\mathrm{We}}{\varepsilon}\int \nabla_j u_i\tau_{jl}\tau_{il} + \frac{\mathrm{We}}{\varepsilon}\int \tau_{ij}\nabla_j u_l\tau_{il} + \int \nabla_j u_i\tau_{ij}\\ &\le C(n)\Big(\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{L^\infty}\|\tau\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\tau\|_{L^2}\Big),\end{aligned} \tag{3.3}$$

where $C(n)$ is a constant depending only on n. Thus by the Cauchy inequality, we have

$$\begin{aligned}&\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \frac{1-\varepsilon}{2}\|\nabla u\|_{L^2}^2 + \frac{\mathrm{We}}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{L^2}^2\\ &\le C(n)\Big(\|\nabla u\|_{L^\infty} + \frac{1}{2(1-\varepsilon)}\frac{\varepsilon}{\mathrm{We}}\Big)\Big(\mathrm{Re}\|u\|_{L^2}^2 + \frac{\mathrm{We}}{\varepsilon}\|\tau\|_{L^2}^2\Big).\end{aligned} \tag{3.4}$$

Then

$$\mathrm{Re}\|u\|_{L^2}^2 + \frac{\mathrm{We}}{\varepsilon}\|\tau\|_{L^2}^2 \le \Big(\mathrm{Re}\|u_0\|_{L^2}^2 + \frac{\mathrm{We}}{\varepsilon}\|\tau_0\|_{L^2}^2\Big)\mathrm{e}^{C(n)\big(M_0 + \frac{1}{2(1-\varepsilon)}\frac{\varepsilon}{\mathrm{We}}T^*\big)}. \tag{3.5}$$

Next, by differentiating (1.1) we obtain

$$\begin{aligned}&\mathrm{Re}\Big(\frac{\partial\nabla u}{\partial t} + u\cdot\nabla(\nabla u)\Big) = -\nabla\nabla p + (1-\varepsilon)\,\triangle(\nabla u) + \mathrm{div}\,\nabla\tau - \mathrm{Re}\nabla u\cdot\nabla u,\\ &\frac{\partial(\nabla\tau)}{\partial t} + u\cdot\nabla(\nabla\tau) + \nabla u\cdot\nabla\tau = \nabla(\nabla u)\tau + (\nabla u)(\nabla\tau) + \nabla\tau(\nabla^{\mathrm{T}}u)\\ &\qquad + \tau\nabla(\nabla^{\mathrm{T}}u) - \frac{1}{\mathrm{We}}\nabla\tau + \frac{\varepsilon}{\mathrm{We}}(\nabla\nabla u + \nabla\nabla^{\mathrm{T}}u).\end{aligned}$$

In view of $\nabla\cdot u=0$, we get by the standard L^2 estimate

$$\begin{aligned}
&\mathrm{Re}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2+(1-\varepsilon)\int|\nabla^2u|^2+\frac{\mathrm{We}}{\varepsilon}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\tau|^2+\frac{1}{\varepsilon}\int|\nabla\tau|^2\\
&=-\mathrm{Re}\int\nabla_iu\cdot\nabla u_j\nabla_iu_j-\frac{\mathrm{We}}{\varepsilon}\int\nabla_iu\cdot\nabla\tau_{jl}\nabla_i\tau_{jl}+\frac{\mathrm{We}}{\varepsilon}\int\nabla_i\nabla_ku_j\tau_{kl}\nabla_i\tau_{jl}\\
&\quad+\frac{\mathrm{We}}{\varepsilon}\int\nabla_ku_j\nabla_i\tau_{kl}\nabla_i\tau_{jl}+\frac{\mathrm{We}}{\varepsilon}\int\nabla_i\tau_{jk}\nabla_ku_l\nabla_i\tau_{jl}\\
&\quad+\frac{\mathrm{We}}{\varepsilon}\int\tau_{jk}\nabla_i\nabla_ku_l\nabla_i\tau_{jl}+\int\nabla_i\nabla_lu_j\nabla_i\tau_{jl}\\
&\le\mathrm{Re}\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2+\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{L^\infty}\|\nabla\tau\|_{L^2}^2+\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{L^\infty}\|\nabla\tau\|_{L^2}\|\nabla^2u\|_{L^2}\\
&\quad+\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{L^\infty}\|\nabla\tau\|_{L^2}^2+\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{L^\infty}\|\nabla\tau\|_{L^2}^2\\
&\quad+\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{L^\infty}\|\nabla^2u\|_{L^2}\|\nabla\tau\|_{L^2}+\|\nabla^2u\|_{L^2}\|\nabla\tau\|_{L^2}.
\end{aligned}\tag{3.6}$$

By the Cauchy inequality, we have

$$\begin{aligned}
&\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2+\frac{1-\varepsilon}{2}\|\nabla^2u\|_{L^2}^2+\frac{\mathrm{We}}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tau\|_{L^2}^2\\
&\le C(n)\Big(\|\nabla u\|_{L^\infty}+\frac{1}{2(1-\varepsilon)}\frac{\mathrm{We}}{\varepsilon}\|\tau\|_\infty^2+\frac{1}{2(1-\varepsilon)}\Big)\Big(\mathrm{Re}\|\nabla u\|_{L^2}^2+\frac{\mathrm{We}}{\varepsilon}\|\nabla\tau\|_{L^2}^2\Big).
\end{aligned}\tag{3.7}$$

Using the equation for τ

$$\frac{\partial\tau}{\partial t}+(u\cdot\nabla)\tau=(\nabla u)\tau+\tau(\nabla^{\mathrm{T}}u)-\frac{1}{\mathrm{We}}\tau+\frac{\varepsilon}{\mathrm{We}}(\nabla u+\nabla^{\mathrm{T}}u),$$

by [12] one can get

$$\|\tau\|_{L^\infty}\le\Big(\|\tau_0\|_{L^\infty}+\frac{\varepsilon}{\mathrm{We}}M_0\Big)\mathrm{e}^{M_0}\triangleq M_1.\tag{3.8}$$

Hence we obtain

$$\mathrm{Re}\|u\|_{H^1}^2+\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^1}^2\le\Big(\mathrm{Re}\|u_0\|_{H^1}^2+\frac{\mathrm{We}}{\varepsilon}\|\tau_0\|_{H^1}^2\Big)\mathrm{e}^{C(n)\left(M_0+\frac{1}{2(1-\varepsilon)}\frac{\mathrm{We}}{\varepsilon}M_1^2T^*+\frac{1}{2(1-\varepsilon)}T^*\right)}.\tag{3.9}$$

Finally, for any positive integer $s>[\frac{n}{2}]+1$, we have the following equations for $v=\mathrm{D}_x^\alpha u$, $\eta=\mathrm{D}_x^\alpha\tau$ where $\alpha\in Z_+^n$ with $|\alpha|=s$:

$$\mathrm{Re}\Big(\frac{\partial v}{\partial t}+u\cdot\nabla v\Big)=-\mathrm{Re}[\mathrm{D}_x^\alpha(u\cdot\nabla u)-u\cdot\mathrm{D}_x^\alpha u]-\nabla\mathrm{D}_x^\alpha p+(1-\varepsilon)\,\triangle v+\operatorname{div}\eta,\tag{3.10}$$

$$\begin{aligned}
\frac{\partial\eta}{\partial t}+(u\cdot\nabla)\eta&=-[\mathrm{D}_x^\alpha(u\cdot\nabla\tau)-u\cdot\nabla\mathrm{D}_x^\alpha\tau]+\mathrm{D}_x^\alpha(\nabla u\tau)+\mathrm{D}_x^\alpha(\tau\nabla^{\mathrm{T}}u)\\
&\quad-\frac{1}{\mathrm{We}}\eta+\frac{\varepsilon}{\mathrm{We}}(\nabla v+\nabla^{\mathrm{T}}v),
\end{aligned}\tag{3.11}$$

and $\operatorname{div}v=0$. After multiplying both sides of (3.10) and (3.11) by v and η respectively, and integrating by parts, we obtain

$$\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|v|^2+(1-\varepsilon)\int|\nabla v|^2=\int\nabla_i\eta_{ij}v_j-\mathrm{Re}\int[\mathrm{D}_x^\alpha(u\cdot\nabla u)-u\cdot\nabla\mathrm{D}_x^\alpha u]v,\tag{3.12}$$

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\eta|^2+\frac{1}{\mathrm{We}}\int|\eta|^2&=-\int[\mathrm{D}_x^\alpha(u\cdot\tau)-u\cdot\nabla\mathrm{D}_x^\alpha\tau]\eta+\int\mathrm{D}_x^\alpha(\nabla u\tau)\eta\\
&\quad+\int\mathrm{D}_x^\alpha(\tau\nabla^{\mathrm{T}}u)\eta+\frac{\varepsilon}{\mathrm{We}}\int(\nabla_iv_j+\nabla_jv_i)\eta_{ij}
\end{aligned}\tag{3.13}$$

with the help of $\operatorname{div} v = 0$. Combining (3.12) and $\frac{\mathrm{We}}{\varepsilon}\cdot$(3.13) yields

$$\begin{aligned}&\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |v|^2 + (1-\varepsilon)\int |\nabla v|^2 + \frac{\mathrm{We}}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int |\eta|^2 + \frac{1}{\varepsilon}\int |\eta|^2\\ &= -\mathrm{Re}\int [\mathrm{D}_x^\alpha(u\cdot\nabla u) - u\cdot\nabla \mathrm{D}_x^\alpha u]v - \frac{\mathrm{We}}{\varepsilon}\int [\mathrm{D}_x^\alpha(u\cdot\nabla\tau) - u\cdot\nabla\mathrm{D}_x^\alpha\tau]\eta\\ &\quad + \frac{\mathrm{We}}{\varepsilon}\int \mathrm{D}_x^\alpha(\nabla u\tau)\eta + \frac{\mathrm{We}}{\varepsilon}\int \mathrm{D}_x^\alpha(\tau\nabla^{\mathrm{T}}u)\eta + \int(\nabla_i v_j)\eta_{ij}. \end{aligned}\tag{3.14}$$

Using (2.1), (2.3) and (2.7), we obtain

$$\begin{aligned}&\frac{\mathrm{Re}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 + (1-\varepsilon)\|\nabla u\|_{H^{s-1}}^2 + \frac{\mathrm{We}}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{H^s}^2 + \frac{1}{\varepsilon}\|\tau\|_{H^s}^2\\ &\le C(n)\mathrm{Re}\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2 + C(n)\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{H^s}\|\tau\|_{L^\infty}\|\tau\|_{H^s}\\ &\quad + \frac{\mathrm{We}}{\varepsilon}\delta\|\nabla u\|_{H^s}^2 + \frac{C(n)}{\delta}\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2(\|\nabla u\|_{L^\infty} + \|\tau\|_{L^\infty}^2 + 1)\\ &\quad + C(n)\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{L^\infty}\|\tau\|_{H^s}^2 + \|\nabla u\|_{H^s}\|\tau\|_{H^s}. \end{aligned}\tag{3.15}$$

By the Cauchy inequality and choosing δ small enough, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Re}\|u\|_{H^s}^2 + \frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2\Big) + \frac{1-\varepsilon}{2}\|\nabla u\|_{H^s}^2\\ &\le C(n)\Big[\|\nabla u\|_{L^\infty} + \frac{\mathrm{We}}{\varepsilon}\frac{1}{2(1-\varepsilon)}\|\tau\|_{L^\infty}^2 + \frac{1}{2(1-\varepsilon)}\Big]\Big(\mathrm{Re}\|u\|_{H^s}^2 + \frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2\Big). \end{aligned}\tag{3.16}$$

With the aid of the Gronwall's inequality, we get that $\|u\|_{H^s}$ and $\|\tau\|_{H^s}$ are bounded as $t \le T^*$ if (3.2) is satisfied. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Similarly to the proof of Theorem 1.1, we are going to present a global priori estimate of the solutions constructed in Theorem 1.1. Let (u,τ) be the unique solution constructed in Theorem 1.1 with the initial data (u_0,τ_0) which satisfies all the requirements in Theorem 1.1.

Actually, combining (3.3), (3.6) and (3.14), we can get

$$\begin{aligned}&\mathrm{Re}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^2}^2 + \frac{\mathrm{We}}{\varepsilon}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{H^2}^2 + (1-\varepsilon)\|\nabla u\|_{H^2}^2 + \frac{1}{\varepsilon}\|\tau\|_{H^2}^2\\ &= \frac{\mathrm{We}}{\varepsilon}\int \nabla_j u_i\tau_{jl}\tau_{il} + \frac{\mathrm{We}}{\varepsilon}\int \tau_{ij}\nabla_j u_l\tau_{il} + \int \nabla_j u_i\tau_{ij}\\ &\quad - \mathrm{Re}\int \nabla_i u\cdot\nabla u_j\nabla_i u_j - \frac{\mathrm{We}}{\varepsilon}\int \big(\nabla_i u\cdot\nabla\tau_{jl} - \nabla_i\nabla_k u_j\tau_{kl}\\ &\quad - \nabla_k u_j\nabla_i\tau_{kl} - \nabla_i\tau_{jk}\nabla_k u_l - \tau_{jk}\nabla_i\nabla_k u_l\big)\nabla_i\tau_{jl} + \int \nabla_i\nabla_l u_j\nabla_i\tau_{jl}\\ &\quad - \mathrm{Re}\int \big(\nabla_{ij}^2 u\cdot\nabla u_k + \nabla_i u\cdot\nabla\nabla_j u_k + \nabla_j u\cdot\nabla\nabla_i u_k\big)\nabla_{ij}^2 u_k\\ &\quad - \frac{\mathrm{We}}{\varepsilon}\int \big(\nabla_i u\cdot\nabla\nabla_j\tau_{kl} + \nabla_j u\cdot\nabla\nabla_i\tau_{kl} + \nabla_{ij}^2 u\cdot\nabla\tau_{kl} - \nabla_{ij}^2\nabla_r u_k\tau_{rl}\\ &\quad - \nabla_i\nabla_r u_k\nabla_j\tau_{rl} - \nabla_j\nabla_r u_k\nabla_i\tau_{kl} - \nabla_r u_k\nabla_{ij}^2\tau_{rl} - \nabla_{ij}^2\tau_{kr}\nabla_r u_l\\ &\quad - \nabla_i\tau_{kr}\nabla_j\nabla_r u_l - \nabla_j\tau_{kr}\nabla_i\nabla_r u_l - \tau_{kr}\nabla_{ij}^2\nabla_r u_l\big)\nabla_{ij}^2\tau_{kl} + \int \nabla_{ij}^2\nabla_l u_k\nabla_{ij}^2\tau_{kl}, \end{aligned}$$

so

$$\begin{aligned}&\mathrm{Re}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^2}^2+\frac{\mathrm{We}}{\varepsilon}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{H^2}^2+(1-\varepsilon)\|\nabla u\|_{H^2}^2+\frac{1}{\varepsilon}\|\tau\|_{H^2}^2\\ \le{}& C\Big(\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{L^\infty}\|\tau\|_{L^2}^2+\mathrm{Re}\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2+\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{L^\infty}\|\nabla\tau\|_{L^2}^2\\ &+\frac{\mathrm{We}}{\varepsilon}\|\nabla^2 u\|_{L^2}\|\tau\|_{L^\infty}\|\nabla\tau\|_{L^2}+\mathrm{Re}\|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2+\frac{\mathrm{We}}{\varepsilon}\|\nabla u\|_{L^\infty}\|\nabla^2\tau\|_{L^2}^2\\ &+\frac{\mathrm{We}}{\varepsilon}\|\nabla^2 u\|_{L^4}\|\nabla\tau\|_{L^4}\|\nabla^2\tau\|_{L^2}+\frac{\mathrm{We}}{\varepsilon}\|\nabla^3 u\|_{L^2}\|\nabla^2\tau\|_{L^2}\|\tau\|_{L^\infty}\Big)\\ &+\|\nabla u\|_{L^2}\|\tau\|_{L^2}+\|\nabla^2 u\|_{L^2}\|\nabla\tau\|_{L^2}+\|\nabla^3 u\|_{L^2}\|\nabla^2\tau\|_{L^2}.\end{aligned}$$

Using the Gagliardo-Nirenberg interpolation inequalities, we have

$$\|\nabla\tau\|_{L^4}\le C(n)\|\tau\|_{H^2},\quad \|\nabla^2 u\|_{L^4}\le C(n)\|\nabla u\|_{H^2}.$$

Therefore, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Re}\|u\|_{H^2}^2+\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^2}^2\Big)+(1-\varepsilon)\|\nabla u\|_{H^2}^2+\frac{1}{\varepsilon}\|\tau\|_{H^2}^2\\ \le{}& C(n)\Big(\|\nabla u\|_{H^2}\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^2}^2+\mathrm{Re}\|\nabla u\|_{H^2}^2\|u\|_{H^2}\Big)+\|\nabla u\|_{H^2}\|\tau\|_{H^2}.\end{aligned}\tag{4.1}$$

Similarly, for gerneral positive integer $s>[\frac{n}{2}]+1$, we can get the following estimate

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Re}\|u\|_{H^s}^2+\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2\Big)+(1-\varepsilon)\|\nabla u\|_{H^s}^2+\frac{1}{\varepsilon}\|\tau\|_{H^s}^2\\ \le{}& C(n)\Big(\|\nabla u\|_{H^s}\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2+\mathrm{Re}\|\nabla u\|_{H^s}^2\|u\|_{H^s}\Big)+\|\nabla u\|_{H^s}\|\tau\|_{H^s}.\end{aligned}\tag{4.2}$$

When $0<\varepsilon<\frac{4}{5}$, we can choose $k_1,k_2>0$ satisfying $k_1+k_2<1-\varepsilon$ and $\frac{1}{4k_2}<\frac{1}{\varepsilon}$ such that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Re}\|u\|_{H^s}^2+\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2\Big)+(1-\varepsilon)\|\nabla u\|_{H^s}^2+\frac{1}{\varepsilon}\|\tau\|_{H^s}^2\\ \le{}& k_1\|\nabla u\|_{H^s}^2+\frac{1}{4k_1}\Big(C(n)\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2\Big)^2+C(n)\mathrm{Re}\|\nabla u\|_{H^s}^2\|u\|_{H^s}+k_2\|\nabla u\|_{H^s}^2+\frac{1}{4k_2}\|\tau\|_{H^s}^2,\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Re}\|u\|_{H^s}^2+\frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2\Big)\le{}&-[(1-\varepsilon)-k_1-k_2-C(n)\mathrm{Re}\|u\|_{H^s}]\|\nabla u\|_{H^s}^2\\ &-\Big[\frac{1}{\varepsilon}-\frac{1}{4k_2}-\frac{1}{4k_1}C(n)^2\frac{\mathrm{We}^2}{\varepsilon^2}\|\tau\|_{H^s}^2\Big]\|\tau\|_{H^s}^2.\end{aligned}\tag{4.3}$$

Set

$$A_0=\max\Big\{\|u_0\|_{H^s}^2+\frac{\mathrm{We}}{\mathrm{Re}\varepsilon}\|\tau_0\|_{H^s}^2,\ \frac{\varepsilon}{\mathrm{We}}\mathrm{Re}\|u_0\|_{H^s}^2+\|\tau_0\|_{H^s}^2\Big\}.$$

If A_0 is so small that

$$(1-\varepsilon)-k_1-k_2-C(n)\mathrm{Re}\sqrt{A_0}>0\tag{4.4}$$

and

$$\frac{1}{\varepsilon}-\frac{1}{4k_2}-\frac{1}{4k_1}C(n)^2\frac{\mathrm{We}^2}{\varepsilon^2}A_0>0,\tag{4.5}$$

from (4.3) it turns out that

$$\mathrm{Re}\|u\|_{H^s}^2 + \frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2 \le \mathrm{Re}\|u_0\|_{H^s}^2 + \frac{\mathrm{We}}{\varepsilon}\|\tau_0\|_{H^s}^2.$$

Hence $\|u\|_{H^s}^2 \le A_0$ and $\|\tau\|_{H^s}^2 \le A_0$. In view of the choice of A_0 in (4.4) and (4.5), (4.3) tells us $\mathrm{Re}\|u\|_{H^s}^2 + \frac{\mathrm{We}}{\varepsilon}\|\tau\|_{H^s}^2$ is monotone-decreasing. Therefore, if the initial data are small enough, then $\|u\|_{H^s}$ and $\|\tau\|_{H^s}$ will be bounded for all $t < \infty$. Using the blow up principle in Theorem 1.1, we finish the proof of Theorem 1.2.

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References

[1] Beale, J. T., Kato, T. and Majda, A., Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Comm. Math. Phys.*, **94**(1), 1984, 61–66.

[2] Bird, R. B., Armstrong, R. C. and Hassager, O., Fluid Mechanics, Dynamics of Polymeric Liquids, Vol. 1, John Wiley and Sons, New York, 1987.

[3] Chemin, J. Y. and Masmoudi, N., About lifespan of regular solutions of equations related to viscoelastic fluids, *SIAM J. Math. Anal.*, **33**(1), 2001, 84–112.

[4] Chen, Y. and Zhang, P., The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions, *Comm. Part. Diff. Eqs.*, **31**(10–12), 2006, 1793–1810.

[5] de Gennes, P. G. and Prost, J., The Physics of Liquid Crystals, Oxford University Press, New York, 1993.

[6] E, W. N., Li, T. J. and Zhang, P. W., Well-posedness for the dumbbell model of polymeric fluids, *Comm. Math. Phys.*, **248**(2), 2004, 409–427.

[7] Friendman, A., Partial Differential Equations, Holt, Rinehart and Winston, New York, 1969.

[8] Guillopé, C. and Saut, J.-C., Existence results for the flow of viscoelastic fluids with a differential constitutive law, *Nonlinear Anal.*, **15**(9), 1990, 849–869.

[9] Heywood, J. G., The Navier-Stokes equations: on the existence, regularity and decay of solutions, *Indiana Univ. Math. J.*, **29**(5), 1980, 639–681.

[10] Larson, R. G., The Structure and Rheology of Complex Fluids, Oxford University Press, New York, 1995.

[11] Lei, Z., Liu, C. and Zhou, Y., Global solutions for incompressible viscoelastic fluids, *Arch. Rational Mech. Anal.*, **188**, 2008, 371–398.

[12] Li, T. T., Global Classical Solutions for Quasilinear Hyperbolic Systems, Wiley, Chichester, New York, Paris, 1994.

[13] Lin, F. H., Liu, C. and Zhang, P., On hydrodynamics of viscoelastic fluids, *Comm. Pure Appl. Math.*, **58**(11), 2005, 1437–1471.

[14] Lin, F. H., Liu, C. and Zhang, P., On a micro-macro model for polymeric fluids near equilibrium, *Comm. Pure Appl. Math.*, **60**(6), 2007, 838–866.

[15] Lin, F. H. and Zhang, P., On the initial boundary value problem of the incompressible viscoelastic fluid system, *Comm. Pure Appl. Math.*, **61**(4), 2008, 539–558.

[16] Nirenberg, L., On elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa*, **13**, 1959, 115–162.

[17] Oldroyd, J. G., Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids, *Proc. Roy. Soc. London. Ser. A*, **245**, 1958, 278–297.

[18] Renardy, M., Hrusa, W. J. and Nohel, W. J., Mathematics Problems in Viscoelasticity, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 35, Longman Scientific and Technical, Harlow, 1987.

[19] Zheng, S. M., Nonlinear Evolution Equations, Monographs and Surveys in Pure and Applied Mathematics, Vol. 133, Chapman & Hall/CRC, Boca Raton, Florida, 2004.