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The Second Type Singularities of Symplectic and Lagrangian Mean Curvature Flows*

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Abstract This paper mainly deals with the type II singularities of the mean curvature flow from a symplectic surface or from an almost calibrated Lagrangian surface in a Kähler surface. The relation between the maximum of the Kähler angle and the maximum of $|H|^2$ on the limit flow is studied. The authors also show the nonexistence of type II blow-up flow of a symplectic mean curvature flow which is normal flat or of an almost calibrated Lagrangian mean curvature flow which is flat.

Keywords Symplectic surface, Lagrangian surface, Mean curvature flow
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1 Introduction

Suppose that M is a compact Kähler surface. Let Σ be a smooth surface in M and ω, $\langle\cdot,\cdot\rangle$ be the Kähler form and the Kähler metric on M respectively. The Kähler angle α of Σ in M is defined by Chern-Wolfson [6]

$$\omega|_\Sigma = \cos\alpha \mathrm{d}\mu_\Sigma,$$

where $\mathrm{d}\mu_\Sigma$ is the area element of Σ of the induced metric from $\langle\,,\,\rangle$. We call Σ a symplectic surface if $\cos\alpha > 0$, a Lagrangian surface if $\cos\alpha \equiv 0$, a holomorphic curve if $\cos\alpha \equiv 1$. If we assume in addition that M is a Calabi-Yau complex surface with a complex structure J, we consider a parallel holomorphic $(2,0)$ form Ω for a Lagrangian surface Σ we have (see [13])

$$\Omega|_\Sigma = \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\mu_\Sigma,$$

where θ is a multivalued function called Lagrangian angle. If $\cos\theta > 0$, then Σ is called almost calibrated. If $\theta \equiv$ constant, then Σ is a special Lagrangian.

It is proved in [2, 22] that, if the initial surface is symplectic, then along the mean curvature flow, at each time t the surface Σ_t is still symplectic. Thus we speak of symplectic mean curvature flow. It is proved in [19] that, if the initial surface is Lagrangian, then along the mean curvature flow, at each time t the surface Σ_t is still Lagrangian. Thus we speak of Lagrangian mean curvature flow. The symplectic mean curvature flow was studied in [2–4, 10, 11, 22]. There are many references for Lagrangian mean curvature flows (see [8, 16–21]).

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In [10], we showed that, if the scalar curvature of the compact Kähler-Einstein surface M is positive and the initial surface is sufficiently close to a holomorphic curve, then the mean curvature flow has a global solution and converges to a holomorphic curve.

In general, the mean curvature flow may produce singularities. The singularities of the mean curvature flow of convex hypersurfaces were studied by Huisken-Sinestrari [14, 15] and White [23]. For symplectic mean curvature flow or almost calibrated Lagrangian mean curvature flow, Chen-Li [2, 3] and Wang [22] proved that there is no Type I singularity.

We consider the strong convergence of the rescaled surfaces Σ_s^k in $B_R(0)$ around a type II singular point X_0. Let $|A_k|$ be the norm of the second fundamental forms of Σ_s^k in $B_R(0)$. Then we have that $|A_k|^2 \le 4$ in $B_R(0)$ during the rescaling process. Thus by Arzela-Ascoli theorem, $\Sigma_s^k \to \Sigma_s^\infty$ in $C^2(B_R(0) \times [-R, R])$ for any $R > 0$ and any $B_R(0) \subset \mathbb{C}^2$. By the definition of the type II singularity, we know that Σ_s^∞ is defined on $(-\infty, +\infty)$ and Σ_s^∞ also evolves along the mean curvature flow in $\mathbb{C}^2$ with the Euclidean metric. We call Σ_s^∞ the limit flow or the blow-up flow at X_0. See Section 2 for details.

An important example of type II singularity is the translating soliton (see [9, 15]). Symplectic or Lagrangian translating solitons were studied in [11, 12, 16, 18] recently. In [11, 12, 18], some kinds of Liouville theorems were proved, and in [16], the authors constructed Lagrangian translating solitons.

In this paper, we mainly study the nature of the general limit flow Σ_s^∞. For this purpose, we consider a general mean curvature flow Σ_t in $\mathbb{R}^4$ which exists globally with bounded second fundamental forms and the following property:

$$\mu_t(\Sigma_t \cap B_R(0)) \le CR^2, \tag{1.1}$$

where $0 < C < \infty$ is a constant independent of t and R.

Theorem 1.1 *Suppose that Σ_t $(t \in (-\infty, 0])$ is a complete symplectic mean curvature flow with $\cos\alpha \ge \delta > 0$ in $\mathbb{C}^2$ which satisfies* (1.1). *Assume that* $\sup_{t\in(-\infty,0]} \sup_{\Sigma_t} |A|^2 = 1$. *Then we have*

$$h^2 = \sup_{t\in(-\infty,0]} \sup_{\Sigma_t} |H|^2 \le 4 \sup_{t\in(-\infty,0]} \sup_{\Sigma_t} \log \frac{1}{1 - 2\sin^2\frac{\alpha}{2}}.$$

For the almost calibrated Lagrangian mean curvature flow, we have the following result.

Theorem 1.2 *Suppose that Σ_t $(t \in (-\infty, 0])$ is a complete almost calibrated Lagrangian mean curvature flow with $\cos\theta \ge \delta > 0$ in $\mathbb{C}^2$ which satisfies* (1.1). *Assume further that* $\sup_{t\in(-\infty,0]} \sup_{\Sigma_t} |A|^2 = 1$. *Then we have*

$$h^2 = \sup_{t\in(-\infty,0]} \sup_{\Sigma_t} |H|^2 \le \Big(\sup_{t\in(-\infty,0]} \sup_{\Sigma_t} \theta - \inf_{t\in(-\infty,0]} \inf_{\Sigma_t} \theta\Big)^2.$$

On the other hand, applying the techniques used in [12], we can rule out the existence of type II blow-up flows for a symplectic mean curvature flow which are normal flat. More precisely, we prove the theorem below.

Theorem 1.3 *Suppose that Σ_t $(t \in (-\infty, 0])$ is a complete symplectic mean curvature flow with $\cos\alpha \geq \delta > 0$ in $\mathbb{C}^2$ which satisfies* (1.1). *Assume that $\sup\limits_{t\in(-\infty,0]} \sup\limits_{\Sigma_t} |A|^2 = 1$. Then $\{\Sigma_t\}_{t\in(-\infty,0]}$ can not be normal flat all the time.*

Analogously for the almost calibrated Lagrangian mean curvature flow, we show the result as follows.

Theorem 1.4 *Suppose that Σ_t $(t \in (-\infty, 0])$ is a complete almost calibrated Lagrangian mean curvature flow with $\cos\theta \geq \delta > 0$ in $\mathbb{C}^2$ which satisfies* (1.1). *Assume further that*

$$\sup_{t\in(-\infty,0]} \sup_{\Sigma_t} |A|^2 = 1.$$

Then $\{\Sigma_t\}_{t\in(-\infty,0]}$ can not be flat all the time.

Theorems 1.3 and 1.4 imply that it is important to know whether or under what condition, the blow-up flow of a symplectic mean curvature flow is normal flat or an almost calibrated Lagrangian mean curvature flow is flat. In fact, as we know (see [1]), the type II blow-up flow of a curve shrinking flow for space curves is a planar curve.

2 Preparations

In this section, we define the rescaled surfaces and study the strong convergence of the rescaled sequence at a type II singular point, which is more or less standard. However, we can not find it in a reference, so we give all details here. It may be interesting in its own right. Suppose that T is discrete singular time, that means there exists an $\varepsilon > 0$ such that the mean curvature flow is smooth in $[T-\varepsilon, T)$. Assume that the mean curvature flow develops a type II singularity at time T. Let X_0 be a type II singular point of the mean curvature flow in M, that means,

$$\max_{B_r(X_0)\cap\Sigma_t} |A|^2 \geq \frac{C}{T-t} \quad \text{for any } i_M > r > 0,\ C > 0,$$

where i_M is the injective radius of M. Then for any sequence $\{r_k\}$ with $r_k \to 0$,

$$\begin{aligned}
&\max_{\sigma\in(0,\frac{r_k}{2}]} \sigma^2 \max_{[T-(r_k-\sigma)^2, T-(\frac{r_k}{2})^2]} \max_{\Sigma_t\cap B_{r_k-\sigma}(X_0)} |A|^2 \\
&\geq \Big(\frac{r_k}{2}\Big)^2 \max_{\Sigma_{T-(\frac{r_k}{2})^2}\cap B_{\frac{r_k}{2}}(X_0)} |A|^2 \\
&= \Big(T - \Big(T - \Big(\frac{r_k}{2}\Big)^2\Big)\Big) \max_{\Sigma_{T-(\frac{r_k}{2})^2}\cap B_{\frac{r_k}{2}}(X_0)} |A|^2 \\
&\to +\infty.
\end{aligned}$$

We choose $\sigma_k \in (0, \frac{r_k}{2}]$ such that

$$\sigma_k^2 \max_{[T-(r_k-\sigma_k)^2, T-(\frac{r_k}{2})^2]} \max_{\Sigma_t\cap B_{r_k-\sigma_k}(X_0)} |A|^2 = \max_{\sigma\in(0,\frac{r_k}{2}]} \sigma^2 \max_{[T-(r_k-\sigma)^2, T-(\frac{r_k}{2})^2]} \max_{\Sigma_t\cap B_{r_k-\sigma}(X_0)} |A|^2.$$

Let $t_k \in [T-(r_k-\sigma_k)^2, T-(\frac{r_k}{2})^2]$ and $F(x_k, t_k) = X_k \in \overline{B}_{r_k-\sigma_k}(X_0)$ satisfy

$$\lambda_k^2 = |A|^2(X_k) = |A|^2(x_k, t_k) = \max_{[T-(r_k-\sigma_k)^2, T-(\frac{r_k}{2})^2]} \max_{\Sigma_t\cap B_{r_k-\sigma_k}(X_0)} |A|^2.$$

Obviously, we have $(X_k, t_k) \to (X_0, T)$ and $\lambda_k^2\sigma_k^2 \to \infty$. In particular,

$$\max_{[T-(r_k-\frac{\sigma_k}{2})^2, T-(\frac{r_k}{2})^2]} \max_{\Sigma_t \cap B_{r_k-\frac{\sigma_k}{2}}(X_0)} |A|^2 \le 4\lambda_k^2, \tag{2.1}$$

and hence

$$\max_{[t_k-(\frac{\sigma_k}{2})^2, t_k]} \max_{\Sigma_t \cap B_{r_k-\frac{\sigma_k}{2}}(X_0)} |A|^2 \le 4\lambda_k^2. \tag{2.2}$$

We now describe the rescaling process around (X_0, T) in details. The argument is discussed with Chen. In the following, we denote the points of the image of F or F_k in M by capital letters. We choose a normal coordinates in $B_r(X_0)$ using the exponential map, where $B_r(X_0)$ is a metric ball in M centered at X_0 with radius r $(0 < r < \frac{i_M}{2})$. We express F in its coordinates functions. Consider the following sequences:

$$F_k(x,s) = \lambda_k(F(x_k + x, t_k + \lambda_k^{-2}s) - F(x_k, t_k)), \quad s \in \Big[-\lambda_k^2\frac{\sigma_k^2}{4}, \lambda_k^2(T-t_k)\Big]. \tag{2.3}$$

We denote the rescaled surfaces by Σ_s^k, in which $d\mu_s^k$ is the induced area element from M. For any $R > 0$, let $B_R(0)$ be a ball in $\mathbb{R}^4$ with radius R in the Euclidean metric and centered at 0. Then

$$\Sigma_s^k \cap B_R(0) = \{|F_k(x,s)| \le R\},$$

it is clear that for any fixed $R > 0$, $\lambda_k^{-1}R < \frac{r}{2}$, $r_k < \frac{r}{2}$ as k sufficiently large. Then the surface Σ_s^k is defined in $B_R(0)$ because

$$\begin{aligned}\exp_{X_0}(\lambda_k^{-1}\{|F_k(x,s)| \le R\}) &\subset \exp_{X_0}(|F - X_0| \le \lambda_k^{-1}R + r_k)\\ &\subset B_{\lambda_k^{-1}R+r_k}(X_0) \subset B_r(X_0).\end{aligned}$$

Moreover, we pull back the metric on $B_r(X_0) \subset M$ via $\exp_{X_0}$ so that we get a metric h on the Euclidean ball $B_r(0)$. Then for any fixed $R > 0$ such that $\lambda_k^{-1}R < \frac{r}{2}$, we can define a metric $h_{k,R}$ on $B_R(0)$,

$$(h_{k,R})_{ij}(X) = \lambda_k^2 h(\lambda_k^{-1}X + X_k).$$

With respect to this metric Σ_s^k evolves along the mean curvature flow, which is derived as follows.

If g_s^k is the metric on Σ_s^k which is induced from the metric $g(\cdot, t_k + \lambda_k^{-1}s)$ on $\Sigma_{t_k+\lambda_k^{-1}s}$, it is clear that

$$(g_s^k)_{ij}(X) = \lambda_k^2 g_{ij}(\lambda_k^{-1}X + X_k, t_k + \lambda_k^{-2}s)$$

and

$$(g_s^k)^{ij}(X) = \lambda_k^{-2} g^{ij}(\lambda_k^{-1}X + X_k, t_k + \lambda_k^{-2}s).$$

In this setting, (Σ_s^k, g_s^k) is an isometric immersion in $(B_R(0), h_{k,R})$. Let A_k, H_k be the second fundamental form and the mean curvature vector of (Σ_s^k, g_s^k) in $(B_R(0), h_{k,R})$ respectively. Let $\overline{\Gamma}^k$, Γ_s^k be the Christoffel symbols of $h_{k,R}$ on $B_R(0)$ and the Christoffel symbols of g_s^k on Σ_s^k. Since F_k is an isometric immersion in $(B_R(0), h_{k,R})$ with respect to the induced metric, by the Gaussian equation, we have

$$(A_k)_{ij} = \sum_{\alpha=1,2} (h_k)_{ij}^{\alpha}\nu_{s\alpha}^k = -\partial_{ij}^2 F_k + \sum_{l=1,2} (\Gamma_s^k)_{ij}^l \partial_l F_k - \sum_{\alpha,\beta,\gamma=1,4} (\overline{\Gamma}^k)_{\beta\gamma}^{\alpha} \partial_i F_k^{\beta} \partial_j F_k^{\gamma} \nu_{s\alpha}^k, \tag{2.4}$$

where $\{\nu^k_{s\alpha}, \alpha = 1,2\}$ are bases of the normal space of Σ^k_s in $(B_R(0), h_{k,R})$. Let $\Gamma_{t_k+\lambda_k^{-2}s}$ be the Christoffel symbols on $\Sigma_{t_k+\lambda_k^{-2}s}$ and $\overline{\Gamma}$ be the Christoffel symbols on M. It is not hard to check that

$$\overline{\Gamma}^k(X) = \overline{\Gamma}(\lambda_k^{-1}X + X_k), \quad \Gamma^k_s(X) = \Gamma_{t_k+\lambda_k^{-2}s}(\lambda_k^{-1}X + X_k).$$

Thus from (2.4), we get that

$$(A_k)_{ij} = \lambda_k\Big(-\partial^2_{ij}F + \sum_{l=1,2}(\Gamma_{t_k+\lambda_k^{-2}s})^l_{ij}\partial_l F_k - \sum_{\alpha,\beta,\gamma=1,4}\overline{\Gamma}^\alpha_{\beta\gamma}\partial_i F^\beta_k \partial_j F^\gamma_k \nu_\alpha\Big) = \lambda_k A_{ij}, \tag{2.5}$$

where $\{v_\alpha, \alpha = 1,2\}$ are bases of the normal space of $\Sigma_{t_k+\lambda_k^{-2}s}$ in M. Therefore,

$$|A_k|^2 = \lambda_k^{-2}|A|^2, \quad H_k = \lambda_k^{-1}H, \quad |H_k|^2 = \lambda_k^{-2}|H|^2.$$

Set $t = t_k + \lambda_k^{-2}s$. It is easy to check that

$$\frac{\partial F_k}{\partial s} = \lambda_k^{-1}\frac{\partial F}{\partial t}.$$

Therefore, it follows that the rescaled surface also evolves by a mean curvature flow

$$\frac{\partial F_k}{\partial s} = H_k \tag{2.6}$$

in $B_{\lambda_k\sigma_k}(0)$, where $s \in [-\lambda_k^2\frac{\sigma_k^2}{4}, \lambda_k^2(T - t_k)]$.

By (2.1) and (2.2), we see that

$$|A_k|(0,0) = 1, \quad |A_k|^2 \le 4$$

in $B_{\lambda_k\sigma_k}(0)$ and $s \in [-\lambda_k^2\frac{\sigma_k^2}{4}, \lambda_k^2(T - t_k)]$. Since (X_0, T) is a type II singularity, we have $\lambda_k^2\sigma_k^2 \to \infty$ and $\lambda_k^2(T - t_k) \to \infty$. Thus by Arzela-Ascoli theorem, $\Sigma^k_s \to \Sigma^\infty_s$ in $C^2(B_R(0) \times [-R, R])$ for any $R > 0$ and any $B_R(0) \subset \mathbb{C}^2$. By (2.3), we know that Σ^∞_s is defined on $(-\infty, +\infty)$. Since for each fixed $R > 0$, $\lambda_k^{-1}X + X_k \to X_0$ for $X \in B_R(0)$ as $k \to \infty$, we get that $h_{k,R}$ converges uniformly in $B_R(0)$ to the Euclidean metric as $k \to \infty$, and the Christoffel symbols $(\overline{\Gamma}^k)$ of $h_{k,R}$ converge uniformly in $B_R(0)$ to 0 as $k \to \infty$. We see that Σ^∞_s also evolves along the mean curvature flow in $\mathbb{C}^2$ with the Euclidean metric. We call Σ^∞_s the limit flow or the blow-up flow at X_0.

In the rest part of this section, we estimate the difference of A_k, H_k and A^0_k, H^0_k, where A^0_k and H^0_k are the second fundamental form and the mean curvature vector of Σ^k_s in the Euclidean metric on $B_R(0)$ respectively. Although it is not needed in this paper, it is interesting in its own right.

Let Γ^{0k}_s be the Christoffel symbols of Σ^k_s for the Euclidean metric on $B_R(0)$, and $\{\nu^{0k}_{s\alpha} : \alpha = 1,2\}$ be bases of the normal space of Σ^k_s with respect to the Euclidean metric on $B_R(0)$. Similarly, considering F_k as an isometric immersion in $B_R(0)$ with the Euclidean metric, we have

$$(A^0_k)_{ij} = \sum_{\alpha=1,2}(h_0)^\alpha_{ij}(\nu^{0k}_s)_\alpha = -\partial^2_{ij}F_k + \sum_{l=1,2}(\Gamma^{0k}_s)^l_{ij}\partial_l F_k. \tag{2.7}$$

Note that the induced metric on Σ_s^k from $h_{k,R}$ is given by $\langle \partial F_k, \partial F_k\rangle_{h_{k,R}}$, so it holds that

$$|\partial F_k|^2_{h_{k,R}} = 2,$$

which in turn implies that, for k sufficiently large and R fixed, $|\partial F_k^\alpha|$ is uniformly bounded in $B_R(0)$ with the Euclidean metric.

Using the Euclidean metric on $B_R(0)$, we decompose the tangent bundle of $B_R(0)$ along Σ_s^k into the tangential component $T\Sigma_s^k$ and the normal component $T^\perp\Sigma_s^k$. Let $A_k^\perp : T\Sigma_s^k \times T\Sigma_s^k \to T^\perp\Sigma_s^k$ be the normal component of A_k. Noticing that $A_k^\perp - A_k^0$ lies in $T^\perp\Sigma_s^k$ and $\partial_i F_k$ lies in $T\Sigma_s^k$, it follows from (2.4) and (2.5) that

$$\sup_{B_R(0)} |A_k^\perp - A_k^0| \le C \sup_{B_R(0)} |\overline{\Gamma}^k| \to 0,$$

as $k \to \infty$ for any fixed $R > 0$. From the uniform convergence of the metrics $h_{k,R}$ to the Euclidean metric, we have

$$|A_k^\perp| \le |A_k| \le 2|A_k|_{h_{k,R}}$$

for any fixed $R > 0$ and sufficiently large k. Hence, there exist positive constants $\delta_{k,R}$ which tend to 0 as $k \to \infty$ such that

$$|A_k^0| = |A_k^\perp| + \delta_{k,R} \le 2|A_k|_{h_{k,R}} + \delta_{k,R}$$

for all sufficiently large k and any fixed $R > 0$; and similarly there exist constants $\delta'_{k,R} > 0$ with $\delta'_{k,R} \to 0$ as $k \to \infty$ such that

$$|H_k^0| \le 2|H_k|_{h_{k,R}} + \delta'_{k,R}$$

for sufficiently large k and any given $R > 0$.

3 Proofs of Theorem 1.1 and Theorem 1.2

Now we begin to prove our main theorems. We first prove Theorem 1.2. Let $H(X, X_0, t, t_0)$ be the backward heat kernel on $\mathbb{R}^4$. Let Σ_t be a smooth family of surfaces in $\mathbb{R}^4$ defined by $F_t : \Sigma \to \mathbb{R}^4$. Define

$$\rho(X,t) = (4\pi(t_0 - t))H(X, X_0, t, t_0) = \frac{1}{4\pi(t_0 - t)} \exp -\frac{|X - X_0|^2}{4(t_0 - t)}$$

for $t < t_0$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\Delta\rho - \rho\Big(\Big|H + \frac{(X - X_0)^\perp}{2(t_0 - t)}\Big|^2 - |H|^2\Big),$$

where $(X - X_0)^\perp$ is the normal component of $X - X_0$.

Define

$$\Psi_{X_0,t_0}(X,t) = \int_{\Sigma_t} \frac{1}{\cos\theta}\rho(X,t)\mathrm{d}\mu_t.$$

Proposition 3.1 *Along the almost calibrated Lagrangian mean curvature flow Σ_t in $\mathbb{R}^4$, we have*

$$\frac{\partial}{\partial t}\Psi_{X_0,t_0}(X,t) = -\Big(\int_{\Sigma_t}\frac{1}{\cos\theta}\rho(F,t)\Big|H+\frac{(F-X_0)^\perp}{2(t_0-t)}\Big|^2\mathrm{d}\mu_t + \int_{\Sigma_t}\frac{1}{\cos\theta}\rho(F,t)|H|^2\mathrm{d}\mu_t + \int_{\Sigma_t}\frac{2}{\cos^3\theta}\left|\nabla\cos\theta\right|^2\rho(F,t)\mathrm{d}\mu_t\Big).$$

Proof From the evolution equation of Lagrangian angle (see [19, 20]),

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\cos\theta = |H|^2\cos\theta, \tag{3.1}$$

we know

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\frac{1}{\cos\theta} = -\frac{|H|^2}{\cos\theta}-2\frac{|\nabla\cos\theta|^2}{\cos^3\theta}. \tag{3.2}$$

Recall the general formula (7) in [7], for a smooth function $f=f(x,t)$ on Σ_t with polynomial growth at infinity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}f\rho\mathrm{d}\mu_t = \int_{\Sigma_t}\Big(\frac{\mathrm{d}}{\mathrm{d}t}f-\Delta f\Big)\rho\mathrm{d}\mu_t - \int_{\Sigma_t}f\rho\Big|H+\frac{(X-X_0)^\perp}{2(t_0-t)}\Big|\mathrm{d}\mu_t. \tag{3.3}$$

Choosing $f=\frac{1}{\cos\theta}$ in (3.3) and putting (3.2) into (3.3), we get our monotonicity formula.

Proof of Theorem 1.2 Without loss of generality, we may assume

$$\inf_{t\in(-\infty,0]}\inf_{\Sigma_t}\theta = 0.$$

If $h=0$, or $\eta := \sup\limits_{t\in(-\infty,0]}\sup\limits_{\Sigma_t}\theta = 0$, it is evident that the result holds. Now we assume that $h>0$, $\eta>0$.

Fix any $R>0$ and set $X_0=0$. First we claim that there exists a sequence $\{s_i\}$ such that $s_i\to-\infty$ as $i\to\infty$ and $\lim\limits_{i\to\infty}\max\limits_{\Sigma_{s_i}\cap B_R(X_0)}|H|^2=0$. Integrating the monotonicity formula in Proposition 3.1 with $t_0=0$ from $2s$ to s for $s<0$, we get

$$\int_{\Sigma_{2s}}\frac{1}{\cos\theta(x,2s)}\frac{1}{-2s}\mathrm{e}^{\frac{|F|^2}{8s}}\mathrm{d}\mu_{2s} - \int_{\Sigma_s}\frac{1}{\cos\theta(x,s)}\frac{1}{-s}\mathrm{e}^{\frac{|F|^2}{4s}}\mathrm{d}\mu_s \ge \int_{2s}^{s}\int_{\Sigma_t}\frac{1}{\cos\theta}\rho(F,t)|H|^2\mathrm{d}\mu_t\mathrm{d}t.$$

By Proposition 3.1, we know that $\int_{\Sigma_s}\frac{1}{\cos\theta}\rho(F,s)$ is nonincreasing in s. Since $\cos\theta$ is bounded below by δ, for any $t<0$, we have

$$\begin{aligned}\int_{\Sigma_t}\frac{1}{\cos\theta}\rho(X,t)\mathrm{d}\mu_t &\le \frac{1}{\delta}\int_{\Sigma_t}\rho(X,t)\mathrm{d}\mu_t\\ &\le \frac{C}{\delta}\int_0^\infty\int_{\Sigma_t\cap\partial B_r(0)}\frac{1}{0-t}\mathrm{e}^{\frac{r^2}{4t}}\mathrm{d}\sigma_t\mathrm{d}r\\ &\le \frac{C}{-t}\int_0^\infty\mathrm{e}^{\frac{r^2}{4t}}\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{vol}(B_r(0)\cap\Sigma_t)\mathrm{d}r\\ &\le \frac{C}{-t}\Big[\mathrm{e}^{\frac{r^2}{4t}}\mathrm{vol}(B_r(0)\cap\Sigma_t)\big|_{r=0}^\infty - \int_0^\infty\mathrm{vol}(B_r(0)\cap\Sigma_t)\mathrm{e}^{\frac{r^2}{4t}}\frac{2r}{4t}\mathrm{d}r\Big],\end{aligned}$$

where we denote by $C>0$ the constant which does not depend on t and may change from one line to another line. Since we have assumed that $\mu_t(B_R(0)\cap\Sigma_t)\le CR^2$ in (1.1), we have

$$\begin{aligned}\int_{\Sigma_t}\frac{1}{\cos\theta}\rho(X,t)\mathrm{d}\mu_t &\le C\Big[\frac{1}{-t}\mathrm{e}^{\frac{r^2}{4t}}r^2\Big|_{r=0}^{\infty}+\int_0^\infty\frac{2r^3}{4t^2}\mathrm{e}^{\frac{r^2}{4t}}\mathrm{d}r\Big]\\ &\le C\Big[\frac{1}{-t}\mathrm{e}^{\frac{r^2}{4t}}r^2+\mathrm{e}^{\frac{r^2}{4t}}\frac{r^2}{t}-4\mathrm{e}^{\frac{r^2}{4t}}\Big]\Big|_{r=0}^{\infty}\\ &\le C.\end{aligned}$$

Thus the quantity $\int_{\Sigma_s}\frac{1}{\cos\theta}\rho(F,s)$ is uniformly bounded above. Moreover, by the mean value theorem, there is $s'\in[2s,s]$ such that

$$\begin{aligned}\int_{2s}^{s}\int_{\Sigma_t}\frac{1}{\cos\theta}\frac{1}{-t}\mathrm{e}^{\frac{|F|^2}{t}}|H|^2\mathrm{d}\mu_t\mathrm{d}t &= -s\int_{\Sigma_{s'}}\frac{1}{\cos\theta}\frac{1}{-s'}\mathrm{e}^{\frac{|F|^2}{s'}}|H|^2\mathrm{d}\mu_{s'}\\ &\ge C\mathrm{e}^{\frac{R^2}{s'}}\int_{\Sigma_{s'}\cap B_R(0)}|H|^2\mathrm{d}\mu_{s'},\end{aligned}$$

where C is independent of s. Thus we can find a sequence $\{s_i\}$ such that $s_i\to-\infty$ as $i\to\infty$ and

$$\int_{\Sigma_{s_i}\cap B_R(0)}|H|^2\mathrm{d}\mu_{s_i}\to 0,\quad \text{as } i\to\infty.$$

Since the second fundamental forms of Σ_{s_i} are bounded above and Σ_s satisfy the mean curvature flow equation, we have that Σ_{s_i} strongly converges to a smooth limit surface $\Sigma_{-\infty}$ in $B_R(0)$. Therefore,

$$\lim_{i\to\infty}\max_{\Sigma_{s_i}\cap B_R(0)}|H|^2=0. \tag{3.4}$$

This can also be proved by Moser iteration.

Now we use gradient estimate to prove our theorem. For this purpose we introduce a new function $f(X,t)=|H|^2+p\theta^2$, where $p>1$, $t\in[s_i,0]$, $\{s_i\}$ is the sequence in (3.4). Using the evolution equation for $|H|^2$ in $\mathbb{R}^4$

$$\Big(\Delta-\frac{\partial}{\partial t}\Big)|H|^2=2|\nabla H|^2-2(H^\alpha h_{ij}^\alpha)^2$$

and the evolution equation for θ

$$\Big(\Delta-\frac{\partial}{\partial t}\Big)\theta=0,$$

we get

$$\Big(\Delta-\frac{\partial}{\partial t}\Big)f\ge 2(p-1)|H|^2. \tag{3.5}$$

Here, we have used the fact $|\nabla\theta|=|H|$.

Let $\psi(r)$ be a C^2 function on $[0,\infty)$ such that

$$\psi(r)=\begin{cases}1, & \text{if } r\in\Big[0,\dfrac12\Big],\\ 0, & \text{if } r\ge 1,\end{cases}$$

$$0\le\psi(r)\le1,\quad \psi'(r)\le 0,\quad \psi''(r)\ge -C\quad\text{and}\quad \frac{|\psi'(r)|^2}{\psi(r)}\le C,$$

where C is an absolute constant.

Let

$$g(X,t)=\psi\Big(\frac{|X|^2}{R^2}\Big).$$

Using the fact that $|\nabla X|^2=2$, a straightforward computation shows that

$$\begin{aligned}\Big(\Delta-\frac{\partial}{\partial t}\Big)g&=4\psi''\frac{\langle X,\nabla X\rangle^2}{R^4}+2\psi'\frac{\langle\nabla X,\nabla X\rangle}{R^2}\geq-\frac{C_1}{R^2},\\ \frac{|\nabla g|^2}{g}&\leq\frac{C_2}{R^2}.\end{aligned}\tag{3.6}$$

Let $(X(s_i),t(s_i))$ be the point where $g\cdot f$ achieves its maximum in $\overline{B_R(0)}\times[s_i,0]$. It is clear that, if the maximum of $g\cdot f$ is achieved at s_i as $i\to\infty$, the claim follows.

Indeed, if $\Sigma_{s_i}\cap B_R(0)=\emptyset$ as $i\to\infty$, then $(g\cdot f)(X,s_i)\to 0$ as $i\to\infty$, and the claim holds. If $\Sigma_{s_i}\cap B_R(0)\neq\emptyset$ as $i\to\infty$, by (3.4), we have

$$\lim_{i\to\infty}(g\cdot f)(X,s_i)\leq p\eta^2.$$

We see that the claim also holds.

Now we assume $(X(s_i),t(s_i))\in B_R(0)\times(s_i,0]$. By the maximum principle, at $(X(s_i),t(s_i))$, we have

$$\nabla(g\cdot f)=0,\quad \frac{\partial}{\partial t}(g\cdot f)\geq 0\tag{3.7}$$

and

$$\Delta(g\cdot f)\leq 0.$$

Hence

$$\Big(\Delta-\frac{\partial}{\partial t}\Big)g\cdot f\leq 0,\tag{3.8}$$

$$\nabla g=-\frac{g}{f}\nabla f.\tag{3.9}$$

Substituting (3.5) and (3.6) into (3.8) and using (3.9), we get

$$\begin{aligned}0\geq\Big(\Delta-\frac{\partial}{\partial t}\Big)g\cdot f&=f\Big(\Delta-\frac{\partial}{\partial t}\Big)g+g\Big(\Delta-\frac{\partial}{\partial t}\Big)f+2\nabla g\cdot\nabla f\\ &\geq-\frac{C_1}{R^2}f-2\frac{|\nabla g|^2}{g}f+g\Big(\Delta-\frac{\partial}{\partial t}\Big)f\\ &\geq-\frac{C_1+2C_2}{R^2}f+2g\cdot|H|^2(p-1).\end{aligned}\tag{3.10}$$

Since $p>1$, we get

$$g|H|^2(X(s_i),t(s_i))\leq\frac{C_3}{(p-1)R^2}.$$

Therefore,

$$\sup_{B_{\frac{R}{2}}\times[s_i,0]}f(X,t)\leq\frac{C_3}{(p-1)R^2}+p\sup_{B_R\times[s_i,0]}\theta^2.$$

Letting $i \to \infty$ and $R \to \infty$, we obtain

$$h^2 \le p\eta^2.$$

Letting $p \to 1$, we get the desired inequality. This completes the proof of Theorem 1.2.

Now we turn to the proof of Theorem 1.1.

Recall the evolution equation of the Kähler angle in $\mathbb{C}^2$ (see [2]),

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\cos\alpha = |\overline{\nabla} J_{\Sigma_t}|^2 \cos\alpha, \tag{3.11}$$

where J_{Σ_t} is an almost complex structure in a tubular neighborhood of Σ_t in $\mathbb{C}^2$ with

$$\begin{cases} J_{\Sigma_t} e_1 = e_2, \\ J_{\Sigma_t} e_2 = -e_1, \\ J_{\Sigma_t} v_1 = v_2, \\ J_{\Sigma_t} v_2 = -v_1. \end{cases} \tag{3.12}$$

It is shown in [2, 5] that

$$|\overline{\nabla} J_{\Sigma_t}|^2 \ge \frac{1}{2}|H|^2, \tag{3.13}$$

which implies

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\cos\alpha \ge \frac{1}{2}|H|^2\cos\alpha.$$

Using equation (3.11), we can prove one monotonicity formula along the symplectic mean curvature flow in $\mathbb{R}^4$ by the same argument as the one used in the proof of Proposition 3.1.

Proposition 3.2 *Along the symplectic mean curvature flow Σ_t in $\mathbb{C}^2$, we have*

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\int_{\Sigma_t} \frac{1}{\cos\alpha}\rho(F,t)\mathrm{d}\mu_t\Big) \\ &= -\Big(\int_{\Sigma_t} \frac{1}{\cos\alpha}\rho(F,t)\Big|H + \frac{(F-X_0)^\perp}{2(t_0-t)}\Big|^2 \mathrm{d}\mu_t \\ &\quad + \int_{\Sigma_t} \frac{1}{\cos\alpha}\rho(F,t)|\overline{\nabla} J_{\Sigma_t}|^2 \mathrm{d}\mu_t + \int_{\Sigma_t} \frac{2}{\cos^3\alpha}|\nabla\cos\alpha|^2 \rho(F,t)\mathrm{d}\mu_t\Big). \end{aligned}$$

Proof of Theorem 1.1 Set $\delta := \inf\limits_{t\in(-\infty,0]} \inf\limits_{\Sigma_t} \cos\alpha$, and we only need to show that $\delta \mathrm{e}^{\frac{h^2}{4}} \le 1$. If $h = 0$ or $\delta = 0$ or $\delta = 1$, it is evident that the result holds. Now we assume that $h > 0$, $0 < \delta < 1$ and argue by contradiction. Suppose that $\delta > \mathrm{e}^{-\frac{h^2}{4}}$, i.e., $\frac{1}{\delta^2} < \mathrm{e}^{\frac{h^2}{2}}$. Then there exists a constant $p' \in (0, \frac{1}{2})$ such that $\frac{1}{\delta^2} \le \mathrm{e}^{p'h^2} < \mathrm{e}^{\frac{h^2}{2}}$.

By the definition of h^2 and the fact that $h > 0$, we know that, for any $\varepsilon > 0$, there exist $R_0 > 0$ and $T_0 > 0$ such that

$$\sup_{[-T_0,0]} \sup_{\Sigma_t \cap \overline{B_{R_0}(X_0)}} |H|^2 > (1-\varepsilon)h^2.$$

Now we choose $\varepsilon \in (0, 1-2p')$, and suppose that

$$|H|^2(\overline{X},\overline{t}) = \sup_{[-T_0,0]} \sup_{\Sigma_t \cap \overline{B_{R_0}(X_0)}} |H|^2 > (1-\varepsilon)h^2$$

for $(\overline{X},\overline{t}) \in \overline{B_{R_0}(X_0)} \times [-T_0, 0]$.

Fix $R > 2R_0$ and set $X_0 = 0$. By the monotonicity formula (see Proposition 3.2) and proceeding as in the proof of Theorem 1.2, we can find a sequence $\{s_i\}$ such that $s_i \to -\infty$ and

$$\int_{\Sigma_{s_i} \cap B_R(0)} |\overline{\nabla} J_{\Sigma_t}|^2 \to 0, \quad \text{as } i \to \infty.$$

By (3.13), we get

$$\lim_{i\to\infty} \max_{\Sigma_{s_i} \cap B_R(0)} |H|^2 = 0. \tag{3.14}$$

Now we use gradient estimate to prove our theorem. For this purpose, we introduce a new function $f(X,t) = \frac{\mathrm{e}^{p|H|^2}}{\cos^2\alpha}$, where $t \in [s_i, 0]$, $\{s_i\}$ is the sequence in (3.14), and p is constant with $0 < p < \frac{1}{2}$ to be determined later.

$$\Big(\Delta - \frac{\partial}{\partial t}\Big) f = \frac{1}{\cos^2\alpha}\Big(\Delta - \frac{\partial}{\partial t}\Big)\mathrm{e}^{p|H|^2} + \mathrm{e}^{p|H|^2}\Big(\Delta - \frac{\partial}{\partial t}\Big)\frac{1}{\cos^2\alpha} + 2\nabla \mathrm{e}^{p|H|^2} \cdot \nabla \frac{1}{\cos^2\alpha}.$$

Using the evolution equation for $|H|^2$ in $\mathbb{R}^4$

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)|H|^2 = 2|\nabla H|^2 - 2(H^\alpha h^\alpha_{ij})^2,$$

we get

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)\mathrm{e}^{p|H|^2} &= \mathrm{e}^{p|H|^2}(4p^2|H|^2|\nabla|H||^2 + 2p|\nabla H|^2 - 2p|H^\alpha h^\alpha_{ij}|^2)\\ &\geq \mathrm{e}^{p|H|^2}(4p^2|H|^2|\nabla|H||^2 + 2p|\nabla H|^2 - 2p|H|^2|A|^2)\\ &\geq \mathrm{e}^{p|H|^2}(4p^2|H|^2|\nabla|H||^2 + 2p|\nabla H|^2 - 2p|H|^2).\end{aligned}$$

Since

$$\nabla \mathrm{e}^{p|H|^2} = \nabla(f\cos^2\alpha) = \cos^2\alpha\nabla f + 2f\cos\alpha\nabla\cos\alpha,$$

we have

$$\nabla \mathrm{e}^{p|H|^2} \cdot \nabla\frac{1}{\cos^2\alpha} = \cos^2\alpha\nabla f \cdot \nabla\frac{1}{\cos^2\alpha} - \frac{4f}{\cos^2\alpha}|\nabla\cos\alpha|^2.$$

Using the evolution equation (3.11), we get

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\frac{1}{\cos^2\alpha} = 6\frac{|\nabla\cos\alpha|^2}{\cos^4\alpha} + 2\frac{|\overline{\nabla} J_{\Sigma_t}|^2}{\cos^2\alpha} \geq 6\frac{|\nabla\cos\alpha|^2}{\cos^4\alpha} + \frac{|H|^2}{\cos^2\alpha}.$$

So,

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)f \geq f\Big(&4p^2|H|^2|\nabla|H||^2 + 2p|\nabla H|^2 + 2\Big(\frac{1}{2} - p\Big)|H|^2 - 2\frac{|\nabla\cos\alpha|^2}{\cos^2\alpha}\Big)\\ &+ 2\cos^2\alpha\nabla f\cdot\nabla\frac{1}{\cos^2\alpha}.\end{aligned} \tag{3.15}$$

Choose g the same as in the proof of Theorem 1.2, such that (3.6) is satisfied. Let $(X(s_i), t(s_i))$ be the point where $g \cdot f$ achieves its maximum in $\overline{B_R(0)} \times [s_i, 0]$. We claim that the maximum of $g \cdot f$ can not be achieved at s_i as $i \to \infty$.

Indeed, if $\Sigma_{s_i}\cap B_R(0)=\emptyset$ as $i\to\infty$, then $(g\cdot f)(X,s_i)\to 0$ as $i\to\infty$, and the claim holds.

If $\Sigma_{s_i}\cap B_R(0)\neq\emptyset$ as $i\to\infty$, we denote $\varepsilon_i=\max\limits_{\Sigma_{s_i}\cap B_R(0)}|H|^2$. Then by (3.14), we know that $\lim\limits_{i\to\infty}\varepsilon_i=0$. Since $s_i\to-\infty$ as $i\to\infty$, we choose i sufficiently large such that $s_i<-T_0$. Then

$$(g\cdot f)(X(s_i),t(s_i))\geq (g\cdot f)(\overline{X},\overline{t})=f(\overline{X},\overline{t})=\frac{\mathrm{e}^{p|H|^2(\overline{X},\overline{t})}}{\cos^2\alpha(\overline{X},\overline{t})}>\mathrm{e}^{(1-\varepsilon)ph^2}.$$

On the other hand,

$$f(X,s_i)=\frac{\mathrm{e}^{p|H|^2(X,s_i)}}{\cos^2\alpha(X,s_i)}\leq\frac{\mathrm{e}^{p\varepsilon_i}}{\delta^2}\leq \mathrm{e}^{p'h^2+p\varepsilon_i}.$$

Note $1-\varepsilon>2p'$. Therefore we can choose $p\in(0,\frac{1}{2})$ such that $p(1-\varepsilon)>p'$. Now for the fixed p',ε and p, there exists an $N>0$, such that for each $i>N$, $p'h^2+p\varepsilon_i<(1-\varepsilon)ph^2$. And for these i, the claim holds.

By the maximum principle, at $(X(s_i),t(s_i))$ we have

$$\nabla(g\cdot f)=0,\quad \frac{\partial}{\partial t}(g\cdot f)\geq 0 \tag{3.16}$$

and

$$\Delta(g\cdot f)\leq 0.$$

Hence

$$\Big(\Delta-\frac{\partial}{\partial t}\Big)g\cdot f\leq 0, \tag{3.17}$$

$$\nabla g=-\frac{g}{f}\nabla f. \tag{3.18}$$

Substituting (3.15) and (3.16) into (3.17) and using (3.18) twice, we get

$$\begin{aligned}
0&\geq\Big(\Delta-\frac{\partial}{\partial t}\Big)g\cdot f=f\Big(\Delta-\frac{\partial}{\partial t}\Big)g+g\Big(\Delta-\frac{\partial}{\partial t}\Big)f+2\nabla g\cdot\nabla f\\
&\geq-\frac{C_1}{R^2}f-2\frac{|\nabla g|^2}{g}f+g\Big(\Delta-\frac{\partial}{\partial t}\Big)f\\
&\geq-\frac{C_1+2C_2}{R^2}f+2g\cdot f|H|^2\Big(\frac{1}{2}-p\Big)\\
&\quad+g\cdot f\Big(2p|\nabla H|^2+4p^2|H|^2|\nabla|H||^2-2\frac{|\nabla\cos\alpha|^2}{\cos^2\alpha}\Big)\\
&\quad+2g\cos^2\alpha\nabla f\cdot\nabla\frac{1}{\cos^2\alpha}\\
&\geq-\frac{C_1+2C_2}{R^2}f+2g\cdot f|H|^2\Big(\frac{1}{2}-p\Big)\\
&\quad+g\cdot f\Big(2p|\nabla H|^2+4p^2|H|^2|\nabla|H||^2-2\frac{|\nabla\cos\alpha|^2}{\cos^2\theta}\Big)\\
&\quad-2\cos^2\alpha f\nabla\frac{1}{\cos^2\alpha}\cdot\nabla g.
\end{aligned} \tag{3.19}$$

Using equation (3.18), we have

$$\nabla g=g\Big(2\frac{\nabla\cos\alpha}{\cos\alpha}-p\nabla|H|^2\Big).$$

Thus,

$$4gp^2|\nabla|H||^2|H|^2 = \frac{|\nabla g|^2}{g} + 4g\frac{|\nabla\cos\alpha|^2}{\cos^2\alpha} - 4\nabla g\cdot\frac{\nabla\cos\alpha}{\cos\alpha}.$$

Putting this equation into (3.19), we get

$$\begin{aligned}0 &\geq -\frac{C_1+2C_2}{R^2}f + 2gf\Big(\frac{1}{2}-p\Big)|H|^2 + 2pgf|\nabla H|^2 + \frac{f}{g}|\nabla g|^2 + 2gf\frac{|\nabla\cos\alpha|^2}{\cos^2\alpha}\\ &\geq -\frac{C_4}{R^2}f + 2gf\Big(\frac{1}{2}-p\Big)|H|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{C_4}{R^2} &\geq 2g\Big(\frac{1}{2}-p\Big)|H|^2 = 2gf\Big(\frac{1}{2}-p\Big)\frac{\cos^2\alpha|H|^2}{\mathrm{e}^{p|H|^2}}\\ &\geq 2gf\delta^2\mathrm{e}^{-ph^2}\Big(\frac{1}{2}-p\Big)|H|^2.\end{aligned}$$

By the assumption that $\sup\limits_{t\in(-\infty,0]}\sup\limits_{\Sigma_t}|A|^2=1$, we have $h^2\leq 2$. So

$$\frac{C_5}{R^2} \geq \delta^2 2gf\Big(\frac{1}{2}-p\Big)|H|^2.$$

Since $\frac{1}{2}-p>0$, we get

$$|H|^2(X(s_i),t(s_i))(g\cdot f)(X(s_i),t(s_i)) \leq \frac{C_5}{(\frac{1}{2}-p)R^2}.$$

So

$$|H|^2(X(s_i),t(s_i))f(0,0) \leq |H|^2(X(s_i),t(s_i))(g\cdot f)(X(s_i),t(s_i)) \leq \frac{C_5}{(\frac{1}{2}-p)R^2}.$$

Notice $f(0,0)\geq 1$. Thus

$$|H|^2(X(s_i),t(s_i)) \leq \frac{C_5}{(\frac{1}{2}-p)R^2}.$$

Therefore,

$$\sup_{B_{\frac{R}{2}}\times[s_i,0]} f(X,t) \leq \frac{1}{\delta^2}\mathrm{e}^{p|H|^2(x(s_i),t(s_i))} \leq \frac{1}{\delta^2}\mathrm{e}^{\frac{pC_5}{(\frac{1}{2}-p)R^2}}.$$

Letting $i\to\infty$ and $R\to\infty$, we get

$$\mathrm{e}^{p'h^2} \geq \frac{1}{\delta^2} \geq \sup f \geq \mathrm{e}^{ph^2},$$

which is a contradiction because $p>p(1-\varepsilon)>p'$ and $h>0$. This completes the proof of Theorem 1.1.

4 Proofs of Theorem 1.3 and Theorem 1.4

We first prove Theorem 1.3.

Proof of Theorem 1.3 Without loss of generality, we assume $|A|^2(0,0) = 1$. We prove the theorem by contradiction. Suppose that the symplectic mean curvature flow $\{\Sigma_t\}_{t\in(-\infty,0]}$ is normal flat at every time. Then we have

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)|A|^2 = 2|\nabla A|^2 - 2\sum_{i,j,m,k}\Big(\sum_\alpha h^\alpha_{ij}h^\alpha_{mk}\Big)^2 \geq 2|\nabla A|^2 - 2|A|^4 \tag{4.1}$$

and

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\cos\alpha = -|A|^2\cos\alpha.$$

Thus, we obtain

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\frac{1}{\cos\alpha} = \frac{|A|^2}{\cos\alpha} + 2\frac{|\nabla\cos\alpha|^2}{\cos^3\alpha}. \tag{4.2}$$

Because Σ_t is normal flat at each t, we have

$$|\overline{\nabla} J_{\Sigma_t}|^2 = |A|^2.$$

Applying Proposition 3.2 with $|\overline{\nabla} J_{\Sigma_t}|^2 = |A|^2$, by the same argument used to derive (3.4), we obtain that there is a sequence s_i such that $s_i \to -\infty$, and

$$\lim_{i\to\infty}\max_{\Sigma_{s_i}\cap B_R(0)}|A|^2 = 0 \tag{4.3}$$

for any fixed $R > 0$.

Assume that f is a positive increasing function which will be defined later. Using (4.1) and (4.2), we have

$$\begin{aligned}
&\Big(\Delta - \frac{\partial}{\partial t}\Big)\Big(|A|^2 f\Big(\frac{1}{\cos\alpha}\Big)\Big)\\
&= \Big(\Delta - \frac{\partial}{\partial t}\Big)|A|^2 f\Big(\frac{1}{\cos\alpha}\Big) + |A|^2\Big(\Delta - \frac{\partial}{\partial t}\Big)\Big(f\Big(\frac{1}{\cos\alpha}\Big)\Big) + 2\nabla|A|^2\cdot\nabla f\Big(\frac{1}{\cos\alpha}\Big)\\
&\geq f(2|\nabla A|^2 - 2|A|^4) + |A|^2\Big(f'\frac{|A|^2}{\cos\alpha} + 2f'\frac{|\nabla\cos\alpha|^2}{\cos^3\alpha} + f''\frac{|\nabla\cos\alpha|^2}{\cos^4\alpha}\Big)\\
&\quad + 2\frac{\nabla(f|A|^2) - |A|^2\nabla f}{f}\cdot\nabla f\Big(\frac{1}{\cos\alpha}\Big)\\
&= |A|^2 f\Big(2\frac{|\nabla A|^2}{|A|^2} - 2|A|^2 + \frac{f'}{f}\frac{|A|^2}{\cos\alpha}\Big) + |A|^2\Big(f'' - 2\frac{(f')^2}{f} + 2f'\cos\alpha\Big)\frac{|\nabla\cos\alpha|^2}{\cos^4\alpha}\\
&\quad + 2|A|^2\frac{\nabla(f|A|^2)}{f|A|^2}\cdot\nabla f\Big(\frac{1}{\cos\alpha}\Big).
\end{aligned} \tag{4.4}$$

Set $\phi = f|A|^2$. At the point where $\phi \neq 0$, it is easy to see that

$$\nabla\phi = f\nabla|A|^2 + |A|^2\nabla f = f\nabla|A|^2 - |A|^2 f'\frac{\nabla\cos\alpha}{\cos^2\alpha},$$

i.e,

$$\frac{\nabla\cos\alpha}{\cos^2\alpha} = \frac{f}{f'}\Big(\frac{\nabla|A|^2}{|A|^2} - \frac{\nabla\phi}{\phi}\Big). \tag{4.5}$$

Plugging (4.5) into (4.4), we obtain

$$\begin{aligned}
\Big(\Delta - \frac{\partial}{\partial t}\Big)\phi \geq\ & \phi\Big(2\frac{|\nabla A|^2}{|A|^2} - 2|A|^2 + \frac{f'}{f}\frac{|A|^2}{\cos\alpha}\Big) \\
& + \frac{\phi f}{(f')^2}\Big(f'' - 2\frac{(f')^2}{f} + 2f'\cos\alpha\Big)\Big(\frac{|\nabla|A|^2|^2}{|A|^4} - 2\frac{\nabla|A|^2}{|A|^2}\cdot\frac{\nabla\phi}{\phi} + \frac{|\nabla\phi|^2}{\phi^2}\Big) \\
& - 2|A|^2 f'\frac{\nabla\phi}{\phi}\cdot\frac{\nabla\cos\alpha}{\cos^2\alpha} \\
=\ & \phi\Big(\frac{f'}{f}\frac{|A|^2}{\cos\alpha} - 2|A|^2\Big) + \phi\Big(2\frac{|\nabla A|^2}{|A|^2} + 4\frac{ff''}{(f')^2}\frac{|\nabla|A||^2}{|A|^2} - 8\frac{|\nabla|A||^2}{|A|^2} \\
& + 8\frac{f}{f'}\cos\alpha\frac{|\nabla|A||^2}{|A|^2}\Big) - 2|A|^2 f'\frac{\nabla\phi}{\phi}\cdot\frac{\nabla\cos\alpha}{\cos^2\alpha} \\
& + \phi\Big(\frac{ff''}{(f')^2} + 2\frac{f}{f'}\cos\alpha - 2\Big)\Big(\frac{|\nabla\phi|^2}{\phi^2} - 2\frac{\nabla|A|^2}{|A|^2}\cdot\frac{\nabla\phi}{\phi}\Big) \\
\geq\ & \phi\Big(\frac{f'}{f}\frac{|A|^2}{\cos\alpha} - 2|A|^2\Big) + \phi\Big(4\frac{ff''}{(f')^2} + 8\frac{f}{f'}\cos\alpha - 6\Big)\frac{|\nabla|A||^2}{|A|^2} \\
& + \phi\Big(\frac{ff''}{(f')^2} + 2\frac{f}{f'}\cos\alpha - 2\Big)\Big(\frac{|\nabla\phi|^2}{\phi^2} - 2\frac{\nabla|A|^2}{|A|^2}\cdot\frac{\nabla\phi}{\phi}\Big) \\
& - 2|A|^2 f'\frac{\nabla\phi}{\phi}\cdot\frac{\nabla\cos\alpha}{\cos^2\alpha} \\
=\ & \phi|A|^2\Big(\frac{f'}{f}\frac{1}{\cos\alpha} - 2\Big) + \phi\Big(4\frac{ff''}{(f')^2} + 8\frac{f}{f'}\cos\alpha - 6\Big)\frac{|\nabla|A||^2}{|A|^2} \\
& - \phi\Big(\frac{ff''}{(f')^2} + 2\frac{f}{f'}\cos\alpha - 2\Big)\Big(\frac{|\nabla\phi|^2}{\phi^2} + 2\frac{f'}{f}\frac{\nabla\cos\alpha}{\cos^2\alpha}\cdot\frac{\nabla\phi}{\phi}\Big) \\
& - 2|A|^2 f'\frac{\nabla\phi}{\phi}\cdot\frac{\nabla\cos\alpha}{\cos^2\alpha}.
\end{aligned} \tag{4.6}$$

Following the ideas in [12], we choose

$$f(x) = \frac{(2-\delta)^2x^2}{(2-\delta x)^2}, \quad x \in \Big[1, \frac{1}{\delta}\Big],$$

such that

$$4\frac{ff''}{(f')^2} + 8\frac{f}{f'}\cos\alpha - 6 = 0.$$

It is evident that for $x \in [1, \frac{1}{\delta}]$,

$$1 \leq f(x) \leq \frac{(2-\delta)^2}{\delta^2}.$$

By (4.6), we have

$$\begin{aligned}
\Big(\Delta - \frac{\partial}{\partial t}\Big)\phi \geq\ & 2\phi|A|^2\Big(\frac{1}{1-\frac{\delta}{2\cos\alpha}} - 1\Big) + \frac{\phi}{2}\Big(\frac{|\nabla\phi|^2}{\phi^2} + 2\frac{f'}{f}\frac{\nabla\cos\alpha}{\cos^2\alpha}\cdot\frac{\nabla\phi}{\phi}\Big) \\
& - 2|A|^2 f'\frac{\nabla\phi}{\phi}\cdot\frac{\nabla\cos\alpha}{\cos^2\alpha} \\
\geq\ & \delta\phi|A|^2 + \frac{|\nabla\phi|^2}{2\phi} - \Big(2|A|^2 f'\frac{\nabla\cos\alpha}{\cos^2\alpha} - \phi\frac{f'}{f}\frac{\nabla\cos\alpha}{\cos^2\alpha}\Big)\cdot\frac{\nabla\phi}{\phi}
\end{aligned}$$

$$\geq \delta\phi|A|^2 - \mathbf{b}\cdot\frac{\nabla\phi}{\phi}, \tag{4.7}$$

where $\mathbf{b} = 2|A|^2 f' \frac{\nabla\cos\alpha}{\cos^2\alpha} - \phi\frac{f'}{f}\frac{\nabla\cos\alpha}{\cos^2\alpha}$ is bounded.

Now we choose g as in the proof of Theorem 1.2. Recall that

$$|\nabla g| \leq \frac{C_6}{R}.$$

Let $(X(s_i), t(s_i))$ be the point where ϕg achieves its maximum in $\overline{B_R(0)} \times [s_i, 0]$. If $\Sigma_{s_i} \cap B_R(0) = \emptyset$ as $i \to \infty$, then $\phi g \to 0$ as $i \to \infty$. If $\Sigma_{s_i} \cap B_R(0) \neq \emptyset$ as $i \to \infty$, then by (4.3), we have

$$\begin{aligned}(\phi g)(X, s_i) &= |A|^2(X, s_i) f(X, s_i) g(X, s_i)\\ &\leq \frac{(2-\delta)^2}{\delta^2}|A|^2(X, s_i)g(X, s_i) \to 0, \quad \text{as } i \to \infty.\end{aligned}$$

On the other hand,

$$(\phi g)(X(s_i), t(s_i)) \geq (\phi g)(0,0) = |A|^2(0,0) f\Big(\frac{1}{\cos\alpha(0,0)}\Big) g(0,0) = f\Big(\frac{1}{\cos\alpha(0,0)}\Big) \geq 1. \tag{4.8}$$

This implies that the maximum of ϕg can not be achieved at s_i as $i \to \infty$. By the maximum principle, at $(X(s_i), t(s_i))$, we have

$$\nabla(g\phi) = 0, \quad \frac{\partial}{\partial t}(g\phi) \geq 0, \quad \Delta(g\phi) \leq 0.$$

Hence,

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)(g\phi) \leq 0, \quad \nabla g = -\frac{g}{\phi}\nabla\phi.$$

Using (4.7) and (3.6), we obtain

$$\begin{aligned}0 &\geq \Big(\Delta - \frac{\partial}{\partial t}\Big)(g\phi)\\ &= \Big(\Delta - \frac{\partial}{\partial t}\Big)g\phi + g\Big(\Delta - \frac{\partial}{\partial t}\Big)\phi + 2\nabla g\cdot\nabla\phi\\ &\geq -\frac{C_1}{R^2}\phi + \delta|A|^2\phi g - \mathbf{b}\cdot\frac{\nabla\phi}{\phi}g + 2\nabla g\cdot\Big(-\frac{\phi}{g}\Big)\nabla g\\ &= -\frac{C_1}{R^2}\phi + \delta|A|^2\phi g + \mathbf{b}\cdot\nabla g - 2\frac{\phi}{g}|\nabla g|^2\\ &\geq -\frac{C_1}{R^2}\phi + \delta|A|^2\phi g - |\mathbf{b}|\frac{C_6}{R} - 2\frac{C_2}{R^2}\phi\\ &\geq \delta|A|^2(X(s_i), t(s_i)) - \frac{C_7}{R^2} - \frac{C_8}{R} \quad (\text{by } (4.8)),\end{aligned}$$

i.e.,

$$|A|^2(X(s_i), t(s_i)) \leq \frac{C_7}{\delta R^2} + \frac{C_8}{\delta R}. \tag{4.9}$$

Here we have used (4.8) and the fact that

$$\phi = |A|^2 f \leq f \leq \frac{(2-\delta)^2}{\delta^2}.$$

The constants $C_7,\ C_8$ depend only on δ.

On the other hand, we have

$$\begin{aligned}1 &\le f\Big(\frac{1}{\cos\alpha(0,0)}\Big) = |A|^2(0,0) f\Big(\frac{1}{\cos\alpha(0,0)}\Big) g(0,0)\\ &= (\phi g)(0,0) \le (\phi g)(X(s_i), t(s_i))\\ &= |A|^2(X(s_i), t(s_i)) f\Big(\frac{1}{\cos\alpha(X(s_i), t(s_i))}\Big) g(X(s_i), t(s_i))\\ &\le \frac{(2-\delta)^2}{\delta^2}|A|^2(X(s_i), t(s_i)),\end{aligned}$$

i.e.,

$$|A|^2(X(s_i), t(X_{s_i})) \ge \frac{\delta^2}{(2-\delta)^2}. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\frac{\delta^2}{(2-\delta)^2} \le \frac{C_7}{\delta R^2} + \frac{C_8}{\delta R}.$$

Letting $R \to \infty$, we get a contradiction.

The proof of Theorem 1.4 is similar. Note that

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\cos\theta = -|H|^2\cos\theta.$$

Suppose that the Lagrangian mean curvature flow $\{\Sigma_t\}_{t\in(-\infty,0]}$ is flat at every time. Then we have

$$|A|^2 = |H|^2 \quad \text{and} \quad \Big(\Delta - \frac{\partial}{\partial t}\Big)\cos\theta = -|A|^2\cos\theta.$$

Therefore

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\frac{1}{\cos\theta} = \frac{|A|^2}{\cos\theta} + 2\frac{|\nabla\cos\theta|^2}{\cos^3\theta}.$$

Also by (3.4), we have

$$\lim_{i\to\infty}\max_{\Sigma_{s_i}\cap B_R(0)}|A|^2 = 0.$$

The remaining part of the proof is the same as that of Theorem 1.3 with $\cos\alpha$ replaced by $\cos\theta$. We leave the details to readers.

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