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Operator-Valued Fourier Multipliers on Multi-dimensional Hardy Spaces*

Shangquan BU[1]

Abstract The author establishes operator-valued Fourier multiplier theorems on multi-dimensional Hardy spaces $H^p(\mathbb{T}^d; X)$, where $1 \leq p < \infty$, $d \in \mathbb{N}$, and X is an AUMD Banach space having the property (α). The sufficient condition on the multiplier is a Marcinkiewicz type condition of order 2 using Rademacher boundedness of sets of bounded linear operators. It is also shown that the assumption that X has the property (α) is necessary when $d \geq 2$ even for scalar-valued multipliers. When the underlying Banach space does not have the property (α), a sufficient condition on the multiplier of Marcinkiewicz type of order 2 using a notion of d-Rademacher boundedness is also given.

Keywords H^p-Spaces, Fourier multiplier, Rademacher boundedness, d-Rademacher boundedness
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1 Introduction

The aim of this paper is to establish operator-valued Fourier multiplier theorems on the multi-dimensional Hardy spaces $H^p(\mathbb{T}^d; X)$, where $1 \leq p < \infty$, $d \in \mathbb{N}$, $\mathbb{T} = [0, 2\pi]$ and X is a complex Banach space. Recall that the Hardy space $H^p(\mathbb{T}^d; X)$ is the space of all functions $f \in L^p(\mathbb{T}^d; X)$ such that $\widehat{f}(n) = 0$ for all $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d \setminus \mathbb{N}_0^d$, where $\widehat{f}(n)$ is the Fourier coefficient of f given by

$$\widehat{f}(n) := \int_0^{2\pi} \cdots \int_0^{2\pi} f(t_1, \cdots, t_d)\, \mathrm{e}^{-\mathrm{i}(n_1 t_1 + \cdots + n_d t_d)} \frac{\mathrm{d}t_1}{2\pi} \cdots \frac{\mathrm{d}t_d}{2\pi}$$

and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \geq 0$, we denote the function $t \to \mathrm{e}^{\mathrm{i}nt}$ on $\mathbb{T}$ by e_n. $H^p(\mathbb{T}^d; X)$ is equipped with the induced norm $\|\cdot\|_p$ by $L^p(\mathbb{T}^d; X)$ so that it becomes a Banach space. A sequence $(M(n))_{n \in \mathbb{N}_0^d} \subset \mathcal{L}(X)$ is said to be a Fourier multiplier on $H^p(\mathbb{T}^d; X)$, if for all $f \in H^p(\mathbb{T}^d; X)$, there exists a unique $g \in H^p(\mathbb{T}^d; X)$ such that $\widehat{g}(n) = M(n)\widehat{f}(n)$ for all $n \in \mathbb{N}_0^d$, where $\mathcal{L}(X)$ is the space of all bounded linear operators on X. In this case we can find a constant $C > 0$ independent of f such that $\|g\|_p \leq C\|f\|_p$ by the closed graph theorem.

When X is an AUMD space (see [2]) and $d = 1$, an operator-valued Fourier multiplier theorem on $H^p(\mathbb{T}; X)$ has been given in [4]: let X be an AUMD space and $1 \leq p < \infty$, let $(M_n)_{n \geq 0} \subset \mathcal{L}(X)$ be such that the sets $\{M_n : n \geq 0\}$, $\{n\Delta^1 M_n : n \geq 0\}$ and $\{n^2 \Delta^2 M_n : n \geq 0\}$ are Rademacher bounded, then $(M_n)_{n \geq 0}$ defines a Fourier multiplier on $H^p(\mathbb{T}; X)$, where $\Delta^1 M_n := M_{n+1} - M_n$ and $\Delta^2 M_n := M_{n+2} - 2M_{n+1} + M_n$ are the first derivative and the second derivative of M_n, respectively. This is the operator-valued analogue of a remarkable

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[1]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
E-mail: sbu@math.tsinghua.edu.cn
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result of Blower giving a characterization of the AUMD space in term of scalar-valued Fourier multipliers on $H^1(\mathbb{T};X)$ (see [2]).

The aim of this paper is to extend the result in [4] to multi-dimensional Hardy spaces $H^p(\mathbb{T}^d;X)$. When the Banach space X has the property (α), the sufficient condition we give is similar to that for $L^p(\mathbb{T}^d;X)$ (see [3]): one requires that the partial derivatives of the multiplier satisfy a Macinkiewicz type condition of order 2 using the Rademacher boundedness (see Theorem 2.2). We also show that the set of bounded operators on $H^p(\mathbb{T}^d;X)$ obtained in this way is Rademacher bounded in $\mathcal{L}(H^d(\mathbb{T}^d;X))$ when the Rademacher boundedness assumption on the multipliers is related to a fixed Rademacher bounded subset of $\mathcal{L}(X)$ (see Theorem 2.2). This strengthens the result in the one dimensional case proved in [4] when X has the property (α). We show that the property (α) is also necessary when considering such type of conditions even for scalar-valued multipliers (see Theorem 2.3).

We also give a sufficient condition for multipliers on $H^p(\mathbb{T}^d;X)$ without assuming that X has the property (α). In this case, the sufficient condition we give involves a new notion of boundedness concerning sequences of operators, we call it the d-Rademacher boundedness. We see that when $(M(n))_{n\in\mathbb{N}_0^d}\subset\mathcal{L}(X)$ is d-Rademacher bounded, then it is also Rademacher bounded, the converse implication is true when the underlying Banach space has the property (α). In particular, we see that part of results obtained when X has the property (α) follow from the general case.

The basic idea to study operator-valued Fourier multipliers on multi-dimensional Hardy spaces $H^p(\mathbb{T}^d;X)$ is the following observation. Let $d>1$ and let $M=M(n)_{n\in\mathbb{N}_0^d}\subset\mathcal{L}(X)$ be fixed. For fixed $n_1\in\mathbb{N}_0$, we let $N_{n_1}(n_2,n_3,\cdots,n_d):=M(n_1,n_2,\cdots,n_d)$ for $n_j\in\mathbb{N}_0$ $(2\le j\le n)$. Then by the Fubini's theorem, M defines a Fourier multiplier on $H^p(\mathbb{T}^d;X)$ if and only if the sequence $(N_{n_1})_{n_1\ge 0}$ defines a Fourier multiplier on $H^p(\mathbb{T};H^p(\mathbb{T}^{d-1};X))$. Thus the result for multi-dimensional Hardy spaces follows from an easy induction argument on $d\in\mathbb{N}$ and the result when $d=1$ proved in [4]. We notice that our result is even stronger than that proved in [4].

The paper is organized as follows. In Section 2, we study the Fourier multipliers on $H^p(\mathbb{T}^d;X)$ when X has the property (α). In Section 3, we treat the case when X has not necessarily the property (α).

2 Multipliers on $H^p(\mathbb{T}^d;X)$ When X Has the Property (α)

The notion of Rademacher boundedness is fundamental in the study of operator-valued Fourier multipliers on L^p spaces or H^p spaces (see [1, 4, 7, 11]). Let X be a Banach space. A subset $\mathcal{M}$ of $\mathcal{L}(X)$ is said to be Rademacher bounded (see [1, 3–5, 7, 11]), if there exists a constant $C>0$, such that for all $n\in\mathbb{N}$, $T_j\in\mathcal{M}$ and $x_j\in X$, we have $\Big\|\sum\limits_{j=1}^n\varepsilon_jT_jx_j\Big\|_p\le C\Big\|\sum\limits_{j=1}^n\varepsilon_jx_j\Big\|_p$ for some $1\le p<\infty$, where $(\varepsilon_j)_{j\ge1}$ are Rademacher functions on $[0,1]$ given by $\varepsilon_j(t)=\mathrm{sign}(\sin(2^{j-1}\pi t))$.

We let Rad be the linear span of ε_j. Then $\mathrm{Rad}\otimes X$ is the space of all finite sums $\sum\limits_{j\ge1}\varepsilon_jx_j$, with $x_j\in X$. For any $1\le p<\infty$, we let $\mathrm{Rad}_p(X)$ be the closed subspace of $L^p(\Omega;X)$ spanned by $\mathrm{Rad}\otimes X$, that we equip with the induced norm. We recall that for any $1\le p,q<\infty$, the two norms $\|\cdot\|_{\mathrm{Rad}_p(X)}$ and $\|\cdot\|_{\mathrm{Rad}_q(X)}$ are equivalent on $\mathrm{Rad}\otimes X$ (see, e.g., [9, Theorem 1.e.13]). Therefore we will simply denote $\mathrm{Rad}_p(X)$ by $\mathrm{Rad}(X)$. One immediate consequence of this fact is that the Rademacher boundedness does not depend on the parameter $1\le p<\infty$ involved in the definition. A sequence $M=(M_n)_{n\ge1}\subset\mathcal{L}(X)$ is Rademacher bounded if and only if the linear operator T_M defined by $T_M\Big(\sum\limits_{j\ge1}\varepsilon_jx_j\Big)=\sum\limits_{j\ge1}\varepsilon_jM_jx_j$ on $\mathrm{Rad}(X)$ is bounded.

Next, we say that an X-valued martingale $(g_j)_{j\ge1}$ is analytic if for any $j\ge1$, $g_j\in L^1(\mathbb{T}^j;X)$

and there exist measurable functions $\Phi_j : \mathbb{T}^{j-1} \to X$ such that

$$\mathrm{d}g_j(\tau) = \Phi_j(t_1, \cdots, t_{j-1})\, \mathrm{e}^{\mathrm{i}t_j}, \quad \tau = (t_1, \cdots, t_j) \in \mathbb{T}^j. \tag{2.1}$$

By definition, X is an AUMD Banach space if for some $1 \le p < \infty$, there is a constant $K_p > 0$ such that for any X-valued analytic martingale $(g_j)_{j\ge1}$, for any bounded sequence $(\alpha_j)_{j\ge1}$ of complex numbers and for any integer $n \ge 1$, we have

$$\Big\| \sum_{j=1}^{n} \alpha_j \,\mathrm{d}g_j \Big\|_{L^p} \le K_p \sup_{j\ge1} |\alpha_j| \Big\| \sum_{j=1}^{n} \mathrm{d}g_j \Big\|_{L^p}. \tag{2.2}$$

This property does not depend on $1 \le p < \infty$, and any UMD Banach space is AUMD. Indeed, by definition, for any $1 < p < \infty$, X is a UMD Banach space if and only if there is constant $K_p > 0$ such that (2.2) holds for any X-valued martingale with respect to the filtration $(\mathcal{F}_j)_{j\ge1}$, where $(\mathcal{F}_j)_{j\ge1}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{T}^j$. Any closed subspace of an AUMD Banach space is AUMD, and the class of AUMD spaces includes L^1-spaces. Indeed, for any measure space Σ and for any $1 \le q < \infty$, the space $L^q(\Sigma; X)$ is AUMD provided that X is AUMD.

A Banach space X is said to have the property (α), if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $x_{i,j} \in X$ and $\alpha_{i,j} \in \mathbb{C}$, we have

$$\Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} \alpha_{i,j} x_{i,j} \Big\|_{L^2} \le C \sup_{1\le i,j\le n} |\alpha_{i,j}| \Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{i,j} \Big\|_{L^2},$$

where $(\varepsilon_j^{(1)})_{j\ge1}$ and $(\varepsilon_j^{(2)})_{j\ge1}$ are two independent sequences of Rademacher functions (see [10]). We notice that the L_2-norm used in the definition may be replaced by any L_p-norm whenever $1 \le p < \infty$. Indeed, by the Kahane's inequality (see [9, Theorem 1.e.13]), there exists a constant C_p depending only on $1 \le p < \infty$, such that

$$\frac{1}{C_p} \Big\| \sum_{j\ge1} \varepsilon_j^{(1)} x_j \Big\|_{L^2} \le \Big\| \sum_{j\ge1} \varepsilon_j^{(1)} x_j \Big\|_{L^p} \le C_p \Big\| \sum_{j\ge1} \varepsilon_j^{(1)} x_j \Big\|_{L^2}$$

for all $x_j \in X$, which implies that

$$\frac{1}{C_p^2} \Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{i,j} \Big\|_{L^2} \le \Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{i,j} \Big\|_{L^p} \le C_p^2 \Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{i,j} \Big\|_{L^2}.$$

It was shown by Pisier [10] that when X has the property (α), then the sequence $(\varepsilon_i^{(1)} \varepsilon_j^{(2)})_{i,j\ge1}$ has the same behavior as a sequence of independent Rademacher functions $(\varepsilon_{i,j})_{i,j\ge1}$, i.e., for all $1 \le p < \infty$, there exists a constant $C > 0$, such that

$$\frac{1}{C} \Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{i,j} \Big\|_{L^p} \le \Big\| \sum_{i,j=1}^{n} \varepsilon_{i,j} x_{i,j} \Big\|_{L^p} \le C \Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{i,j} \Big\|_{L^p}$$

for all $n \in \mathbb{N}$ and $x_{i,j} \in X$. This implies in particular that when X has the property (α), and $(T_{i,j})_{i,j\ge1} \subset \mathcal{L}(X)$ is Rademacher bounded, then for every $1 \le p < \infty$, there exists a constant $C > 0$ depending only on p and the sequence $(T_{i,j})_{i,j\ge1}$, such that

$$\Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} T_{i,j} x_{i,j} \Big\|_{L^p} \le C \Big\| \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{i,j} \Big\|_{L^p} \tag{2.3}$$

for all $x_{i,j} \in X$ and $n \in \mathbb{N}$. This observation is crucial in the proof of our main result in this section.

We first recall the following known result proved in [4]. Our general result will follow from it and an application of the Fubini's theorem.

Theorem 2.1 (see [4]) *Let X be an AUMD space and let $M = (M_n)_{n\geq 0} \subset \mathcal{L}(X)$ be such that the sets*

$$\{M_n : n \geq 0\}, \quad \{n\Delta^1 M_n : n \geq 0\} \quad \text{and} \quad \{n^2\Delta^2 M_n : n \geq 0\}$$

are Rademacher bounded subsets. Then M defines a Fourier multiplier on $H^p(\mathbb{T}; X)$ whenever $1 \leq p < \infty$.

To state our main result of this section, we need to introduce some notations. Let $M = (M(n))_{n\in\mathbb{N}_0^d} \subset \mathcal{L}(X)$. For $1 \leq j \leq d$ and $n \in \mathbb{N}_0^d$, we let

$$(D_j^0)M(n) := M(n), \quad (D_j^1)M(n) := M(n + f_j) - M(n)$$

be the partial derivative of M with respect to the j-th coordinate, where $f_j := (\delta_{j,h})_{1\leq h\leq d}$. We define the second partial derivative of M with respect to the j-th coordinate by $(D_j^2)M(n) = (D_j^1)M(n + f_j) - (D_j^1)M(n)$. It is easy to verify that when $1 \leq k, j \leq d$ and $\alpha_k, \alpha_j \in \{0, 1, 2\}$, we have $(D_k^{\alpha_k})(D_j^{\alpha_j}M)(n) = (D_j^{\alpha_j})(D_k^{\alpha_k}M)(n)$. Thus we can define the expression

$$\Big(\prod_{1\leq j\leq d} D_j^{\alpha_j}\Big)M(n) := (D_d^{\alpha_d}D_{d-1}^{\alpha_{d-1}}\cdots D_1^{\alpha_1})M(n)$$

without any confusion whenever $\alpha_j \in \{0, 1, 2\}$ $(1 \leq j \leq d)$. For $n = (n_1, n_2, \cdots, n_d) \in \mathbb{N}_0^d$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \{0, 1\}^d$, we let $n^\alpha := n_1^{\alpha_1}\cdots n_d^{\alpha_d}$.

Now we are ready to state the operator-valued Fourier multiplier theorem on $H^p(\mathbb{T}^d; X)$ when X has the property (α).

Theorem 2.2 *Let X be an AUMD space having the property (α), $1 \leq p < \infty$ and let $\mathcal{M} \subset \mathcal{L}(X)$ be a Rademacher bounded subset. Then every sequence $M = (M(n))_{n\in\mathbb{N}_0^d} \subset \mathcal{L}(X)$ satisfying*

$$n^\alpha\Big(\prod_{1\leq j\leq d} D_j^{\alpha_j}\Big)M(n) \in \mathcal{M} \tag{2.4}$$

for $\alpha_j \in \{0, 1, 2\}$ $(1 \leq j \leq d)$ and $n \in \mathbb{N}_0^d$, defines a Fourier multiplier on $H^p(\mathbb{T}^d; X)$. Moreover, if we denote by T_M the corresponding bounded linear operator on $H^p(\mathbb{T}^d; X)$, then the set $\{T_M : M = (M(n))_{n\in\mathbb{N}_0^d}$ satisfies (2.4)$\}$ is Rademacher bounded in $\mathcal{L}(H^p(\mathbb{T}^d; X))$.

We need the following lemma which is exactly Theorem 2.2 when $d = 1$. Theorem 2.2 will follow from this lemma and an easy induction argument on $d \in \mathbb{N}$. We notice that it strengthens Theorem 2.1 when X has the property (α).

Lemma 2.1 *Let X be an AUMD space having the property (α), $1 \leq p < \infty$ and let $\mathcal{M} \subset \mathcal{L}(X)$ be a Rademacher bounded subset. If we denote by T_M the corresponding bounded linear operator on $H^p(\mathbb{T}; X)$ given by Theorem 2.1 for $M = (M_n)_{n\geq 0} \subset \mathcal{L}(X)$ satisfying*

$$M_n \in \mathcal{M}, \quad n\Delta^1 M_n \in \mathcal{M}, \quad n^2\Delta^2 M_n \in \mathcal{M}, \tag{2.5}$$

whenever $n \geq 0$, then the set $\{T_M : M = (M_n)_{n\geq 0}$ satisfies (2.5)$\}$ is Rademacher bounded in $\mathcal{L}(H^p(\mathbb{T}; X))$.

Proof For $j \in \mathbb{N}$, let $M^{(j)} = (M^{(j)}(k))_{k\geq 0}$ be a sequence in $\mathcal{L}(X)$ satisfying the condition (2.5). We need to show that there exists a constant $C > 0$ depending only on $\mathcal{M}$ such that for all $f_j \in H^p(\mathbb{T}; X)$, we have

$$\Big\|\sum_{j\geq 1} \varepsilon_j T_{M^{(j)}} f_j\Big\|_{L^p} \leq C\Big\|\sum_{j\geq 1} \varepsilon_j f_j\Big\|_{L^p}, \tag{2.6}$$

where $T_{M^{(j)}}$ is the bounded linear operator on $H^p(\mathbb{T};X)$ given by Theorem 2.1. By the Fubini's theorem, we have

$$\Big\|\sum_{j\geq 1}\varepsilon_j T_{M^{(j)}} f_j\Big\|_{L^p} = \Big\|\sum_{k\geq 0}\Big[\sum_{j\geq 1}\varepsilon_j M_k^{(j)}\widehat{f}_j(k)\Big]e_k\Big\|_{L^p},$$
$$\Big\|\sum_{j\geq 1}\varepsilon_j f_j\Big\|_{L^p} = \Big\|\sum_{k\geq 0}\Big[\sum_{j\geq 1}\varepsilon_j \widehat{f}_j(k)\Big]e_k\Big\|_{L^p}.$$

Hence to show (2.6), it will suffice to show that the sequence $(S_k)_{k\geq 0}\in\mathcal{L}(\mathrm{Rad}(X))$ defined by

$$S_k\Big(\sum_{j\geq 1}\varepsilon_j x_j\Big) = \sum_{j\geq 1}\varepsilon_j M_k^{(j)} x_j$$

defines a Fourier multiplier on $H^p(\mathbb{T};\mathrm{Rad}(X))$. We notice that $\mathrm{Rad}(X)$ is an AUMD space having the property (α) as X is an AUMD space having the property (α). Therefore, it will suffice to verify that the sequences $(S_k)_{k\geq 0}$, $(k\Delta^1 S_k)_{k\geq 0}$ and $(k^2\Delta^2 S_k)_{k\geq 0}$ are Rademacher bounded in $\mathcal{L}(\mathrm{Rad}(X))$ by Theorem 2.1. For the Rademacher boundedness of $(S_k)_{k\geq 1}$, we need to show that there exists a constant $C>0$ such that for $\sum_{j\geq 1}\varepsilon_j x_{k,j}\in\mathrm{Rad}(X)$,

$$\Big\|\sum_{k\geq 0}\varepsilon'_k S_k\Big(\sum_{j\geq 1}\varepsilon_j x_{k,j}\Big)\Big\|_{L^p}\leq C\Big\|\sum_{k\geq 0}\varepsilon'_k\sum_{j\geq 1}\varepsilon_j x_{k,j}\Big\|_{L^p}$$

or equivalently

$$\Big\|\sum_{k\geq 0}\sum_{j\geq 1}\varepsilon'_k\varepsilon_j M_k^{(j)} x_{k,j}\Big\|_{L^p}\leq C\Big\|\sum_{k\geq 0}\sum_{j\geq 1}\varepsilon'_k\varepsilon_j x_{k,j}\Big\|_{L^p}, \tag{2.7}$$

where $(\varepsilon'_k)_{k\geq 0}$ is another Rademacher function sequence independent of $(\varepsilon_j)_{j\geq 1}$. (2.7) follows from the Rademacher boundedness assumption of the set $\{M_k^{(j)}: j\geq 1,\ k\geq 0\}$ and inequality (2.3). We have shown that $(S_k)_{k\geq 0}$ is Rademacher bounded. Similar arguments show that the sequences $(k\Delta^1 S_k)_{k\geq 0}$ and $(k^2\Delta^2 S_k)_{k\geq 0}$ are also Rademacher bounded. This finishes the proof.

Proof of Theorem 2.2 When $d=1$, the claim is just Lemma 2.1. Now assuming that the statement is true for some $d\in\mathbb{N}$, we are going to show that it remains true for $d+1$. To this end, we let $\mathcal{M}\subset\mathcal{L}(X)$ be Rademacher bounded. By the induction hypothesis, each $N=(N(n))_{n\in\mathbb{N}_0^d}\subset\mathcal{L}(X)$ such that

$$\Big\{n^\alpha\Big(\prod_{1\leq j\leq d}D_j^{\alpha_j}\Big)N(n): n\in\mathbb{N}_0^d,\ \alpha_j\in\{0,1,2\},\ 1\leq j\leq d\Big\}\subset\mathcal{M} \tag{2.8}$$

defines a Fourier multiplier on $H^p(\mathbb{T}^d;X)$. Moreover, if we denote by T_N the corresponding bounded linear operator on $H^p(\mathbb{T}^d;X)$, the set

$$\mathcal{M}' := \{T_N : N \text{ satisfies } (2.8)\}$$

is Rademacher bounded in $\mathcal{L}(H^p(\mathbb{T}^d;X))$. Now let $M=(M(n))_{n\in\mathbb{N}_0^{d+1}}\subset\mathcal{L}(X)$ satisfy

$$\Big\{n^\alpha\Big(\prod_{1\leq j\leq d+1}D_j^{\alpha_j}\Big)M(n): n\in\mathbb{N}_0^{d+1},\ \alpha_j\in\{0,1,2\},\ 1\leq j\leq d+1\Big\}\subset\mathcal{M}. \tag{2.9}$$

For fixed $\alpha_1\in\{0,1,2\}$ and $n_1\in\mathbb{N}_0$, we consider the sequence $N^{\alpha_1,n_1}=(N^{\alpha_1,n_1}(n))_{n\in\mathbb{N}_0^d}$ given by

$$N^{\alpha_1,n_1}(n_2,\cdots,n_{d+1}) = n_1^{\alpha_1}D_1^{\alpha_1}M(n_1,n_2,\cdots,n_{d+1}),\quad n_j\in\mathbb{N}_0,\ 2\leq j\leq d+1.$$

It is clear that N^{α_1,n_1} satisfies condition (2.8) by assumption (2.9), and thus N^{α_1,n_1} defines a Fourier multiplier on $H^p(\mathbb{T}^d;X)$ and $T_{N^{\alpha_1,n_1}} \in \mathcal{M}'$. This implies that M defines a Fourier multiplier on $H^p(\mathbb{T};H^p(\mathbb{T}^d;X))$ (which is the same as $H^p(\mathbb{T}^{d+1};X)$ by the Fubini's theorem) by the known result in the one dimensional case (see Theorem 2.2). Here we have used the fact that the set $\mathcal{M}'$ is Rademacher bounded. Let S_M be the corresponding bounded linear operator on $H^p(\mathbb{T}^{d+1};X)$. It remains to show that the set $\{S_M : M \text{ satisfies } (2.9)\}$ is Rademacher bounded in $\mathcal{L}(H^p(\mathbb{T}^{d+1};X))$. This follows from the fact that for $M = (M_n)_{n\in\mathbb{N}_0^{d+1}}$ satisfying (2.9) and $\alpha_1 \in \{0,1,2\}$, $n_1 \in \mathbb{N}_0$, the corresponding bounded linear operator $T_{N^{\alpha_1,n_1}}$ on $H^p(\mathbb{T}^d;X)$ belongs to $\mathcal{M}'$, and $\mathcal{M}'$ is Rademacher bounded. Therefore, we can apply Lemma 2.1 on $H^p(\mathbb{T};Z)$ when Z is an AUMD space having the property (α) (we take $Z = H^p(\mathbb{T}^d;X)$). We notice that when X is an AUMD space having the property (α), $H^p(\mathbb{T}^d;X)$ is also an AUMD space having the property (α). This completes the proof.

The next result shows that the assumption that X has the property (α) is necessary in Theorem 2.2 when $d \geq 2$ even for scalar-valued Fourier multipliers.

Theorem 2.3 *Let X be a Banach space and $1 \leq p < \infty$. We assume that each sequence $M(m,n)_{m,n\geq 0} \subset \mathbb{C}$ such that*

$$\sup_{\substack{m,n\geq 0\\ \alpha,\beta\in\{0,1,2\}}} |m^\alpha n^\beta \Delta_1^\alpha \Delta_2^\beta M(m,n)| < \infty \tag{2.10}$$

defines a Fourier multiplier on $H^p(\mathbb{T}^2;X)$. Then X has the property (α).

Proof Let $M = M(m,n)_{m,n\geq 0} \subset \mathbb{C}$, and

$$\eta(M) := \sup_{\substack{m,n\geq 0\\ \alpha,\beta\in\{0,1,2\}}} |m^\alpha n^\beta \Delta_1^\alpha \Delta_2^\beta M(m,n)|.$$

If we denote by T_M the bounded linear operator on $H^p(\mathbb{T}^d;X)$ defined by M, then by the closed graph theorem, there exists a constant $C_1 > 0$ independent of M, such that $\|T_M\| \leq C_1\eta(M)$. Let $\phi \in C_c^\infty(\mathbb{R}^2)$ be fixed such that $\phi(1,1) = 1$, $0 \leq \phi \leq 1$ and $\operatorname{supp}(\phi) \subset [\frac{3}{4},\frac{5}{4}]^2$. For $k,j \geq 0$, we let $\phi_{k,j}(s,t) := \phi(2^{-k}s, 2^{-j}t)$ for $s,t \in \mathbb{R}$. It is clear that $\phi_{k,j} \in C_c^\infty(\mathbb{R}^2)$ and the supports of $\phi_{k,j}$'s are pairwisely disjoint. For any fixed choice of signs $\varepsilon_{k,j} = \pm 1$, we let $\varphi(s,t) = \sum_{k,j\geq 0} \varepsilon_{k,j}\phi_{k,j}(s,t)$. It is easy to see that φ is C^∞ and the sequence $M = (\varphi(k,j))_{k,j\geq 0}$ verifies condition (2.10). By assumption, $(\varphi(k,j))_{k,j\geq 0}$ defines a Fourier multiplier on $H^p(\mathbb{T}^2;X)$. Hence, there exists a constant $C_2 > 0$ such that for every $f \in H^p(\mathbb{T}^2;X)$, one has

$$\Big\| \sum_{k,j\geq 0} \varphi(k,j)\widehat{f}(k,j)e_k e'_j \Big\|_{L^p} \leq C_2\|f\|_{L^p}, \tag{2.11}$$

where $e_k(s) = \mathrm{e}^{\mathrm{i}ks}$ and $e'_j(t) = \mathrm{e}^{\mathrm{i}jt}$ for $(s,t) \in \mathbb{T}^2$. Here we use the fact that there exists a constant $C > 0$ independent of the $\varepsilon_{k,j}$'s, such that $\eta(M) \leq C$. For $x_{k,j} \in X$, substituting $f = \sum_{k,j\geq 0} e_{2^k}e'_{2^j}x_{k,j}$ in (2.11), we deduce that

$$\Big\| \sum_{k,j\geq 0} \varepsilon_{k,j}x_{k,j}e_{2^k}e'_{2^j} \Big\|_{L^p} \leq C_2 \Big\| \sum_{k,j\geq 0} x_{k,j}e_{2^k}e'_{2^j} \Big\|_{L^p},$$

as it is clear that $\phi(2^k,2^j) = \varepsilon_{k,j}$. This is equivalent to say that

$$\Big\| \sum_{k,j\geq 0} \alpha_{k,j}x_{k,j}\varepsilon_k\varepsilon'_j \Big\|_{L^p} \leq C_2 \Big\| \sum_{k,j\geq 0} x_{k,j}\varepsilon_k\varepsilon'_j \Big\|_{L^p}$$

for any $|\alpha_{k,j}| \leq 1$ by the Pisier's result (see [10]). Thus X has the property (α).

Remark 2.1 In [12, Proposition 2], Zimmermann gave a Marcinkiewicz type condition of order 1 for a scalar sequence to be a Fourier multiplier on $L^p(T^d; X)$ when X is a UMD space with l.u.st. and $1 < p < \infty$. It is not hard to verify that the Zimmermann's result remains true if we replace the assumption that X has l.u.st. by the weaker assumption that X has the property (α). The same argument used in the proof of Theorem 2.3 shows that when $d \geq 2$, the assumption that X has the property (α) is also necessary in the corresponding Zimmermann's result.

In the last part of this section, we give an operator-valued Fourier multiplier theorem on $H^p(\mathbb{R}^d; X)$. Let X be a Banach space. Given $f \in L^1(\mathbb{R}^d; X)$, the Fourier transform $\mathcal{F}f$ of f is given by $\mathcal{F}f(t) = \int_{\mathbb{R}} f(\xi)\mathrm{e}^{-\mathrm{i}\xi\cdot t}\mathrm{d}\xi$ $(t \in \mathbb{R}^d)$. The inverse Fourier transform is denoted by $\mathcal{F}^{-1}f$ for $f \in L^1(\mathbb{R}^d; X)$. By definition, the Hardy space $H^1(\mathbb{R}^d; X)$ is the closed subspace of $L^1(\mathbb{R}^d; X)$ consisting of all f such that $\mathcal{F}f(t) = 0$ for $t \in \mathbb{R}^d \setminus \mathbb{R}^d_+$. If $1 \leq p < \infty$, then we denote by $H^p(\mathbb{R}^d; X)$ the closure of $H^1(\mathbb{R}^d; X) \cap L^p(\mathbb{R}^d; X)$ in $L^p(\mathbb{R}; X)$.

Let $\mathcal{S}_+ = \mathcal{S}(\mathbb{R}^d) \cap H^p(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^d$. Then an approximating argument shows that $\mathcal{S}_+$ is dense in $H^p(\mathbb{R}^d)$ (see [8] for a similar argument). This implies that the tensor product $\mathcal{S}_+ \otimes X$ is a dense subspace of $H^p(\mathbb{R}^d; X)$. Now let $m : \mathbb{R}^d_+ \to \mathcal{L}(X)$ be a bounded measurable function, m is said to be a Fourier multiplier on $H^p(\mathbb{R}^d; X)$, if there exists a constant $C > 0$ such that for all $f = \sum\limits_{j=1}^{n} f_j \otimes x_j \in \mathcal{S}_+ \otimes X$, we have $\|\mathcal{F}^{-1}(m\mathcal{F}f)\|_p \leq C\|f\|_p$ (note that each term $\mathcal{F}^{-1}(m\mathcal{F}f_j x_j)$ makes sense as $f_j \in \mathcal{S}_+$). In this case there exists a unique bounded linear operator T_m on $H^p(\mathbb{R}^d; X)$ such that for all $f \in \mathcal{S}_+ \otimes X$, we have $T_m f = \mathcal{F}^{-1}(m\mathcal{F}f)$.

In [8, Proposition 4.3], Le Merdy has shown that if $1 \leq p < \infty$ and $m : \mathbb{R}_+ \to \mathbb{C}$ is a bounded continuous function, then m defines a Fourier multiplier on $H^p(\mathbb{R}; X)$ if and only if for each $\epsilon > 0$, the sequence $(m(j\epsilon))_{j\geq 0}$ is a Fourier multiplier on $H^p(\mathbb{T}; X)$ and the corresponding bounded linear operators on $H^p(\mathbb{T}; X)$ are uniformly bounded for $\epsilon > 0$. It is easy to verify that the corresponding result for multi-dimensional Hardy spaces is still valid. This remark together with Theorem 2.2 gives the following Fourier multiplier theorem on $H^p(\mathbb{T}^d; X)$.

Theorem 2.4 *Let X be an AUMD space having the property (α) and let $1 \leq p < \infty$. Then each C^{2d}-function $m : \mathbb{R}^d_+ \to \mathcal{L}(X)$ such that the set*

$$\Big\{\Big(\prod_{j=1}^{d} x_j^{\alpha_j}\Big)\Big(\prod_{j=1}^{d} \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}\Big)m(x_1, \cdots, x_d) : x_j \geq 0,\ \alpha_j \in \{0, 1, 2\},\ 1 \leq j \leq d\Big\}$$

is Rademacher bounded, defines a Fourier multiplier on $H^p(\mathbb{R}^d; X)$.

3 The General Case

In this section, the Banach space X has not necessarily the property (α) when studying Fourier multipliers on $H^p(\mathbb{T}^d; X)$. Therefore, we need a stronger notion of Rademacher boundedness. A sequence $(M(n))_{n\in\mathbb{N}_0^d} \subset \mathcal{L}(X)$ is said to be d-Rademacher bounded, if for some $1 \leq p < \infty$, there exists a constant $C > 0$ such that for all $x_n \in X$, we have

$$\Big\|\sum_{n=(n_1,\cdots,n_d)\in\mathbb{N}_0^d} \varepsilon^{(1)}_{n_1}\varepsilon^{(2)}_{n_2}\cdots\varepsilon^{(d)}_{n_d} M(n)x_n\Big\|_{L^p} \leq C\Big\|\sum_{n=(n_1,\cdots,n_d)\in\mathbb{N}_0^d} \varepsilon^{(1)}_{n_1}\varepsilon^{(2)}_{n_2}\cdots\varepsilon^{(d)}_{n_d} x_n\Big\|_{L^p},$$

where $(\varepsilon^{(j)}_n)_{n\geq 0}$ $(1 \leq j \leq d)$ are d sequences of Rademacher functions pairwisely independent. It turns out that this notion is still independent of the choice of $1 \leq p < \infty$ by the Kahane's inequality (see [9, Theorem 1.e.13]).

We begin with a result concerning the relations between the Rademacher boundedness and the d-Rademacher boundedness.

Lemma 3.1 *Let X be a Banach space, $(M(n))_{n\in\mathbb{N}_0^d}\subset\mathcal{L}(X)$ and let $d\geq 2$. Then*

(1) *If $(M(n))_{n\in\mathbb{N}_0^d}$ is d-Rademacher bounded, then it is Rademacher bounded.*

(2) *If X has the property (α) and $(M(n))_{n\in\mathbb{N}_0^d}$ is Rademacher bounded, then it is d-Rademacher bounded.*

Proof We only give the proof for the case $d=2$, and the proof for the general case is similar. Suppose that $(M(m,n))_{m,n\geq 0}$ is 2-Rademacher bounded. Let $(\varepsilon_m^{(1)})_{m\geq 0}$ and $(\varepsilon_n^{(2)})_{n\geq 0}$ are two independent sequences of Rademacher functions, and let $(\varepsilon_{m,n})_{m,n\geq 0}$ be another sequence of Rademacher functions independent of $(\varepsilon_m^{(1)})_{m\geq 0}$ and $(\varepsilon_n^{(2)})_{n\geq 0}$. By the Kahane's contraction principle, for all $s,t\in[0,1]$ and $x_{m,n}\in X$,

$$\Big\|\sum_{m,n\geq 0}\varepsilon_{m,n}M(m,n)x_{m,n}\Big\|_{L^2}^2\leq 4\Big\|\sum_{m,n\geq 0}\varepsilon_m^{(1)}(s)\varepsilon_n^{(2)}(t)\varepsilon_{m,n}M(m,n)x_{m,n}\Big\|_{L^2}^2.$$

Integrating both sides on $[0,1]^2$, we deduce by the Fubini's theorem that

$$\begin{aligned}
&\Big\|\sum_{m,n\geq 0}\varepsilon_{m,n}M(m,n)x_{m,n}\Big\|_{L^2}^2\\
&\leq 4\int_0^1\int_0^1\int_0^1\Big\|\sum_{m,n\geq 0}\varepsilon_m^{(1)}(s)\varepsilon_n^{(2)}(t)\varepsilon_{m,n}(u)M(m,n)x_{m,n}\Big\|^2\mathrm{d}u\mathrm{d}t\mathrm{d}s\\
&=4\int_0^1\int_0^1\int_0^1\Big\|\sum_{m,n\geq 0}\varepsilon_m^{(1)}(s)\varepsilon_n^{(2)}(t)\varepsilon_{m,n}(u)M(m,n)x_{m,n}\Big\|^2\mathrm{d}t\mathrm{d}s\mathrm{d}u\\
&\leq 4C\int_0^1\int_0^1\int_0^1\Big\|\sum_{m,n\geq 0}\varepsilon_m^{(1)}(s)\varepsilon_n^{2}(t)\varepsilon_{m,n}(u)x_{m,n}\Big\|^2\mathrm{d}t\mathrm{d}s\mathrm{d}u\\
&=4C\int_0^1\int_0^1\int_0^1\Big\|\sum_{m,n\geq 0}\varepsilon_m^{(1)}(s)\varepsilon_n^{2}(t)\varepsilon_{m,n}(u)x_{m,n}\Big\|^2\mathrm{d}u\mathrm{d}t\mathrm{d}s\\
&\leq 16C\Big\|\sum_{m,n\geq 0}\varepsilon_{m,n}x_{m,n}\Big\|_{L^2}^2
\end{aligned}$$

for some constant $C>0$ depending only on $(M(m,n))_{m,n\geq 0}$ by the assumption. This shows that $(M(m,n))_{m,n\geq 0}$ is Rademacher bounded.

Conversely, assume that X has the property (α). It was shown by Pisier [10] that in this case, the sequence $(\varepsilon_m^{(1)}\varepsilon_n^{(2)})_{m,n\geq 0}$ has the same behavior as a sequence of independent Rademacher functions $(\varepsilon_{m,n})_{m,n\geq 0}$, i.e., for all $1\leq p<\infty$, there exists a constant $C>0$, such that

$$\frac{1}{C}\Big\|\sum_{m,n=1}^N\varepsilon_m^{(1)}\varepsilon_n^{(2)}x_{m,n}\Big\|_{L^p}\leq\Big\|\sum_{m,n=1}^N\varepsilon_{m,n}x_{m,n}\Big\|_{L^p}\leq C\Big\|\sum_{m,n=1}^N\varepsilon_m^{(1)}\varepsilon_n^{(2)}x_{m,n}\Big\|_{L^p}$$

for all $N\in\mathbb{N}$ and $x_{m,n}\in X$, where $\varepsilon_m^{(1)}$, $\varepsilon_n^{(2)}$ and $\varepsilon_{m,n}$ are as in the first part of the proof. This implies in particular that when X has the property (α), and $(T_{m,n})_{m,n\geq 0}\subset\mathcal{L}(X)$ is a Rademacher bounded sequence, then it is also 2-Rademacher bounded. The proof is completed.

In [3], the authors have shown that the Rademacher boundedness is necessary for a sequence to be a Fourier multiplier on $L^p(\mathbb{T}^d;X)$. In the next result we show that the stronger condition of d-Rademacher boundedness is also necessary for a sequence to be a Fourier multiplier on $H^p(\mathbb{T}^d;X)$.

Proposition 3.1 *Let X be a Banach space and let $1 \le p < \infty$. Assume that the sequence $(M(n))_{n\in\mathbb{N}_0^d} \subset \mathcal{L}(X)$ defines a Fourier multiplier on $H^p(\mathbb{T}^d;X)$. Then $(M(n))_{n\in\mathbb{N}_0^d}$ is d-Rademacher bounded.*

Proof For $t_j \in [0,2\pi]$ $(1 \le j \le d)$, we have by the Kahane's inequality that

$$\begin{aligned}&\Big\|\sum_{n_j\ge 0}\varepsilon^{(1)}_{n_1}\cdots\varepsilon^{(d)}_{n_d}M(n_1,\cdots,n_d)x_{n_1,\cdots,n_d}\Big\|^p_{L_p}\\ &\le 2^p\Big\|\sum_{n_j\ge 0}\varepsilon^{(1)}_{n_1}\cdots\varepsilon^{(d)}_{n_d}e_{n_1,\cdots,n_d}(t_1,\cdots,t_d)M(n_1,\cdots,n_d)x_{n_1,\cdots,n_d}\Big\|^p_{L_p},\end{aligned}$$

where $\varepsilon^{(j)}_n$ $(1 \le j \le d)$ are d sequences of independent Rademacher functions. Integrating on both sides on $[0,2\pi]^d$, using the Fubini's theorem and the assumption that $(M(n))_{n\in\mathbb{N}_0^d} \subset \mathcal{L}(X)$ defines a Fourier multiplier on $H^p(\mathbb{T}^d;X)$, we deduce that

$$\Big\|\sum_{n_j\ge 0}\varepsilon^{(1)}_{n_1}\cdots\varepsilon^{(d)}_{n_d}M(n_1,\cdots,n_d)x_{n_1,\cdots,n_d}\Big\|^p_{L_p} \le C\Big\|\sum_{n_j\ge 0}\varepsilon^{(1)}_{n_1}\cdots\varepsilon^{(d)}_{n_d}x_{n_1,\cdots,n_d}\Big\|^p_{L_p}$$

for some constant $C>0$ depending only on $(M(n))_{n\in\mathbb{N}_0^d}$. The proof is completed.

Now we are ready to state the operator-valued Fourier multiplier theorem on $H^p(\mathbb{T}^d;X)$ when X has not necessarily the property (α).

Theorem 3.1 *Let X be an AUMD space, $1 \le p < \infty$ and let $(M(n))_{n\in\mathbb{N}_0^d} \subset \mathcal{L}(X)$ be such that the sequences $\Big(n^{(\alpha_1,\cdots,\alpha_d)}\Big(\prod_{j=1}^{d}\Delta_j^{\alpha_j}\Big)M(n)\Big)_{n\in\mathbb{N}_0^d}$ are d-Rademacher bounded for $\alpha_j = 0,1,2$ $(1 \le j \le d)$. Then $(M(n))_{n\in\mathbb{N}_0^d}$ defines a Fourier multiplier on $H^p(\mathbb{T}^d;X)$.*

Proof We only give the proof for the case $d=2$, and the proof for the general case is similar. Let $(M(m,n))_{m,n\ge 0} \subset \mathcal{L}(X)$ be such that when $\alpha,\beta = 0,1,2$, $(m^\alpha n^\beta(\Delta_1^\alpha\Delta_2^\beta)M(m,n))_{m,n\ge 0}$ are 2-Rademacher bounded sequences.

By the Fubini's theorem, the space $H^p(\mathbb{T}^2;X)$ and the space $H^p(\mathbb{T};H^p(\mathbb{T};X))$ may be naturally identified. To show that $(M(m,n))_{m,n\ge 0}$ defines a Fourier multiplier on $H^p(\mathbb{T}^2;X)$, it will suffice to show that the sequence $M_m \in \mathcal{L}(H^p(\mathbb{T};X))$ defines a Fourier multiplier on $H^p(\mathbb{T};H^p(\mathbb{T};X))$, where M_m is defined by

$$M_m\Big(\sum_{n\ge 0}x_ne_n\Big) := \sum_{n\ge 0}M_{m,n}x_ne_n.$$

We notice that for fixed $m \ge 0$, the sequence $(M_{m,n})_{n\ge 0} \subset \mathcal{L}(X)$ verifies the sufficient condition of Theorem 2.1 by the assumptions and Lemma 3.1, thus defines a Fourier multiplier on $H^p(\mathbb{T};X)$. The space $H^p(\mathbb{T};X)$ is still an AUMD space as X is an AUMD space. To show that $(M_m)_{m\ge 0}$ defines a Fourier multiplier on $H^p(\mathbb{T};H^p(\mathbb{T};X))$, it suffice to show that the sets $\{M_m : m \ge 0\}$, $\{m\Delta M_m : m \ge 0\}$ and $\{m^2\Delta^2 M_m : m \ge 0\}$ are Rademacher bounded by Theorem 2.1. In other words, we have to show that there exists a constant $C>0$, such that for $\sum\limits_{n\ge 0}x_{m,n}e_n \in H^p(\mathbb{T};X)$,

$$\Big\|\sum_{m\ge 0}\varepsilon_m\sum_{n\ge 0}W_{m,n}x_{m,n}e_n\Big\|_{L^p} \le C\Big\|\sum_{m\ge 0}\varepsilon_m\sum_{n\ge 0}x_{m,n}e_n\Big\|_{L^p},$$

where $(W_{m,n})_{m,n\ge 0}$ is one of the sequences $(M(m,n))_{m,n\ge 0}$, $(m(M(m+1,n)-M(m,n)))_{m,n\ge 0}$ and $(m^2(M(m+2,n)-2M(m+1,n)+M_{m,n}))_{m,n\ge 0}$. By the Fubini's theorem, this is equivalent to show that there exists $C > 0$, such that for all $x_{m,n} \in X$,

$$\Big\| \sum_{n\geq 0} \Big(\sum_{m\geq 0} \varepsilon_m W_{m,n} x_{m,n} \Big) e_n \Big\|_{L^p} \leq C \Big\| \sum_{n\geq 0} \Big(\sum_{m\geq 0} \varepsilon_m x_{m,n} \Big) e_n \Big\|_{L^p}.$$

Hence, we have to show that the sequence $(V_n)_{n\geq 0} \subset \mathcal{L}(\mathrm{Rad}(X))$ verifies the sufficient condition in Theorem 2.1, where $V_n\Big(\sum\limits_{m\geq 0} \varepsilon_m x_m \Big) = \sum\limits_{m\geq 0} \varepsilon_m W_{m,n} x_m$. This is precisely the 2-Rademacher boundedness of the sequence $(M(m,n))_{m,n\geq 0}$. This completes the proof.

Remark 3.1 (1) When the underlying Banach space X has the property (α), a sequence $(M(n))_{n\in\mathbb{N}^d}$ is d-Rademacher bounded if and only if it is Rademacher bounded by Lemma 3.1. This implies that the first claim of Theorem 2.2 is a consequence of Theorem 3.1.

(2) When X is a UMD space and $1 < p < \infty$, a sequence $(M(n))_{n\in\mathbb{Z}^d}$ is a Fourier multiplier on $L^p(\mathbb{T}^d; X)$ if the sequences

$$\Big((n_1^2 + \cdots + n_d^2)^{\frac{\alpha_1+\cdots+\alpha_d}{2}} \Big(\prod_{j=1}^{d} \Delta_j^{\alpha_j} \Big) M(n) \Big)_{n\in\mathbb{Z}^d}$$

are Rademacher bounded for $\alpha_j = 0, 1$ $(1 \leq j \leq d)$ (see [3, 11]). Almost the same argument used in the proof of Theorem 3.1 shows that if the sequences $\Big(n^{(\alpha_1,\cdots,\alpha_d)} \Big(\prod\limits_{j=1}^{d} \Delta_j^{\alpha_j} \Big) M(n) \Big)_{n\in\mathbb{Z}^d}$ are d-Rademacher bounded for $\alpha_j = 0, 1$ $(1 \leq j \leq d)$, then $(M(n))_{n\in\mathbb{Z}^d}$ defines a Fourier multiplier on $L^p(\mathbb{T}^d; X)$. We do not know whether this sufficient condition is weaker than that given in [3, 11].

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