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Sharp Bounds for Symmetric and Asymmetric Diophantine Approximation

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Abstract In 2004, Tong found bounds for the approximation quality of a regular continued fraction convergent to a rational number, expressed in bounds for both the previous and next approximation. The authors sharpen his results with a geometric method and give both sharp upper and lower bounds. The asymptotic frequencies that these bounds occur are also calculated.

Keywords Continued fractions, Diophantine approximation, Upper and lower bounds
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1 Introduction

In 1894, Hurwitz [5] showed that for every irrational number x there exist infinitely many co-prime integers p and q, with $q > 0$, such that

$$\Big|x - \frac{p}{q}\Big| < \frac{1}{\sqrt{5}}\frac{1}{q^2},$$

where the constant $\frac{1}{\sqrt{5}}$ is "best possible", in the sense that it cannot be replaced by a smaller constant.

Let x be a real irrational number, with regular continued fraction (RCF) expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}} = [a_0; a_1, a_2, \cdots, a_n, \cdots]. \tag{1.1}$$

Here we take $a_0 \in \mathbb{Z}$ such that $x - a_0 \in [0, 1)$, and $a_n \in \mathbb{N}$ for $n \geq 1$. Finite truncation in (1.1) yields the convergents $\frac{p_n}{q_n}$, $n \geq 0$, i.e.,

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \cdots, a_n] \quad \text{for } n \geq 1.$$

The partial coefficients a_n can be found from the regular continued fraction map $T : [0, 1) \to [0, 1)$, defined by

$$T(x) := \Big\{\frac{1}{x}\Big\} = \frac{1}{x} - \Big\lfloor\frac{1}{x}\Big\rfloor, \quad x \neq 0, \quad T(0) := 0,$$

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where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

Borel [2] showed that for all $n \geq 1$,

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{\sqrt{5}}, \tag{1.2}$$

where the approximation coefficients Θ_n of x are defined by

$$\Theta_n = \Theta_n(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right| \quad \text{for } n \geq 0. \tag{1.3}$$

Hurwitz's result is a direct consequence of Borel's result, and a classical theorem by Legendre, which states that if p and q are two co-prime integers with $q > 0$, satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{2} \frac{1}{q^2},$$

then there exists an $n \in \mathbb{N}$, such that $p = p_n$ and $q = q_n$.

Over the last century Borel's result (1.2) has been refined in various ways. For example, in [1, 4, 10], it was shown that

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{\sqrt{a_{n+1}^2 + 4}} \quad \text{for } n \geq 0,$$

while Tong [13] showed that the "conjugate property" holds

$$\max\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} > \frac{1}{\sqrt{a_{n+1}^2 + 4}} \quad \text{for } n \geq 0.$$

Also various other results on Diophantine approximation have been obtained, starting with Dirichlet's observation from [9], that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \quad \text{for } n \geq 0,$$

which lead to various results in symmetric and asymmetric Diophantine approximation (see, e.g., [7, 8, 14, 15]).

Define for x irrational the number C_n by

$$x - \frac{p_n}{q_n} = \frac{(-1)^n}{C_n q_n q_{n+1}} \quad \text{for } n \geq 0. \tag{1.4}$$

Tong [15, 16] derived various properties of the sequence $(C_n)_{n\geq 0}$, and of the related sequence $(D_n)_{n\geq 0}$, where

$$D_n = [a_{n+1}; a_n, \cdots, a_1] \cdot [a_{n+2}; a_{n+3}, \cdots] = \frac{1}{C_n - 1} \quad \text{for } n \geq 0. \tag{1.5}$$

Recently, Tong [17] obtained the following theorem, which covers many previous results.

Theorem 1.1 (see [17]) *Let $x = [\,a_0; a_1,\, a_2, \cdots, a_n, \cdots]$ be an irrational number. If $r > 1$ and $R > 1$ are two real numbers and*

$$M_{\text{Tong}} = \frac{1}{2}\Big(\frac{1}{r} + \frac{1}{R} + a_n a_{n+1}\Big(1 + \frac{1}{r}\Big)\Big(1 + \frac{1}{R}\Big) + \sqrt{\Big[\frac{1}{r} + \frac{1}{R} + a_n a_{n+1}\Big(1 + \frac{1}{r}\Big)\Big(1 + \frac{1}{R}\Big)\Big]^2 - \frac{4}{rR}}\Big),$$

then

(1) $D_{n-2} < r$ *and* $D_n < R$ *imply* $D_{n-1} > M_{\text{Tong}}$;
(2) $D_{n-2} > r$ *and* $D_n > R$ *imply* $D_{n-1} < M_{\text{Tong}}$.

Tong derived a similar result for the sequence C_n, but it is incorrect. We state this result, give a counterexample and present a correct version of it in Section 6.

In Section 3, we prove the following result.

Theorem 1.2 *Let $r, R > 1$ be reals and put*

$$F = \frac{r(a_{n+1} + 1)}{a_n(a_{n+1} + 1)(r + 1) + 1} \quad \text{and} \quad G = \frac{R(a_n + 1)}{(a_n + 1)a_{n+1}(R + 1) + 1}.$$

If $D_{n-2} < r$ and $D_n < R$, then
(1) *if $r - a_n \geq G$ and $R - a_{n+1} < F$, then $D_{n-1} > \frac{a_{n+1}+1}{R-a_{n+1}}$,*
(2) *if $r - a_n < G$ and $R - a_{n+1} \geq F$, then $D_{n-1} > \frac{a_n+1}{r-a_n}$,*
(3) *in all other cases $D_{n-1} > M_{\text{Tong}}$.*
These bounds are sharp.

The outline of this paper is as follows. We derive elementary properties of the sequence D_n in Section 2. In Section 3, we prove Theorem 1.2 that gives a sharp lower bound for the minimum of D_{n-1} in case $D_{n-2} < r$ and $D_n < R$ for real numbers $r, R > 1$. We prove a similar theorem for the case that $D_{n-2} > r$ and $D_n > R$ in Section 4. In Section 5, we calculate the asymptotic frequency that simultaneously $D_{n-2} > r$ and $D_n > R$. Finally, we correct Tong's result for C_n in Section 6 and give the sharp bound in this case.

2 The Natural Extension

Define the space $\Omega = [0, 1) \times [0, 1]$ and define $\mathcal{T} : \Omega \to \Omega$ as

$$\mathcal{T}(x, y) = \Big(\frac{1}{x} - \Big\lfloor \frac{1}{x} \Big\rfloor, \frac{1}{a_1(x) + y}\Big).$$

The following theorem was obtained in 1977 by Nakada et al. [12] (see also [6, 11]).

Theorem 2.1 *Let ν be the probability measure on Ω with density $d(x, y)$, given by*

$$d(x, y) = \frac{1}{\log 2} \frac{1}{(1 + xy)^2}, \quad (x, y) \in \Omega. \tag{2.1}$$

Then ν is the invariant measure for $\mathcal{T}$. Furthermore, the dynamical system $(\Omega, \nu, \mathcal{T})$ is an ergodic system.

The system $(\Omega, \nu, \mathcal{T})$ is the natural extension of the ergodic dynamical system $([0,1), \mu, T)$, where μ is the so-called Gauss-measure, the probability measure on $[0,1)$ with density

$$d(x) = \frac{1}{\log 2}\frac{1}{1+x}, \quad x \in [0,1).$$

This natural extension plays a key role in the proofs of various results in this paper.

We write t_n and v_n for the "future" and "past" of x at time n, respectively,

$$t_n = [0; a_{n+1}, a_{n+2}, \cdots] \quad \text{and} \quad v_n = [0; a_n, \cdots, a_1]. \tag{2.2}$$

Furthermore, $t_0 = x$ and $v_0 = 0$.

The approximation coefficients may be written in terms of t_n and v_n

$$\Theta_n = \frac{t_n}{1+t_n v_n} \quad \text{and} \quad \Theta_{n-1} = \frac{v_n}{1+t_n v_n}, \quad n \geq 1.$$

Lemma 2.1 *Let $x = [a_0; a_1, a_2, \cdots]$ be in $\mathbb{R}\backslash\mathbb{Q}$ and $n \geq 2$ be an integer. The variables D_{n-2}, D_{n-1} and D_n can be expressed in terms of future t_n, past v_n and digits a_n and a_{n+1} by*

$$D_{n-2} = D_{n-2}(t_n, v_n) = \frac{(a_n + t_n)v_n}{1 - a_n v_n}, \tag{2.3}$$

$$D_{n-1} = D_{n-1}(t_n, v_n) = \frac{1}{t_n v_n}, \tag{2.4}$$

$$D_n = D_n(t_n, v_n) = \frac{(a_{n+1} + v_n)t_n}{1 - a_{n+1} t_n}. \tag{2.5}$$

Proof The expression for D_{n-1} follows from the definition in (1.5).

$$\begin{aligned} D_{n-1} &= [\, a_n; a_{n-1}, \cdots, a_1\,][\, a_{n+1}; a_{n+2}, \cdots] \\ &= \frac{1}{[\, 0; a_n, a_{n-1}, \cdots, a_1\,][\, 0; a_{n+1}, a_{n+2}, \cdots]} = \frac{1}{v_n t_n}. \end{aligned}$$

It follows in a similar way that $D_n = \frac{1}{t_{n+1}}\frac{1}{v_{n+1}}$. Using

$$\begin{aligned} t_{n+1} &= \frac{1}{t_n} - a_{n+1}, \\ v_{n+1} &= \frac{q_n}{q_{n+1}} = \frac{q_n}{a_{n+1}q_n + q_{n-1}} = \frac{1}{a_{n+1} + v_n}, \end{aligned}$$

we find (2.5). The formula for D_{n-2} can be derived in a similar way.

Remark 2.1 Of course, D_{n-2}, D_{n-1} and D_n also depend on x, but we suppress this dependence in our notation.

The following result on the distribution of the sequence $(t_n, v_n)_{n\geq 0}$ is a consequence of the Ergodic Theorem, and was originally obtained by Bosma et al. [3] (see also [6, Chapter 4]).

Theorem 2.2 *For almost all $x \in [0,1)$, the two-dimensional sequence*

$$(t_n, v_n) = \mathcal{T}^n(x, 0), \quad n \geq 0$$

is distributed over Ω according to the density-function $d(t,v)$, as given in (2.1).

Consequently, for any Borel measurable set $B \subset \Omega$ with a boundary of Lebesque measure zero, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} I_B(t_n, v_n) = \nu(B), \tag{2.6}$$

where I_B is the indicator function of B. We use this result to derive the following lemma.

Lemma 2.2 *For almost all $x \in [0,1)$, and for all $R \geq 1$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n \,|\, D_j(x) \leq R\}$$

exists, and equals

$$H(R) = 1 - \frac{1}{\log 2}\Big(\log\Big(\frac{R+1}{R}\Big) + \frac{\log R}{R+1}\Big). \tag{2.7}$$

Consequently, for almost all $x \in [0,1)$ one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} D_n(x) = \infty.$$

Proof By (2.4) and (2.6), for almost every x, the asymptotic frequency that $D_{n-1} \leq R$ is given by the measure of those points (t, v) in Ω with $\frac{1}{tv} \leq R$. This measure equals

$$\frac{1}{\log 2} \int_{t=\frac{1}{R}}^{1} \int_{v=\frac{1}{Rt}}^{1} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}$$

(see also Figure 1).

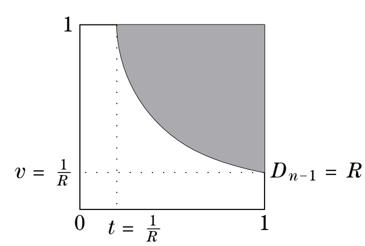


Figure 1 The curve $\frac{1}{tv} = R$ on Ω. For (t_n, v_n) in the gray part, it holds that $D_{n-1} \leq R$.

It follows that

$$H(R) = \frac{1}{\log 2} \int_{\frac{1}{R}}^{1} \Big[\frac{v}{1+tv}\Big]_{\frac{1}{Rt}}^{1} \mathrm{d}t = \frac{1}{\log 2}\Big[\log 2 - \log\frac{R+1}{R} - \frac{1}{R+1}\log R\Big],$$

which may be rewritten as (2.7).

To calculate the expectation of D_n, we use that the density function of D_n is given by $h(x) = H'(x)$, so

$$h(x) = \frac{1}{\log 2} \frac{\log x}{(x+1)^2} \quad \text{for } x \geq 1.$$

We can now easily calculate the expected value of D_n

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}D_j(x)=\int_1^\infty x\,h(x)\,\mathrm{d}x=\lim_{t\to\infty}\int_1^t\frac{1}{\log 2}\frac{x\log x}{(x+1)^2}\,\mathrm{d}x=\infty.$$

Apart for proving metric results on the D_n's, the natural extension $(\Omega,\nu,\mathcal{T})$ is also very handy to obtain various Borel-type results on the D_n's.

For $a,b\in\mathbb{N}$, consider the rectangle $\Delta_{a,b}=\left[\frac{1}{b+1},\frac{1}{b}\right)\times\left[\frac{1}{a+1},\frac{1}{a}\right)\subset\Omega$. On this rectangle, we have $a_n=a$ and $a_{n+1}=b$. So $(t_n,v_n)\in\Delta_{a,b}$ if and only if $a_n=a$ and $a_{n+1}=b$. We use a and b as abbreviation for a_n and a_{n+1}, respectively, if we are working in such a rectangle.

We define two functions from $[\frac{1}{b+1},\frac{1}{b})$ to $\mathbb{R}$ as

$$f_{a,r}(t)=\frac{r}{a(r+1)+t}\quad\text{and}\quad g_{b,R}(t)=\frac{R}{t}-b(R+1).\tag{2.8}$$

From (2.3) and (2.5), it follows for $(t_n,v_n)\in\Delta_{a,b}$ that

$$\begin{aligned}D_{n-2}<r&\quad\text{if and only if}\quad v_n<f_{a,r}(t_n),\\ D_n<R&\quad\text{if and only if}\quad v_n<g_{b,R}(t_n).\end{aligned}$$

We introduce the following notation

$$F=\frac{r(b+1)}{a(b+1)(r+1)+1}\quad\text{and}\quad G=\frac{R(a+1)}{(a+1)b(R+1)+1}.\tag{2.9}$$

We have $F=f_{a,r}\left(\frac{1}{b+1}\right)$ and $g_{b,R}(G)=\frac{1}{a+1}$ (see also Figure 2).

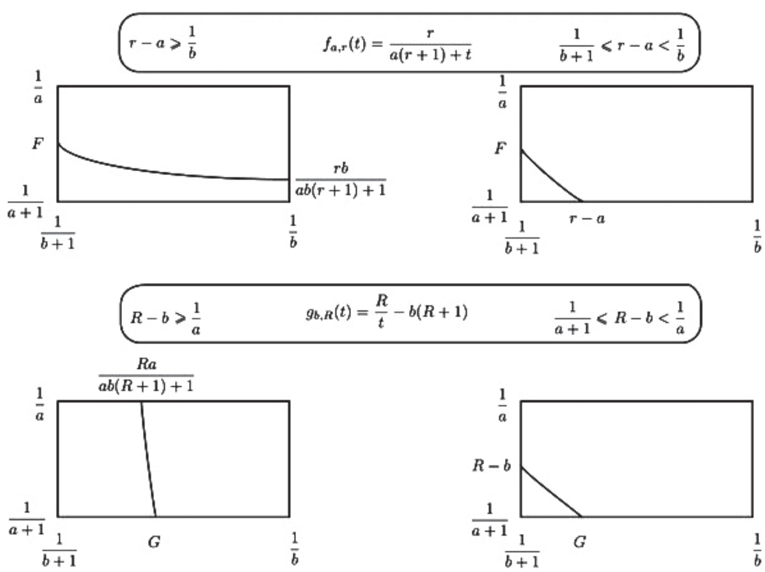


Figure 2 The possible intersection points of the graphs of $f_{a,r}$ and $g_{b,R}$ and the boundary of the rectangle $\Delta_{a,b}$, where $a_n=a$ and $a_{n+1}=b$.

Remark 2.2 The position of the graph of $f_{a,r}$ in $\Delta_{a,b}$ depends on a and r. Obviously, we always have $f_{a,r}\left(\frac{1}{b}\right) < f_{a,r}\left(\frac{1}{b+1}\right) = F < \frac{1}{a}$. Furthermore

$$f_{a,r}\left(\frac{1}{b+1}\right) \geq \frac{1}{a+1} \quad \text{if and only if} \quad r \geq a + \tfrac{1}{b+1},$$
$$f_{a,r}\left(\frac{1}{b}\right) \geq \frac{1}{a+1} \quad \text{if and only if} \quad r \geq a + \tfrac{1}{b}.$$

Similarly, the position of the graph of $g_{b,R}$ in $\Delta_{a,b}$ depends on b and R. We always have $G < \frac{1}{b}$. Furthermore,

$$G \geq \frac{1}{b+1} \quad \text{if and only if} \quad R \geq b + \frac{1}{a+1},$$
$$g_{b,R}\left(\frac{1}{b+1}\right) < \frac{1}{a} \quad \text{if and only if} \quad R < b + \frac{1}{a},$$
$$g_{b,R}\left(\frac{1}{b+1}\right) \geq \tfrac{1}{a+1} \quad \text{if and only if} \quad R \geq b + \frac{1}{a+1}.$$

Compare with Figure 2.

We use the following lemma to determine where D_{n-1} attains it extreme values.

Lemma 2.3 *Let $a, b \in \mathbb{N}$, and $D_{n-1}(t,v) = \frac{1}{tv}$ for points $(t,v) \in (0,1] \times (0,1]$.*
(1) *When t is constant, D_{n-1} is monotonically decreasing as a function of v.*
(2) *When v is constant, D_{n-1} is monotonically decreasing as a function of t.*
(3) *$D_{n-1}(t,v)$ is monotonically decreasing as a function of t on the graph of $f_{a,r}$.*
(4) *$D_{n-1}(t,v)$ is monotonically increasing as a function of t on the graph of $g_{b,R}$.*

Proof The first two statements follow from the trivial observation

$$\frac{\partial D_{n-1}}{\partial t} < 0 \quad \text{and} \quad \frac{\partial D_{n-1}}{\partial v} < 0. \tag{2.10}$$

For points (t,v) on the graph of $f_{a,r}$, we find $D_{n-1}(t,v) = \frac{a(r+1)+t}{rt}$ and

$$\frac{\partial D_{n-1}}{\partial t} = \frac{-a(r+1)}{rt^2} < 0,$$

which proves (3).

Finally, for points (t,v) on the graph of $g_{b,R}$, we find $D_{n-1}(t,v) = \frac{1}{R-b(R+1)t}$. So $\frac{\partial D_{n-1}}{\partial t} > 0$ and (4) is proved.

Lemma 2.4 *On $\Delta_{a,b}$, the infimum of D_{n-1} is attained in the upper right corner, and its maximum in the lower left corner. To be more precise*

$$ab < D_{n-1} \leq (a+1)(b+1).$$

Lemma 2.5 *Let $a, b \in \mathbb{N}$, $r, R > 1$. Set*

$$L = ab(r+1)(R+1), \quad w = \sqrt{4LR + (r - R + L)^2)} \quad \textit{and} \quad S = \frac{-L + R - r + w}{2b(R+1)}.$$

On $\mathbb{R}_+$, the graphs of $f_{a,r}$ and $g_{b,R}$ have one intersection point, which is given by

$$(S, f_{a,r}(S)) = \Big(\frac{-L+R-r+w}{2b(R+1)}, \frac{2br(R+1)}{L+R-r+w}\Big).$$

The corresponding value for D_{n-1} in this point is given by M_{Tong} as defined in Theorem 1.1. For $x < S$ one has $f_{a,r}(x) < g_{b,R}(x)$, while $f_{a,r}(x) > g_{b,R}(x)$ if $x > S$.

Proof Solving

$$\frac{r}{a(r+1)+t} = \frac{R}{t} - b(R+1)$$

yields

$$S = \frac{-L+R-r+w}{2b(R+1)} \quad \text{or} \quad S = \frac{-L+R-r-w}{2b(R+1)}.$$

Since $L > R$, we have the second solution is always negative. So this solution can not be in $\Delta_{a,b}$. The second coordinate follows from substituting $S = \frac{-L+R-r+w}{2b(R+1)}$ in $f_{a,r}(t)$ or $g_{b,R}(t)$.

The corresponding value for D_{n-1} in this point is given by

$$\begin{aligned} & D_{n-1}\Big(\frac{-L+R-r+w}{2b(R+1)}, \frac{2br(R+1)}{L+R-r+w}\Big) \\ &= \frac{-L-R+r-w}{r(L-R+r-w)} = \frac{-L^2+r^2-2Rr+R^2-2Lw-w^2}{r((L-R+r)^2-w^2)} \\ &= \frac{-2L^2-2Lw-2Lr-2LR}{-4RrL} = \frac{1}{2}\Big(\frac{1}{r}+\frac{1}{R}+\frac{L}{Rr}+\frac{w}{Rr}\Big) = M_{\mathrm{Tong}}. \end{aligned}$$

Since $\lim_{x\downarrow 0} f_{a,r}(x) = \frac{r}{a(r+1)}$ and $\lim_{x\downarrow 0} g_{b,R}(x) = \infty$, we immediately have that $f_{a,r}(x) < g_{b,R}(x)$ if $x < S$. And because there is only one intersection point on $\mathbb{R}_+$, it follows that $f_{a,r}(x) > g_{b,R}(x)$ if $x > S$.

Remark 2.3 In view of Remark 2.2 and the last statement of Lemma 2.5, the only possible configurations for $f_{a,r}$ and $g_{b,R}$ in $\Delta_{a,b}$ are given in Figure 3.

3 The Case $D_{n-2} < r$ and $D_n < R$

We assume that both D_{n-2} and D_n are smaller than some given reals r and R. We prove Theorem 1.2 from the Introduction.

Proof of Theorem 1.2 We consider the closure of the region containing all points (t,v) in $\Delta_{a,b}$ with $D_{n-2}(t,v) < r$ and $D_n(t,v) < R$. In Figure 3, we show all possible configurations of this region.

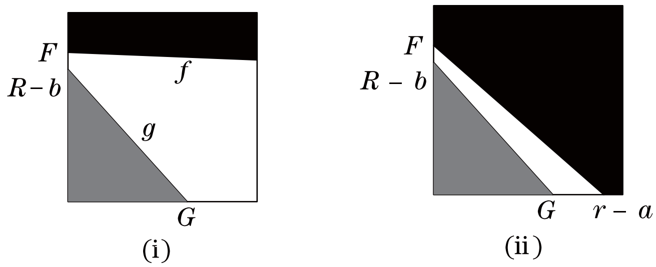
From (2.10), it follows that the extremum of D_{n+1} is attained in a boundary point. Lemma 2.3 implies that we only need to consider the following three points:

(1) The intersection point of the graph of $g_{b,R}$ and the line $t = \frac{1}{b+1}$, given by $\big(\frac{1}{b+1}, R-b\big)$.

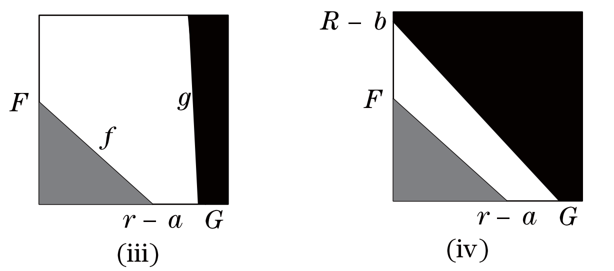
(2) The intersection point of the graph of $f_{a,r}$ and the line $v = \frac{1}{a+1}$, given by $\big(r-a, \frac{1}{a+1}\big)$.

(3) The intersection point of the graphs of $f_{a,r}$ and $g_{b,R}$, given by M_{Tong}.

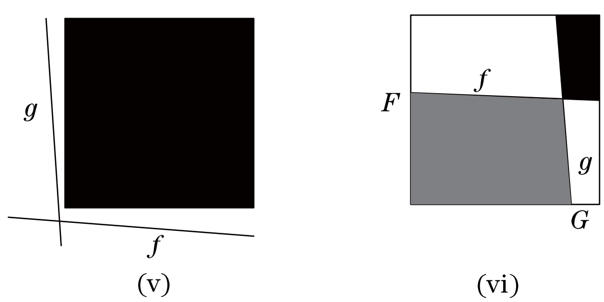
Assume $r - a \geq G$ and $R - b < F$. We know from Lemma 2.5 that the graphs of $f_{a,r}$ and $g_{b,R}$ can not intersect more than once in $\Delta_{a,b}$, thus we are in case (1) (see Figure 3(i) and (ii)).



(a) In case (i) and (ii), we have $r - a \geq G$ and $R - b < F$. It is allowed that $R - b < \frac{1}{a+1}$. In case (i), we have $r - a > \frac{1}{b}$ and in case (ii) $r - a \leq \frac{1}{b}$.



(b) In cases (iii) and (iv), we have $r - a < G$ and $R - b \geq F$. It is allowed that $r - a < \frac{1}{b+1}$. In case (iii) we have $R - b > \frac{1}{a}$ and in case (iv) $R - b \leq \frac{1}{a}$.



(c) In case (v), we have $F < \frac{1}{a+1}$ and $G < \frac{1}{b+1}$. Case (vi) contains all other cases, it can be separated in four subcases (see Figure 6).

Figure 3 The possible configurations of the graphs of $f_{a,r}$ and $g_{b,R}$ on $\Delta_{a,b}$. On the grey parts $D_{n-2} < r$ and $D_n < R$, on the black parts $D_{n-2} > r$ and $D_n > R$.

In this case, the minimum of D_{n-1} is given by $D_{n-1}\left(\frac{1}{b+1}, R-b\right) = \frac{b+1}{R-b}$. The intersection point $(S, f_{a,r}(S))$ lies to the left of $\left(\frac{1}{b+1}, R-b\right)$ and from Lemma 2.3 we know that D_{n-1} increases on the graph of $g_{a,r}$. We conclude that $M_{\text{Tong}} = D_{n-1}(S, f_{a,r}(S))$ is smaller than $\frac{b+1}{R-b}$.

Assume $r - a < G$ and $R - b \geq F$. Then we are in case (2) (see Figure 3(iii) and (iv)), and the minimum is given by $D_{n-1} = \left(r - a, \frac{1}{a+1}\right) = \frac{a+1}{r-a}$. A similar argument as before shows $M_{\text{Tong}} < \frac{a+1}{r-a}$.

Otherwise, still assuming there are points $(t, v) \in \Delta_{a,b}$ with $D_{n-2}(t, v) < r$ and $D_n(t, v) < R$, we must be in case (3) (see Figure 3(vi)). The minimum follows from Lemma 2.5.

These bounds are sharp since the minimum is attained in the extreme point.

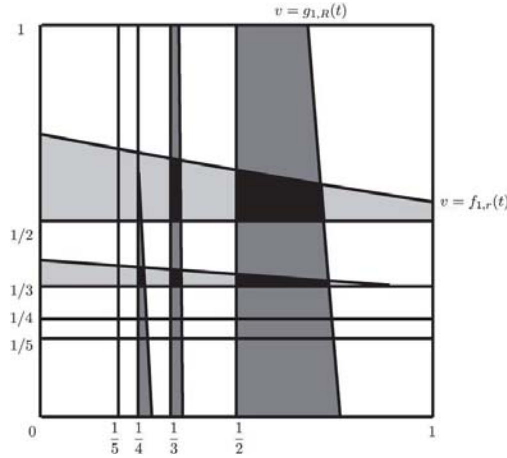


Figure 4 Example with $r = 2.9$ and $R = 3.6$. The regions where $D_{n-2} < 2.9$ are light grey, the regions where $D_n < 3.6$ are dark grey. The intersection where both $D_{n-2} < 2.9$ and $D_n < 3.6$ is black. The horizontal and vertical black lines are drawn to identify the strips and have no meaning for the value of D_{n-2} and D_n.

Example 3.1 Take $r = 2.9$ and $R = 3.6$ (see Figure 4).

If $a_n = a_{n+1} = 1$, then $r - a_n = 1.9$, $R - a_{n+1} = 2.6$, $F \approx 0.66$ and $G \approx 0.71$. Since $R - a_{n+1} > F$, we do not have the case Theorem 1.2(i). Since $r - a_n > G$, we are not in case (ii) either. So in this case $D_{n-1} > M_{\text{Tong}} \approx 2.30$. For the following combinations the minimum is also given by M_{Tong}:

$$a_n = 1 \quad \text{and} \quad a_{n+1} = 2: \quad D_{n-1} > M_{\text{Tong}} \approx 4.04.$$
$$a_n = 2 \quad \text{and} \quad a_{n+1} = 1: \quad D_{n-1} > M_{\text{Tong}} \approx 4.04.$$
$$a_n = 2 \quad \text{and} \quad a_{n+1} = 2: \quad D_{n-1} > M_{\text{Tong}} \approx 7.48.$$
$$a_n = 2 \quad \text{and} \quad a_{n+1} = 3: \quad D_{n-1} > M_{\text{Tong}} \approx 10.92.$$

If $a_n = 1$ and $a_{n+1} = 3$, then $F \approx 0.70$ and $G \approx 0.25$. So $r - a_n > G$ and $\frac{1}{a_{n+1}} < R - a_{n+1} < F$. Thus

$$D_{n-1} > \frac{a_{n+1} + 1}{R - a_{n+1}} \approx 6.67 > M_{\text{Tong}} \approx 5.76.$$

For all other values of a_n and a_{n+1} either $D_{n-2} > r$ or $D_n > R$, or both.

4 The Case $D_{n-2} > r$ and $D_n > R$

In this section, we study the case that D_{n-2} and D_n are larger than given reals r and R, respectively.

Theorem 4.1 *Let $r, R > 1$ be reals, $n \geq 1$ be an integer, and F and G be given as in* (2.9). *If $D_{n-2} > r$ and $D_n > R$, then*
(1) *if $r - a_n \geq G$ and $R - a_{n+1} < F$, then $D_{n-1} < \frac{a_{n+1}+1}{F}$,*
(2) *if $r - a_n < G$ and $R - a_{n+1} \geq F$, then $D_{n-1} < \frac{a_n+1}{G}$,*
(3) *if $r - a_n < \frac{1}{a_{n+1}+1}$ and $R - a_{n+1} < \frac{1}{a_n+1}$, then $D_{n-1} < (a_n + 1)(a_{n+1} + 1)$,*

(4) *in all other cases* $D_{n-1} < M_{\text{Tong}}$.
The bounds are sharp.

Proof The proof is very similar to that of Theorem 1.2. The only "new" case is the one where $r - a < \frac{1}{b+1}$ and $R - b < \frac{1}{a+1}$ (see Figure 3(v)). If $r - a < \frac{1}{b+1}$, then the graph of $f_{a,r}$ lies below $\Delta_{a,b} \subset \Omega$. Similarly, if $R - b < \frac{1}{a+1}$ the graph $g_{b,R}$ lies left of $\Delta_{a,b} \subset \Omega$. In this case, we have that $D_{n-2} > r$ and $D_n > R$ for all $(t_n, v_n) \in \Delta_{a,b}$. In this case, D_{n-1} attains its maximum in the lower left corner $\left(\frac{1}{b+1}, \frac{1}{a+1}\right)$. For the intersection point $(S, f_{a,r}(S))$ either $S < \frac{1}{b+1}$ or $f_{a,r}(S) < \frac{1}{a+1}$ and from Lemma 2.3, we conclude $(a+1)(b+1) < M_{\text{Tong}}$.

Example 4.1 We again use $r = 2.9$ and $R = 3.6$ (see Figure 5 and Table 1).

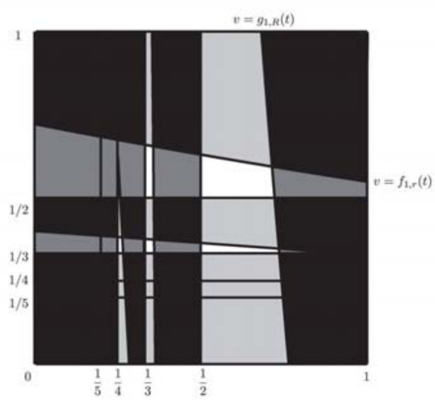


Figure 5 Example with $r = 2.9$ and $R = 3.6$. The regions where $D_{n-2} > 2.9$ are light grey, the regions where $D_n > 3.6$ are dark grey. The intersection where both $D_{n-2} > 2.9$ and $D_n > 3.6$ is black.

Table 1 The sharp upper bounds and the Tong bounds for D_{n-1} for $r = 2.9$ and $R = 3.6$. See Figure 3 for cases (i)–(v) and Figure 6 for (vi_a) and (vi_c).

a_n	a_{n+1}	Case	Upper bound for D_{n-1}	Tong's upper bound
1	1	(vi_a)	2.30	2.30
1	2	(vi_a)	4.04	4.04
1	3	(i)	5.72	5.76
1	4	(i)	7.07	7.48
1	$5, 6, \cdots$	(i)	$\cdots$	$\cdots$
1	37	(i)	51.44	64.20
2	1	(vi_c)	4.04	4.04
2	2	(vi_a)	7.48	7.48
2	3	(vi_a)	10.92	10.92
2	4	(i)	13.79	14.36
2	$5, 6, \cdots$	(i)	$\cdots$	$\cdots$
2	42	(i)	116.00	144.97
3	1	(iii)	4.04	5.76
3	2	(iii)	7.48	10.92
3	3	(iii)	10.92	16.08
3	4	(v)	13.79	21.23
$4, 5, 6 \cdots$	$1, 2, 3$	(iii)	$\cdots$	$\cdots$
$3, 4, 5 \cdots$	$4, 5, 6, \cdots$	(v)	$\cdots$	$\cdots$
17	29	(v)	540.00	847.79

5 Asymptotic Frequencies

Due to Theorem 2.1 and the ergodic theorem, the asymptotic frequency that an event occurs is equal to the measure of the area of this event in the natural extension. We calculate the measure of the region where $D_{n-2} > r$ and $D_n > R$. The same calculations can be done in the easier case where $D_{n-2} < r$ and $D_n < R$.

5.1 The measure of the region where $D_{n-2} > r$ and $D_n > R$ in a rectangle $\Delta_{a,b}$

We calculate the measure in $\Delta_{a,b}$ above the graphs of $f_{a,r}$ and $g_{b,R}$ in the six cases from Figure 3. We denote $\log 2$ times the measure for case $(*)$ in $\Delta_{a,b}$ by $m_{a,b}^{(*)}$.

$$\begin{aligned}
m_{a,b}^{(\mathrm{i})} &= \int_{\frac{1}{b+1}}^{\frac{1}{b}} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} = \int_{\frac{1}{b+1}}^{\frac{1}{b}} \left[\frac{-1}{t}\frac{1}{1+tv}\right]_{\frac{r}{a(r+1)+t}}^{\frac{1}{a}} \mathrm{d}t \\
&= \int_{\frac{1}{b+1}}^{\frac{1}{b}} \left(\frac{-1}{t}\frac{a}{a+t} + \frac{1}{t}\frac{a(r+1)+t}{(a+t)(r+1)}\right) \mathrm{d}t \\
&= \int_{\frac{1}{b+1}}^{\frac{1}{b}} \left(\frac{-1}{t} + \frac{1}{a+t} + \frac{1}{t} - \frac{r}{(a+t)(r+1)}\right) \mathrm{d}t \\
&= \int_{\frac{1}{b+1}}^{\frac{1}{b}} \frac{1}{(a+t)(r+1)} \mathrm{d}t = \frac{1}{(r+1)} [\log(a+t)]_{\frac{1}{b+1}}^{\frac{1}{b}} \\
&= \frac{1}{(r+1)} \log \frac{(ab+1)(b+1)}{(ab+a+1)b}.
\end{aligned}$$

Next we compute $m_{a,b}^{(\mathrm{v})}$, because it is handy for finding $m_{a,b}^{(\mathrm{ii})}$.

$$m_{a,b}^{(\mathrm{v})} = \int_{\frac{1}{b+1}}^{\frac{1}{b}} \int_{\frac{1}{a+1}}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} = \log \frac{(ab+1)(ab+a+b+2)}{(ab+a+1)(ab+b+1)}.$$

For $m_{a,b}^{(\mathrm{ii})}$, we subtract the measure of the region in $\Delta_{a,b}$ below the graph of $f_{a,r}$ from $m_{a,b}^{(\mathrm{v})}$.

$$\begin{aligned}
m_{a,b}^{(\mathrm{ii})} &= m_{a,b}^{(\mathrm{v})} - \int_{\frac{1}{b+1}}^{r-a} \int_{\frac{1}{a+1}}^{f_{a,r}(t)} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} \\
&= \log \frac{(ab+1)(ab+a+b+2)}{(ab+b+1)(ab+a+1)} - \frac{r}{r+1} \log \frac{r(b+1)}{ab+a+1} - \log \frac{ab+a+b+2}{(b+1)(r+1)} \\
&= \log \frac{(ab+1)(b+1)(r+1)}{(ab+b+1)(ab+a+1)} - \frac{r}{r+1} \log \frac{r(b+1)}{ab+a+1}.
\end{aligned}$$

In the computation of $m_{a,b}^{(\mathrm{iii})}$, we use that $v = g_{b,R}(t)$ if and only if $t = \frac{R}{v+b(R+1)}$, so

$$m_{a,b}^{(\mathrm{iii})} = \int_{\frac{1}{a+1}}^{\frac{1}{a}} \int_{\frac{R}{b(R+1)+v}}^{\frac{1}{b}} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2} = \frac{1}{(R+1)} \log \frac{(ab+1)(a+1)}{(ab+b+1)a}.$$

Note that $m_{a,b}^{(\mathrm{iii})}$ is $m_{a,b}^{(\mathrm{i})}$ with a interchanged with b and r replaced by R.

For $m_{a,b}^{(\mathrm{iv})}$, we find using the same techniques as before

$$m_{a,b}^{(\mathrm{iv})} = m_{a,b}^{(\mathrm{v})} - \int_{\frac{1}{a+1}}^{R-b} \int_{\frac{1}{b+1}}^{\frac{R}{b(R+1)+v}} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2}$$
$$= \log \frac{(ab+1)(a+1)(R+1)}{(ab+a+1)(ab+b+1)} - \frac{R}{R+1} \log \frac{R(a+1)}{ab+b+1},$$

which is $m_{a,b}^{(\mathrm{ii})}$ where a is interchanged with b and r replaced by R.

In case (vi), there are four possibilities for the measure of the part above the graphs of $f_{a,r}$ and $g_{b,R}$, depending on where the graphs intersect with $\Delta_{a,b}$ (see Figure 6).

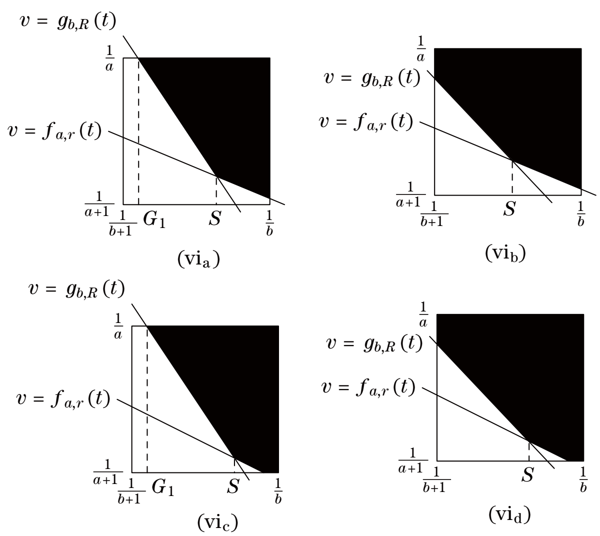


Figure 6 The four possible configurations for case (vi).

Denote $G_1 = \frac{Ra}{ab(R+1)+1}$ (found from solving $g_{b,R}(G_1) = \frac{1}{a}$) and recall from Lemma 2.5 that S is the first coordinate of the intersection point of the graphs of $f_{a,r}$ and $g_{b,R}$. In this case, we have that $(S, f_{a,r}(S)) \in \Delta_{a,b}$.

(vi$_\mathrm{a}$) If $r - a \geq \frac{1}{b}$ and $R - b \geq \frac{1}{a}$, then

$$m_{a,b}^{(\mathrm{vi_a})} = \int_{G_1}^{S} \int_{g_{b,R}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{S}^{\frac{1}{b}} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}.$$

(vi$_\mathrm{b}$) If $r - a \geq \frac{1}{b}$ and $R - b < \frac{1}{a}$, then

$$m_{a,b}^{(\mathrm{vi_b})} = \int_{\frac{1}{b+1}}^{S} \int_{g_{b,R}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{S}^{\frac{1}{b}} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}.$$

(vi$_\mathrm{c}$) If $r - a < \frac{1}{b}$ and $R - b \geq \frac{1}{a}$, then

$$m_{a,b}^{(\mathrm{vi_c})} = \int_{G_1}^{S} \int_{g_{b,R}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{S}^{r-a} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{r-a}^{\frac{1}{b}} \int_{\frac{1}{a+1}}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}.$$

(vi$_\mathrm{d}$) If $r-a<\frac{1}{b}$ and $R-b<\frac{1}{a}$, then

$$m_{a,b}^{(\mathrm{vi_d})}=\int_{\frac{1}{b+1}}^{S}\int_{g_{b,R}(t)}^{\frac{1}{a}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}+\int_{S}^{r-a}\int_{f_{a,r}(t)}^{\frac{1}{a}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}+\int_{r-a}^{\frac{1}{b}}\int_{\frac{1}{a+1}}^{\frac{1}{a}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}.$$

Using the following intergrals:

$$\int_{x}^{S}\int_{g_{b,R}(t)}^{\frac{1}{a}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}=\frac{1}{R+1}\log\frac{S(1-bx)}{x(1-bS)}+\log\frac{x(S+a)}{S(x+a)},$$

$$\int_{S}^{y}\int_{f_{a,r}(t)}^{\frac{1}{a}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}=\frac{1}{r+1}\log\frac{a+y}{a+S},$$

$$\int_{r-a}^{\frac{1}{b}}\int_{\frac{1}{a+1}}^{\frac{1}{a}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}=\log\frac{(ab+1)(r+1)}{(ab+b+1)r},$$

we find that

$$m_{a,b}^{(\mathrm{vi_a})}=\frac{1}{R+1}\log\frac{S(1-bG_1)}{G_1(1-bS)}+\frac{1}{r+1}\log\frac{ab+1}{(a+S)b}+\log\frac{G_1(S+a)}{S(G_1+a)},$$

$$m_{a,b}^{(\mathrm{vi_b})}=\frac{1}{R+1}\log\frac{S}{1-bS}+\frac{1}{r+1}\log\frac{ab+1}{(a+S)b}+\log\frac{S+a}{S(ab+a+1)},$$

$$m_{a,b}^{(\mathrm{vi_c})}=\frac{1}{R+1}\log\frac{S(1-bG_1)}{G_1(1-bS)}+\frac{1}{r+1}\log\frac{r}{a+S}+\log\frac{G_1(S+a)(ab+1)(r+1)}{S(G_1+a)(ab+b+1)r},$$

$$m_{a,b}^{(\mathrm{vi_d})}=\frac{1}{R+1}\log\frac{S}{1-bS}+\frac{1}{r+1}\log\frac{r}{a+S}+\log\frac{(S+a)(ab+1)(r+1)}{S(ab+a+1)(ab+b+1)r}.$$

5.2 The total measure of the region where $D_{n-2}>r$ and $D_n>R$ in the natural extension

For every $r>1$ and $R>1$ the asymptotic frequency that $D_{n-2}>r$ and $D_n>R$ can be found by adding a finite number of integrals. Let $\{x\}=x-\lfloor x\rfloor$ and 1_A be the indicator function of A, i.e.,

$$1_A=\begin{cases}1, & \text{if condition } A \text{ is satisfied,}\\ 0, & \text{else.}\end{cases}$$

Theorem 5.1 *For almost all $x\in[0,1)$, and for all $r,R\geq 1$, we have that*

$$\log 2\lim_{n\to\infty}\frac{1}{n}\#\{2\leq j\leq n+1; D_{j-2}>r \text{ and } D_j>R\}$$

exists and equals

$$\sum_{a=1}^{\lfloor r\rfloor-1}\sum_{b=\lfloor R\rfloor+1}^{\infty}m_{a,b}^{(\mathrm{i})}+\sum_{a=1}^{\lfloor r\rfloor-1}\left(1_{(\{R\}\leq F)}m_{a,\lfloor R\rfloor}^{(\mathrm{i})}+1_{(\{R\}\geq\frac{1}{a})}m_{a,\lfloor R\rfloor}^{(\mathrm{vi_a})}+1_{(F<\{R\}<\frac{1}{a})}m_{a,\lfloor R\rfloor}^{(\mathrm{vi_b})}\right)$$
$$+\sum_{a=1}^{\lfloor r\rfloor-1}\sum_{b=1}^{\lfloor R\rfloor-1}m_{a,b}^{(\mathrm{vi_a})}+\sum_{b=\lfloor R\rfloor+1}^{\infty}\left(1_{(\{r\}\geq\frac{1}{b})}m_{\lfloor r\rfloor,b}^{(\mathrm{i})}+1_{(\frac{1}{b+1}<\{r\}<\frac{1}{b})}m_{\lfloor r\rfloor,b}^{(\mathrm{ii})}\right)+M_{r,R}$$
$$+\sum_{b=1}^{\lfloor R\rfloor-1}\left(1_{(\{r\}\leq G)}m_{\lfloor r\rfloor,b}^{(\mathrm{iii})}+1_{(\{r\}\geq\frac{1}{b})}m_{\lfloor r\rfloor,b}^{(\mathrm{vi_a})}+1_{(G<\{r\}<\frac{1}{b})}m_{\lfloor r\rfloor,b}^{(\mathrm{vi_c})}\right)+\sum_{a=\lfloor r\rfloor+1}^{\infty}\sum_{b=\lfloor R\rfloor+1}^{\infty}m_{a,b}^{(\mathrm{v})}$$

$$+\sum_{a=\lfloor r\rfloor+1}^{\infty}(1_{(\{R\}\ge\frac{1}{a})}m^{(\mathrm{iii})}_{a,\lfloor R\rfloor}+1_{(\{R\}\ge\frac{1}{a})}m^{(\mathrm{vi_a})}_{a,\lfloor R\rfloor}+1_{(F<\{R\}<\frac{1}{a})}m^{(\mathrm{vi_b})}_{a,\lfloor R\rfloor})+\sum_{a=\lfloor r\rfloor+1}^{\infty}\sum_{b=1}^{\lfloor R\rfloor-1}m^{(\mathrm{iii})}_{a,b},$$

where $M_{r,R}$ is the measure of the regions where $D_{n-2}>r$ and $D_n>R$ in $\Delta_{\lfloor r\rfloor,\lfloor R\rfloor}$.

Proof Let $a,b\ge 1$ be integers. We denote strips with constant a_n or a_{n+1} by

$$H_a=[0,1]\times\left[\frac{1}{a+1},\frac{1}{a}\right]\quad\text{and}\quad V_b=\left[\frac{1}{b+1},\frac{1}{b}\right]\times[0,1].$$

For $a<\lfloor r\rfloor$, the curve $v=f_{a,r}(t)$ is entirely inside the rectangle H_a and (depending on the position of the curve $v=g_{b,R}(t)$) we are either in case (i) or (vi) (see Figure 3 and Remark 2.2). If $a>\lfloor r\rfloor$ the curve $v=f_{a,r}(t)$ is entirely underneath H_a and we are in case (iii), (iv) or (v). For $a=\lfloor r\rfloor$ the curve $v=f_{a,r}(t)$ is partially inside and partially underneath $H_{\lfloor r\rfloor}$. In this strip, we can have each of the six cases.

Similarly, for $b<\lfloor R\rfloor$, the curve of $v=g_{b,R}(t)$ is entirely inside the rectangle V_b and (depending on the position of the curve $v=g_{b,R}(t)$) we are in case (iii) or (vi). For $b>\lfloor R\rfloor$ the curve $v=g_{b,R}(t)$ is left of V_b and we are in case (i), (ii) or (v) . For $b=\lfloor R\rfloor$, the curve $v=g_{b,R}(t)$ is partially inside and partially left of $V_{\lfloor R\rfloor}$ and we can have each of the six cases.

We use the strips $H_{\lfloor r\rfloor}$ and $V_{\lfloor R\rfloor}$ to divide Ω in nine rectangles. Each of the nine terms in the sum in the proposition gives the measure of the region where $D_{n-2}>r$ and $D_n>R$ on one of those rectangles. We work from left to right and from top to bottom. The results follow from (2.6), Remark 2.2, Theorem 4.1 and the above. For instance, the first rectangle is given by $\left[0,\frac{1}{\lfloor R+1\rfloor}\right)\times\left[\frac{1}{\lfloor r\rfloor},1\right)$ and we see that for every $\Delta_{a,b}$ in this rectangle we are in case (i).

Remark 5.1 All the infinite sums are just finite integrals, for example

$$\sum_{a=1}^{\lfloor r\rfloor-1}\sum_{b=\lfloor R\rfloor+1}^{\infty}m^{(\mathrm{i})}_{a,b}=\int_0^{\frac{1}{\lfloor R\rfloor+1}}\int_{f_{a,r}(t)}^{\frac{1}{a}}\frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}. \tag{5.1}$$

Example 5.1 In this example, we compute the asymptotic frequency that simultaneously $D_{n-2}>2.9$ and $D_n>3.6$ (see Figure 5 and Table 2). Also compare with Table 1 where some of the upper bounds for this case are listed.

Table 2 The probabilities that $D_{n-2}>2.9$ and $D_n>3.6$ in the various cases.

a_n	a_{n+1}	Case	asymptotic frequency
1	1	(vi$_\mathrm{a}$)	0.047
1	2	(vi$_\mathrm{a}$)	0.025
1	>2	(i)	0.106
2	1	(vi$_\mathrm{c}$)	0.025
2	2	(vi$_\mathrm{a}$)	0.013
2	3	(vi$_\mathrm{a}$)	0.090
2	>3	(i)	0.044
>2	1	(iii)	0.097
>2	2	(iii)	0.050
>2	3	(iii)	0.034
>2	>3	(v)	0.115

Summing over the cases yields that for almost all $x \in [0,1) \setminus \mathbb{Q}$ the asymptotic frequency that simultaneously $D_{n-2} > 2.9$ and $D_n > 3.6$ is 0.64.

We can also compute the conditional probability that M_{Tong} is the sharp bound. Given $D_{n-2} > 2.9$ and $D_n > 3.6$, the conditional probability that M_{Tong} is the sharp bound is 0.31.

6 Results for C_n

In [17], Tong states the following result as theorem without a proof.

Let $t > 1$, $T > 1$ be two real numbers and

$$K = \frac{1}{2}\Big(\frac{1}{t-1} + \frac{1}{T-1} + a_n a_{n+1} t\,T + \sqrt{\Big(\frac{1}{t-1} + \frac{1}{T-1} + a_n a_{n+1} tT\Big)^2 - \frac{4}{(t-1)(T-1)}}\,\Big).$$

Then

(1) $C_{n-2} < t,\ C_n < T$ *imply* $C_{n-1} > K$;

(2) $C_{n-2} > t,\ C_n > T$ *imply* $C_{n-1} < K$.

This statement is not correct; assume for instance that $C_{n-2} < 1.1$ and $C_n < 1.4$, and that $a_n = a_{n+1} = 1$. Part (1) of Tong's result then implies that $C_{n-1} > 11.94$. However, by definition $C_{n-1} \in (1,2)$, so this bound is clearly wrong.

In this section, we give the correct result. The bounds in our theorems are sharp. We start with the case that both C_{n-2} and C_n are larger than given reals, this is related to the case where D_{n-2} and D_n are smaller than given numbers.

Theorem 6.1 *Let $t, T \in (1,2)$ and put*

$$F' = \frac{a_{n+1}+1}{(a_n a_{n+1} + a_n + 1)t - 1}, \quad G' = \frac{a_n + 1}{(a_n a_{n+1} + a_{n+1} + 1)T - 1},$$
$$L' = t + T + a_n a_{n+1} tT - 2.$$

Assume $C_{n-2} > t$ and $C_n > T$. We have

(1) *if* $\frac{1}{t-1} - a_n \geq G'$ *and* $\frac{1}{T-1} - a_{n+1} < F'$, *then* $C_{n-1} < \frac{T}{(a_{n+1}+1)(T-1)}$,

(2) *if* $\frac{1}{t-1} - a_n < G'$ *and* $\frac{1}{T-1} - a_{n+1} \geq F'$, *then* $C_{n-1} < \frac{t}{(a_n+1)(t-1)}$,

(3) *in all other cases,* $C_{n-1} < 1 + \frac{L' - \sqrt{L'^2 - 4(t-1)(T-1)}}{2(t-1)(T-1)}$.

The bounds are sharp.

Proof The proof follows from the fact that $C_n = 1 + \frac{1}{D_n}$ and Theorem 1.2. If $C_{n-2} > t$, then $D_{n-2} = \frac{1}{C_{n-2}-1} < \frac{1}{t-1}$ and likewise if $C_n > T$, then $D_n < \frac{1}{T-1}$. Set $r = \frac{1}{t-1}$ and $R = \frac{1}{T-1}$. It directly follows from (2.9) that $F = F'$ and $G = G'$.

Consider case (1). The condition $\frac{1}{t-1} - a_n \geq G'$ is equivalent to $r - a_n \geq G$ and $\frac{1}{a_n+1} \leq \frac{1}{T-1}$ $-a_{n+1} < F'$ is equivalent to $\frac{1}{a_n+1} \leq R - a_{n+1} < F$ in part (1) of Theorem 1.2. We find that

$$C_{n-1} < \frac{\frac{1}{T-1} - a_{n+1}}{a_{n+1}+1} + 1 = \frac{T}{(a_{n+1}+1)(T-1)}.$$

The proof of the second case is similar. For the third case we use Theorem 1.1 for M_{Tong}.

$$\begin{aligned}
C_{n-1} &< 1 + \frac{1}{M_{\text{Tong}}} \\
&= 1 + \frac{2}{t + T + a_n a_{n+1} tT - 2 + \sqrt{[t + T + a_n a_{n+1} tT - 2]^2 - 4(t-1)(T-1)}} \\
&= 1 + \frac{2}{L' + \sqrt{L'^2 - 4(t-1)(T-1)}} \cdot \frac{L' - \sqrt{L'^2 - 4(t-1)(T-1)}}{L' - \sqrt{L'^2 - 4(t-1)(T-1)}} \\
&= 1 + \frac{L' - \sqrt{L'^2 - 4(t-1)(T-1)}}{2(t-1)(T-1)}.
\end{aligned}$$

Example 6.1 Take $t = 1.1$, $T = 1.4$ and $a_n = a_{n+1} = 1$. We find that $F' = 0.870$, $G' = 0.625$ and $L' = 2.04$. Since $\frac{1}{T-1} - a_{n+1} = \frac{3}{2} > F'$, the case (1) of Theorem 6.1 does not apply. The second case does not apply either, since $\frac{1}{t-1} - a_n = 9 > G'$. So we are in case (3) and $C_{n-1} < 1.50$.

We state the next theorem without a proof, since it is similar to that of Theorem 6.1. The only difference is that the proof is based on Theorem 4.1 instead of Theorem 1.2.

Theorem 6.2 *Let $t, T \in (1,2)$ and F', G' and L' be as defined in Theorem 6.1. Assume $C_{n-2} < t$ and $C_n < T$. We have*

(1) *if* $\frac{1}{t-1} - a_n \geq G'$ *and* $\frac{1}{T-1} - a_{n+1} < F'$, *then* $C_{n-1} > 1 + \frac{F'}{a_{n+1}+1}$,

(2) *if* $G' \leq \frac{1}{t-1} - a_n$ *and* $\frac{1}{T-1} - a_{n+1} < F'$, *then* $C_{n-1} > 1 + \frac{G'}{a_n+1}$,

(3) *if* $\frac{1}{t-1} - a_n < \frac{1}{a_{n+1}+1}$ *and* $\frac{1}{T-1} - a_{n+1} < \frac{1}{a_n+1}$, *then* $C_{n-1} > 1 + \frac{1}{(a_n+1)(a_{n+1}+1)}$,

(4) *in all other cases,* $C_{n-1} > 1 + \frac{L' - \sqrt{L'^2 - 4(t-1)(T-1)}}{2(t-1)(T-1)}$.

The bounds are sharp.

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