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Chen's Theorem with Small Primes*

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Abstract Let N be a sufficiently large even integer. Let p denote a prime and P_2 denote an almost prime with at most two prime factors. In this paper, it is proved that the equation $N = p + P_2$ $(p \le N^{0.945})$ is solvable.

Keywords Chen's Theorem, Sieve method, Mean value theorem
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1 Introduction

In 1966, Jingrun Chen [4] made great progress in the research of the binary Goldbach conjecture. In 1973, Jingrun Chen [5] proved what is now called the Chen's theorem: Let N be a sufficiently large even integer. Let p denote a prime and P_2 denote an almost prime with at most two prime factors. Then the equation

$$N = p + P_2 \tag{1.1}$$

is solvable. In fact, Chen's theorem can be expressed in a more precise form: Let $S(N)$ be the number of solutions to the equation (1.1). Then

$$S(N) \ge \frac{0.67C(N)N}{\log^2 N},$$

where

$$C(N) = \prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{\substack{p\mid N\\ p>2}}\frac{p-1}{p-2}.$$

Chen's constant 0.67 was improved by many authors. The historical record is as follows: 0.689 by Halberstam and Richert [9], 0.754, 0.81 by Chen [7, 8], 0.828 by Cai and Lu [2], 0.836 by Wu [14], and 0.867 by Cai [3].

Chen's theorem with a small prime p was studied in [1]: Let $S(N,\theta)$ be the number of solutions of the equation

$$N = p + P_2, \quad p \le N^{\theta}. \tag{1.2}$$

For $\theta = 0.95$, we have $S(N,\theta) > \frac{0.01C(N)N^{\theta}}{\log^2 N}$.

The aim of this paper is to propose a better result.

Theorem 1.1 *For $\theta = 0.945$, we have $S(N,\theta) > \frac{0.001C(N)N^{\theta}}{\log^2 N}$.*

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2 Some Lemmas

Let $\mathscr{A}$ denote a finite integral set and $\mathscr{P}$ denote an infinite set of primes. $\overline{\mathscr{P}}$ denotes the set of primes that do not belong to $\mathscr{P}$. Let $z \geq 2$, and put

$$P(z) = \prod_{p<z,\, p\in\mathscr{P}} p, \quad S(\mathscr{A};\mathscr{P},z) = \sum_{a\in\mathscr{A},(a,P(z))=1} 1,$$
$$\mathscr{A}_d = \{a \mid a\in\mathscr{A}, a\equiv 0 \ (\mathrm{mod}\ d)\}, \quad \mathscr{P}(q) = \{p \mid p\in\mathscr{P},(p,q)=1\}.$$

Lemma 2.1 (see [10]) *If*
(A$_1$) $|\mathscr{A}_d| = \frac{\omega(d)}{d}X + r_d,\ \mu(d)\neq 0,\ (d,\overline{\mathscr{P}})=1;$
(A$_2$) $\sum\limits_{z_1\leq p<z_2} \frac{\omega(p)}{p} = \log\frac{\log z_2}{\log z_1} + O\big(\frac{1}{\log z_1}\big), \quad 2\leq z_1<z_2,$
where $\omega(d)$ *is multiplicative with* $0\leq\omega(p)<p$. $X>1$ *is independent of* d. *Then*

$$S(\mathscr{A};\mathscr{P},z) \geq XV(z)\Big\{f(s) + O\Big(\frac{1}{\log^{\frac13} D}\Big)\Big\} - R_D,$$
$$S(\mathscr{A};\mathscr{P},z) \leq XV(z)\Big\{F(s) + O\Big(\frac{1}{\log^{\frac13} D}\Big)\Big\} + R_D,$$

where

$$C(\omega) = \prod_p \Big(1-\frac{\omega(p)}{p}\Big)\Big(1-\frac1p\Big)^{-1}, \quad R_D = \sum_{\substack{d<D\\ d\mid P(z)}} |r_d|,$$
$$V(z) = C(\omega)\frac{\mathrm{e}^{-\gamma}}{\log z}\Big(1+O\Big(\frac{1}{\log z}\Big)\Big), \quad s = \frac{\log D}{\log z}.$$

Here γ *denotes Euler constant.* $f(s)$ *and* $F(s)$ *are determined by the following differential-difference equations:*

$$\begin{cases} F(s) = \dfrac{2\mathrm{e}^\gamma}{s}, \quad f(s)=0, \quad 0<s\leq 2,\\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), \quad s\geq 2.\end{cases}$$

Lemma 2.2 (see [11]) *We have*

$$F(s) = \frac{2\mathrm{e}^\gamma}{s}, \quad 0<s\leq 3,$$
$$F(s) = \frac{2\mathrm{e}^\gamma}{s}\Big(1+\int_2^{s-1}\frac{\log(t-1)}{t}\mathrm{d}t\Big), \quad 3\leq s\leq 5,$$
$$f(s) = \frac{2\mathrm{e}^\gamma\log(s-1)}{s}, \quad 2\leq s\leq 4,$$
$$f(s) = \frac{2\mathrm{e}^\gamma}{s}\Big(\log(s-1)+\int_3^{s-1}\frac{\mathrm{d}t}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\mathrm{d}u\Big), \quad 4\leq s\leq 6.$$

Lemma 2.3 (see [11]) *For any given constant* $A>0$, *there exists a constant* $B=B(A)>0$, *such that*

$$\sum_{d\leq D}\max_{(l,d)=1}\max_{y\leq x}\Big|\sum_{\substack{p\leq y\\ p\equiv l \ (\mathrm{mod}\, d)}} 1 - \frac{\mathrm{Li}y}{\varphi(d)}\Big| \ll \frac{x}{\log^A x},$$

where $\mathrm{Li}x = \int_2^x \frac{\mathrm{d}t}{\log t}$, $D = x^{\frac12}\log^{-B} x$.

Lemma 2.4 (see [13]) *Let $g(n)$ be a number-theoretic function such that $\sum_{n\le x}\frac{g^2(n)}{n}\ll \log^c x$, where $c>0$. For $(al,q)=1$, define*

$$H(z,h,a,q,l)=\sum_{\substack{z\le ap\le z+h\\ ap\equiv l \pmod q}} 1-\frac{1}{\varphi(q)}\Big(\mathrm{Li}\Big(\frac{z+h}{a}\Big)-\mathrm{Li}\Big(\frac{z}{a}\Big)\Big).$$

Then for any constant $A>0$, there exists a constant $B=B(A,c)>0$, such that

$$\sum_{d\le D}\max_{(l,d)=1}\max_{h\le y}\max_{\frac{x}{2}\le z\le x}\Big|\sum_{\substack{a\le x^\beta\\ (a,d)=1}} g(a)H(z,h,a,d,l)\Big|\ll \frac{y}{\log^A x}$$

for $\frac{3}{5}<\theta\le 1,\ y=x^\theta,\ 0\le\beta<\frac{5\theta-3}{2},\ \lambda=\theta-\frac{1}{2},\ D=x^\lambda\log^{-B}x$.

Lemma 2.5 (see [12]) *Suppose that $\omega(u)$ is the solution to the following equations:*

$$\begin{cases}\omega(u)=\dfrac{1}{u}, & 1\le u\le 2,\\ (u\omega(u))'=\omega(u-1), & u>2.\end{cases}$$

Then we have $\omega(u)<\frac{1}{1.763},\ u\ge 2$.

Lemma 2.6 *Let $\omega(u)$ be defined in Lemma 2.5. Let $x>1,\ x^{\frac{19}{24}+\varepsilon}\le y\le\frac{x}{\log x},\ z=x^{\frac{1}{u}},\ P_1(z)=\prod_{p<z}p$. Then for any $u>1$, we have*

$$\sum_{\substack{x-y\le n\le x\\ (n,P_1(z))=1}} 1=\omega(u)\frac{y}{\log z}+O\Big(\frac{y}{\log^2 z}\Big). \tag{2.1}$$

Proof We will prove it by mathematical induction.

Firstly, when $1<u\le 2$, by Huxley's prime number theorem in shorter intervals and the definition of $\omega(u)$ in Lemma 2.5, we have

$$\sum_{\substack{x-y\le n\le x\\ (n,P_1(z))=1}} 1=\sum_{x-y\le p\le x} 1=\frac{y}{\log x}+O\Big(\frac{y}{\log^2 x}\Big)=\omega(u)\frac{y}{\log z}+O\Big(\frac{y}{\log^2 z}\Big).$$

So (2.1) holds for $1<u\le 2$.

Next, we assume that (2.1) is true for $k<u\le k+1$ (k being a natural number). When $k+1<u\le k+2$, let $\mathscr{P}_1$ be the set of all prime numbers and $\mathscr{N}=\{n: x-y\le n\le x\}$. Then we have

$$\sum_{\substack{x-y\le n\le x\\ (n,P_1(z))=1}} 1=S(\mathscr{N};\mathscr{P}_1,z).$$

If $k+1<u\le k+2$, we have

$$\begin{aligned}S(\mathscr{N};\mathscr{P}_1,x^{\frac{1}{u}})&=S(\mathscr{N};\mathscr{P}_1,x^{\frac{1}{k+1}})+\sum_{x^{\frac{1}{u}}\le p<x^{\frac{1}{k+1}}} S(\mathscr{N}_p;\mathscr{P}_1,p)\\ &=\sum_{\substack{x-y\le n\le x\\ (n,P_1(x^{\frac{1}{k+1}}))=1}} 1+\sum_{x^{\frac{1}{u}}\le p<x^{\frac{1}{k+1}}}\ \sum_{\substack{\frac{x-y}{p}\le n_1\le\frac{x}{p}\\ (n_1,P_1(p))=1}} 1.\end{aligned} \tag{2.2}$$

Since $p=\left(\frac{x}{p}\right)^{\frac{1}{\frac{\log \frac{x}{p}}{\log p}}}$ and $k<\frac{\log \frac{x}{p}}{\log p}=\frac{\log x}{\log p}-1\le k+1$, $\frac{y}{p}\ge\left(\frac{x}{p}\right)^{\frac{7}{12}+\varepsilon}$ for $x^{\frac{1}{u}}\le p<x^{\frac{1}{k+1}}$, by assumption, (2.1)–(2.2), the prime number theorem and the definition of $\omega(u)$, we get

$$\begin{aligned}S(\mathscr{N};\mathscr{P}_1,x^{\frac{1}{u}})&=(k+1)\omega(k+1)\frac{y}{\log x}+\int_{x^{\frac{1}{u}}}^{x^{\frac{1}{k+1}}}\omega\Big(\frac{\log x}{\log t}-1\Big)\frac{y}{t\log^2 t}\mathrm{d}t\\&\quad+O\Big(\int_{x^{\frac{1}{u}}}^{x^{\frac{1}{k+1}}}\frac{y}{t\log^3 t}\mathrm{d}t\Big)+O\Big(\frac{y}{(\log x^{\frac{1}{k+1}})^2}\Big)\\&=\omega(u)\frac{y}{\log x^{\frac{1}{u}}}+O\Big(\frac{y}{(\log x^{\frac{1}{u}})^2}\Big).\end{aligned}$$

Hence, (2.1) holds when $k+1<u\le k+2$.

By the principle of mathematical induction, (2.1) is true for all $u>1$. Thus the proof of Lemma 2.6 is completed.

3 Weighted Sieve Method

In the following two sections, we suppose that N is a sufficiently large even integer and $p,\ p_1,\ p_2,\ p_3,\ p_4$ denote primes. Put

$$\mathscr{A}=\{a\mid a=N-p,\ \ p\le N^{\theta}\},\quad \theta=0.945,\quad \mathscr{P}=\{p\mid (p,N)=1\}.$$

Then

$$X=\mathrm{Li}N^{\theta}\sim\frac{N^{\theta}}{\log N^{\theta}},\quad (d,N)=1,\quad D=\frac{N^{\frac{\theta}{2}}}{\log^{B}N},\quad B=B(5)>0,$$
$$r_d=\pi(N^{\theta};d,N)-\frac{\mathrm{Li}N^{\theta}}{\varphi(d)},\quad \omega(d)=\frac{d}{\varphi(d)},\quad \mu(d)\ne 0,\quad (d,N)=1.$$

Lemma 3.1 (see [5]) *We have*

$$S(N,\theta)>S-\frac{1}{2}S_1-\frac{1}{2}S_2-S_3+O(N^{\frac{9.95}{10.95}}),$$

where

$$S=\sum_{\substack{a\in\mathscr{A},(a,N)=1\\(a,P(N^{\frac{1}{10.95}}))=1}}1,\qquad S_1=\sum_{\substack{N^{\frac{1}{10.95}}\le p<N^{\frac{1}{3.3}}\\(p,N)=1}}S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{10.95}}),$$
$$S_2=\sum_{\substack{a\in\mathscr{A},(a,N)=1\\(a,P(N^{\frac{1}{10.95}}))=1}}\rho_2(a),\quad S_3=\sum_{\substack{a\in\mathscr{A},(a,N)=1\\(a,P(N^{\frac{1}{10.95}}))=1}}\rho_3(a),$$
$$\rho_2(a)=\begin{cases}1, & a=p_1p_2p_3,\ N^{\frac{1}{10.95}}\le p_1<N^{\frac{1}{3.3}}\le p_2<p_3,\ (a,N)=1,\\0, & \textit{otherwise},\end{cases}$$
$$\rho_3(a)=\begin{cases}1, & a=p_1p_2p_3,\ N^{\frac{1}{3.3}}\le p_1<p_2<p_3,\ (a,N)=1,\\0, & \textit{otherwise}.\end{cases}$$

Lemma 3.2 *For S_1, we have*

$$S_1 \le \sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \\ (p,N)=1}} S(\mathscr{A}_p; \mathscr{P}, N^{\frac{1}{10.95}}) + \sum_{\substack{N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} S\Big(\mathscr{A}_p; \mathscr{P}, \Big(\frac{D}{p}\Big)^{\frac{1}{2.5}}\Big)$$
$$= S_4 + S_5.$$

Proof

$$S_1 = \sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \\ (p,N)=1}} S(\mathscr{A}_p; \mathscr{P}, N^{\frac{1}{10.95}}) + \sum_{\substack{N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} S(\mathscr{A}_p; \mathscr{P}, N^{\frac{1}{10.95}})$$
$$\le S_4 + S_5.$$

Lemma 3.3 (see [6]) *We have*

$$S_5 \le S_6 - \frac{1}{2} S_7 + \frac{1}{2} S_8 + O(N^{0.9}), \tag{3.1}$$

where

$$S_6 = \sum_{\substack{N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} S\Big(\mathscr{A}_p; \mathscr{P}, \Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big),$$

$$S_7 = \sum_{\substack{N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\Big(\mathscr{A}_{pp_1}; \mathscr{P}, \Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big),$$

$$S_8 = \sum_{\substack{N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1 p_2 p_3,N)=1}} S(\mathscr{A}_{pp_1p_2p_3}; \mathscr{P}(p_2), p_3).$$

Proof By Buchstab's identity, we have

$$S\Big(A_p; \mathscr{P}, \Big(\frac{D}{p}\Big)^{\frac{1}{2.5}}\Big) = S\Big(A_p; \mathscr{P}, \Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big) - \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\Big(A_{pp_1}; \mathscr{P}, \Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big)$$
$$+ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1 p_2,N)=1}} S(A_{pp_1p_2}; \mathscr{P}, p_2), \tag{3.2}$$

$$S\Big(A_p; \mathscr{P}, \Big(\frac{D}{p}\Big)^{\frac{1}{2.5}}\Big) \le S\Big(A_p; \mathscr{P}, \Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big) - \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\Big(A_{pp_1}; \mathscr{P}(p_1), \Big(\frac{D}{p}\Big)^{\frac{1}{2.5}}\Big)$$
$$- \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < p_2 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1 p_2,N)=1}} S(A_{pp_1p_2}; \mathscr{P}(p_1), p_2) \tag{3.3}$$

and

$$\begin{aligned}
&\sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_2<p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\ (p_1p_2,N)=1}} S(A_{pp_1p_2};\mathscr{P},p_2)-\sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_1<p_2<(\frac{D}{p})^{\frac{1}{2.5}}\\ (p_1p_2,N)=1}} S(A_{pp_1p_2};\mathscr{P}(p_1),p_2)\\
=&\sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_2<p_3<p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\ (p_1p_2p_3,N)=1}} S(A_{pp_1p_2p_3};\mathscr{P}(p_2),p_3)\\
&+\sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_2<p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\ (p_1p_2,N)=1}} S(A_{pp_1p_2^2};\mathscr{P},p_2). \qquad (3.4)
\end{aligned}$$

Now adding (3.2) and (3.3), suming over p in the interval $[N^{\frac{\theta}{2}-\frac{2.5}{10.95}}, N^{\frac{1}{3.3}})$ and by (3.4), we get Lemma 3.3, where the trivial inequality

$$\begin{aligned}
&\sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}\le p<N^{\frac{1}{3.3}}\\ (p,N)=1}}\ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_2<p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\ (p_1p_2,N)=1}} S(A_{pp_1p_2^2};\mathscr{P},p_2)\\
\ll&\sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}\le p<N^{\frac{1}{3.3}}\\ (p,N)=1}}\ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_2<p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\ (p_1p_2,N)=1}} \Big(\frac{N^\theta}{pp_1p_2^2}+1\Big)\ll N^{0.9}
\end{aligned}$$

is used.

Hence, combining Lemmas 3.1–3.3, we get

$$S(N,\theta)>S-\frac{1}{2}S_4-\frac{1}{2}S_6+\frac{1}{4}S_7-\frac{1}{4}S_8-\frac{1}{2}S_2-S_3+O(N^{\frac{9.95}{10.95}}). \qquad (3.5)$$

4 Proof of the Theorem

4.1 Estimation of the lower bound of *S*

Suppose $D=\frac{N^{\frac{\theta}{2}}}{\log^B N}$ with $B=B(5)>0$. By Lemma 2.3, we have

$$R_D=\sum_{d\le D}\Big|\pi(N^\theta;d,N)-\frac{\mathrm{Li}N^\theta}{\varphi(d)}\Big|\le\sum_{d\le D}\max_{y\le N^\theta}\max_{(l,d)=1}\Big|\pi(y;d,l)-\frac{\mathrm{Li}y}{\varphi(d)}\Big|\ll\frac{N^\theta}{\log^5 N}. \qquad (4.1)$$

Since

$$C(\omega)=2\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{p|N,p>2}\Big(\frac{p-1}{p-2}\Big)=2C(N), \qquad (4.2)$$

by Lemmas 2.1–2.2, (4.1) and (4.2), we get

$$\begin{aligned}
S&\ge 8(1+o(1))\frac{C(N)N^\theta}{\theta^2\log^2 N}\Big(\log\Big(\frac{10.95\theta}{2}-1\Big)+\int_2^{\frac{10.95\theta}{2}-2}\frac{\log(s-1)}{s}\log\frac{\frac{10.95\theta}{2}-1}{s+1}\mathrm{d}s\Big)\\
&>12.9972\frac{C(N)N^\theta}{\log^2 N}. \qquad (4.3)
\end{aligned}$$

4.2 Estimation of the upper bounds of S_4 and S_6

Let $R_D(p) = \sum\limits_{d<\frac{D}{p}, d|P(N^{\frac{1}{10.95}})} |r_{dp}|$. By Lemma 2.3, we get

$$\sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \\ (p,N)=1}} R_D(p) \le \sum_{d\le D} \max_{y\le N^\theta} \max_{(l,d)=1} \left| \pi(y;d,l) - \frac{\mathrm{Li} y}{\varphi(d)} \right| \ll \frac{N^\theta}{\log^5 N}. \tag{4.4}$$

By Lemmas 2.1–2.2, (4.2), (4.4), the prime number theorem and partial integration, we have

$$\begin{aligned} S_4 &\le 21.9(1+o(1))\mathrm{e}^{-\gamma} \frac{C(N)N^\theta}{\theta \log^2 N} \sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \\ (p,N)=1}} \frac{1}{p} F\Big(\frac{10.95}{2}\theta - 10.95 \frac{\log p}{\log N}\Big) \\ &\le 21.9(1+o(1))\mathrm{e}^{-\gamma} \frac{C(N)N^\theta}{\theta \log^2 N} \int_{N^{\frac{1}{10.95}}}^{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}} \frac{1}{u\log u} F\Big(\frac{10.95}{2}\theta - 10.95\frac{\log u}{\log N}\Big)\mathrm{d}u \\ &\le 8(1+o(1)) \frac{C(N)N^\theta}{\theta^2 \log^2 N} \Big(\log\Big(\frac{(10.95\theta-2)(10.95\theta-5)}{10}\Big) \\ &\quad + \int_2^{\frac{10.95}{2}\theta - 2} \frac{\log(s-1)}{s} \log \frac{(\frac{10.95}{2}\theta-1)(\frac{10.95}{2}\theta-1-s)}{s+1} \mathrm{d}s \Big) \\ &\le 14.1914 \frac{C(N)N^\theta}{\log^2 N}. \end{aligned} \tag{4.5}$$

Similarly, we have

$$\begin{aligned} S_6 &\le 8(1+o(1)) \frac{C(N)N^\theta}{\theta^2 \log^2 N} \Big(\log\Big(\frac{10}{(3.3\theta-2)(10.95\theta-5)} \Big)\Big) \Big(1 + \int_2^{2.67} \frac{\log(x-1)}{x}\mathrm{d}x\Big) \\ &< 4.9577 \frac{C(N)N^\theta}{\log^2 N}. \end{aligned} \tag{4.6}$$

4.3 Estimation of the upper bounds of S_2 and S_3

Let $D_1 = N^\lambda \log^{-B} N$. Here λ and $B = B(5) > 0$ are determined by Lemma 2.4. By the method in [5] and Huxley's prime number theorem in shorter intervals, we get

$$\begin{aligned} S_2 &\le 4(1+o(1)) \frac{C(N)}{\log D_1} \sum_{N^{\frac{1}{10.95}} \le p_1 < N^{\frac{1}{3.3}} \le p_2 < (\frac{N}{p_1})^{\frac{1}{2}}} \sum_{N - N^\theta \le p_1 p_2 p_3 < N} 1 \\ &\le 8(1+o(1)) \frac{C(N)N^\theta}{(2\theta-1)\log^2 N} \int_{2.3}^{9.95} \frac{\log\big(2.3 - \frac{3.3}{t+1}\big)}{t}\mathrm{d}t \\ &< 6.9078 \frac{C(N)N^\theta}{\log^2 N}. \end{aligned} \tag{4.7}$$

Similarly, we have

$$S_3 \le 8(1+o(1)) \frac{C(N)N^\theta}{(2\theta-1)\log^2 N} \int_2^{2.3} \frac{\log(t-1)}{t}\mathrm{d}t < 0.1682 \frac{C(N)N^\theta}{\log^2 N}. \tag{4.8}$$

4.4 Estimation of the lower bound of S_7

Let $R_D(pp_1) = \sum\limits_{d<\frac{D}{pp_1}, d|P((\frac{D}{p})^{\frac{1}{3.67}})} |r_{dpp_1}|$. By Lemma 2.3, we have

$$\sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1,N)=1}} R_D(pp_1) \le \sum_{d\le D} \max_{y\le N^\theta} \max_{(l,d)=1} \left| \pi(y;d,l) - \frac{\operatorname{Li} y}{\varphi(d)} \right| \ll \frac{N^\theta}{\log^5 N}. \tag{4.9}$$

By Lemmas 2.1–2.2, (4.2), (4.9), the prime number theorem and partial integration, we obtain

$$\begin{aligned} S_7 &\ge 7.34(1+o(1))\mathrm{e}^{-\gamma} \frac{C(N)N^\theta}{\theta \log N} \\ &\quad \times \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1,N)=1}} \frac{1}{pp_1 \log \frac{D}{p}} f\Big(3.67 - 3.67 \frac{\log p_1}{\log \frac{D}{p}}\Big) \\ &\ge 8(1+o(1)) \frac{C(N)N^\theta}{\theta^2 \log^2 N} \Big(\log\Big(\frac{10}{(3.3\theta-2)(10.95\theta-5)}\Big)\Big) \int_{1.5}^{2.67} \frac{\log\big(2.67 - \frac{3.67}{x+1}\big)}{x} \mathrm{d}x \\ &> 0.9625 \frac{C(N)N^\theta}{\log^2 N}. \end{aligned} \tag{4.10}$$

4.5 Estimation of the upper bound of S_8

We set

$$E_1 = \max\Big(\frac{N-N^\theta}{\mathrm{e}}, \frac{D}{p_2^{3.67}}, N^{\frac{\theta}{2}-\frac{2.5}{10.95}}\Big), \quad E_2 = \min\Big(\frac{N}{\mathrm{e}}, \frac{D}{p_1^{2.5}}, N^{\frac{1}{3.3}}\Big),$$
$$E_3 = \frac{N-N^\theta}{p_1p_2p_3N^{\frac{1}{3.3}}}, \quad E_4 = \frac{N}{p_1p_2p_3N^{\frac{\theta}{2}-\frac{2.5}{10.95}}}, \quad E_5 = \Big(\frac{D}{N^{\frac{1}{3.3}}}\Big)^{\frac{1}{3.67}}, \quad E_6 = \Big(\frac{D}{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}}\Big)^{\frac{1}{2.5}}.$$

Then

$$\begin{aligned} S_8 &= \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2p_3,N)=1}} \sum_{\substack{a\in\mathscr{A}, pp_1p_2p_3|a \\ (a,\frac{N}{p_2}P(p_3))=1}} 1 + O(N^{\frac{9}{10}}) \\ &= \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2p_3,N)=1}} \sum_{\substack{p_4 = N - pp_1p_2p_3n \\ (n,\frac{N}{p_2}P(p_3))=1}} 1 + O(N^{\frac{9}{10}}) \\ &= S_8' + O(N^{\frac{9}{10}}), \end{aligned}$$

where

$$S_8' = \sum_{\substack{E_5 \le p_2 < p_3 < p_1 < E_6 \\ (p_1p_2p_3,N)=1}} \sum_{\substack{E_3 \le n \le E_4 \\ (n,\frac{N}{p_2}P(p_3))=1}} \sum_{\substack{p_4 = N - p(p_1p_2p_3n) \\ E_1 \le p < E_2 \\ (p,N)=1}} 1.$$

Now we consider

$$\begin{aligned}\mathscr{E} = \Big\{ & e : e = p_1p_2p_3n, E_5 \le p_2 < p_3 < p_1 < E_6,\ (p_1p_2p_3, N) = 1, \\ & E_3 \le n \le E_4,\ \Big(n, \frac{N}{p_2}P\Big(p_3\Big)\Big) = 1\Big\}, \\ \mathscr{L} = \{ & l : l = N - ep, e \in \mathscr{E}, E_1 \le p < E_2\}.\end{aligned}$$

Obviously, $(\mathscr{E}, N) = 1$. Since

$$N^{\frac{1}{2}} < e < N^{0.76},\ e \in \mathscr{E}; \quad |\mathscr{E}| < \sum_{E_5 \le p_2 < p_3 < p_1 < E_6} \frac{N}{p_1p_2p_3N^{\frac{\theta}{2} - \frac{2.5}{10.95}}} \ll N^{0.76},$$

the number of elements not exceeding $N^{\frac{1}{2}}$ in $\mathscr{L} \ll N^{0.76}$. S_8' does not exceed the number of primes in $\mathscr{L}$, hence

$$S_8 \le S(\mathscr{L}; \mathscr{P}, z) + O(N^{\frac{9}{10}}), \quad z \le N^{\frac{1}{2}}. \tag{4.11}$$

Thus we can choose

$$\begin{aligned}X_1 = \sum_{e \in \mathscr{E}} \sum_{E_1 \le p < E_2} 1 &= \sum_{N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 \le p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2p_3, N) = 1}} \sum_{\substack{\frac{N - N^\theta}{pp_1p_2p_3} \le n \le \frac{N}{pp_1p_2p_3} \\ (n, \frac{N}{p_2}P(p_3)) = 1}} 1 \\ &\le X + O(N^{\frac{9}{10}}),\end{aligned} \tag{4.12}$$

where

$$X = \sum_{N^{\frac{\theta}{2} - \frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}}} \sum_{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 \le p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}}} \sum_{\substack{\frac{N - N^\theta}{pp_1p_2p_3} \le n \le \frac{N}{pp_1p_2p_3} \\ (n, NP(p_3)) = 1}} 1.$$

Let $z^2 = D_1 = N^\lambda \log^{-B} N$. Here λ and $B = B(5) > 0$ are determined by Lemma 2.4. Set $g(a) = \sum_{\substack{e = a \\ e \in \mathscr{E}}} 1$. By Lemma 2.4, we have

$$\begin{aligned}R_{D_1} &= \sum_{\substack{d \le D_1 \\ d | P(D_1^{0.5})}} \Big| \sum_{e \in \mathscr{E}} \Big(\sum_{\substack{E_1 \le p < E_2 \\ ep \equiv N(d)}} 1 - \frac{1}{\varphi(d)} \sum_{E_1 \le p < E_2} 1 \Big) \Big| \\ &\le \sum_{d \le D_1} \max_{(l,d)=1} \max_{h \le N^\theta} \max_{\frac{N}{2} \le z \le N} \Big| \sum_{\substack{a \le N^\beta \\ (a,d)=1}} g(a) H(z, h, a, d, l) \Big| \ll \frac{N^\theta}{\log^5 N}.\end{aligned} \tag{4.13}$$

Hence, by (4.13) and Lemmas 2.1–2.2, we get

$$S(\mathscr{L}; \mathscr{P}, D_1^{0.5}) \le 8(1 + o(1))C(N) \frac{X_1}{(2\theta - 1)\log N} + O\Big(\frac{N^\theta}{\log^5 N}\Big). \tag{4.14}$$

Combining (4.11)–(4.12) and (4.14), we obtain

$$S_8 \le 8(1 + o(1))C(N) \frac{X}{(2\theta - 1)\log N} + O\Big(\frac{N^\theta}{\log^5 N}\Big). \tag{4.15}$$

Since

$$\frac{\log \frac{N}{pp_1p_2p_3}}{\log p_3} > 4, \quad \Big(\frac{N}{pp_1p_2p_3}\Big)^{\frac{19}{24}+\varepsilon} < \frac{N^\theta}{pp_1p_2p_3} < \frac{N}{pp_1p_2p_3},$$

by Lemma 2.6, Lemma 2.5, the prime number theorem and partial integration, we get

$$\begin{aligned} X &\le (1+o(1)) \sum_{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}}} \sum_{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 \le p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}}} \omega\Big(\frac{\log \frac{N}{pp_1p_2p_3}}{\log p_3}\Big) \frac{\frac{N^\theta}{pp_1p_2p_3}}{\log p_3} \\ &< \frac{N^\theta}{1.763}(1+o(1)) \sum_{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}}} \frac{1}{p} \int_{(\frac{D}{p})^{\frac{1}{3.67}}}^{(\frac{D}{p})^{\frac{1}{2.5}}} \frac{\mathrm{d}u}{u\log u} \int_u^{(\frac{D}{p})^{\frac{1}{2.5}}} \frac{\mathrm{d}s}{s\log^2 s} \int_s^{(\frac{D}{p})^{\frac{1}{2.5}}} \frac{\mathrm{d}t}{t\log t} \\ &= \frac{2}{1.763}(1+o(1)) \frac{N^\theta}{\theta \log N} (6.17\log 1.468 - 2.34) \log \frac{10}{(3.3\theta-2)(10.95\theta-5)}. \end{aligned}$$

This, together with (4.15), gives

$$S_8 < 0.159 \frac{C(N)N^\theta}{\log^2 N}. \tag{4.16}$$

4.6 Proof of Theorem 1.1

By (3.5), (4.3), (4.5)–(4.8), (4.10) and (4.16), we obtain

$$\begin{aligned} S(N,\theta) &> \Big(12.9972 - \frac{14.1914}{2} - \frac{4.9577}{2} + \frac{0.9625}{4} - \frac{0.159}{4} - \frac{6.9078}{2} - 0.1682\Big) \frac{C(N)N^\theta}{\log^2 N} \\ &= 0.001425 \frac{C(N)N^\theta}{\log^2 N}. \end{aligned}$$

This completes the proof of Theorem 1.1.

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