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Harnack Estimates for Weak Solutions to a Singular Parabolic Equation*

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Abstract By an interpolation method, an intrinsic Harnack estimate and some sup-estimates are established for nonnegative solutions to a general singular parabolic equation.

Keywords Harnack estimate, Interpolation method, Nonnegative solution, Singular parabolic equation
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1 Introduction and Main Results

It is well-known that the Harnack estimate for nonnegative solutions to the heat equation is an important behavior of solution. The first devotion to this problem is due to Hadamard [1] and Pini [2]. After their pioneering work, one considered the following quasilinear parabolic equation:

$$u_t - \mathrm{div}\mathbf{a}(x,t,u,Du) = \mathbf{b}(x,t,u,Du), \tag{1.1}$$

where $p > 1$, $u \in V^{1,p}(\Omega_T) \equiv L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$, and there are nonnegative functions $\varphi_i(x,t)$ and positive constants C_i $(i = 1,2,3)$ such that

$$\mathbf{a}(x,t,u,Du) \cdot Du \geq C_0|Du|^p - \varphi_0(x,t), \tag{1.2}$$
$$|\mathbf{a}(x,t,u,Du)| \leq C_1|Du|^{p-1} + \varphi_1(x,t), \tag{1.3}$$
$$|b(x,t,u,Du)| \leq C_2|Du|^p + \varphi_2(x,t). \tag{1.4}$$

A typical example of (1.1) which satisfies (1.2)–(1.4) is the well-known evolutionary p-Laplacian equation

$$u_t = \mathrm{div}(|Du|^{p-2}Du). \tag{1.5}$$

For this equation, Moser [3] showed that when $p = 2$ the same conclusion as [1] is true. But when $p \neq 2$, the same conclusion fails to hold (see [4–6]). An intrinsic version takes place. The intrinsic Harnack type inequalities were shown for the degenerate case ($p > 2$) by DiBenedetto [7]. The similar estimates were obtained in the singular case ($1 < p < 2$) by DiBenedetto and Kwong [8]. One can refer to [9] for more details.

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In this paper, we suppose that

$$\mathbf{a}(x,t,u,Du)\cdot Du \geq C_0 u^{(m-1)(p-1)}|Du|^p - \varphi_0(x,t), \tag{1.6}$$

$$|\mathbf{a}(x,t,u,Du)| \leq C_1 u^{(m-1)(p-1)}|Du|^{p-1} + \varphi_1(x,t), \tag{1.7}$$

$$|\mathbf{b}(x,t,u,Du)| \leq u^{(m-1)(p-1)} C_2 |Du|^{p-1} + \varphi_2(x,t). \tag{1.8}$$

A typical example of (1.1) which satisfies (1.6)–(1.8) is the following quasilinear parabolic equation:

$$u_t = \operatorname{div}(|Du^m|^{p-2}Du^m), \tag{1.9}$$

which includes the evolutionary p-Laplacian equation (1.5). Zhao and Xu [10] generalized the result in [6] to (1.9) $(m(p-1)>1)$. Yang [11] generalized Dibenedetto and the result in [7] to (1.9) $\big(\frac{m(N-1)}{mN+1} < m(p-1) < 1\big)$.

When $m=1$, comparing (1.6)–(1.9) with (1.2)–(1.4), we find that there is a difference between (1.4) and (1.8). But this difference is not essential, and we are only for simplification of the calculus to assume the form of (1.8).

In this paper, we will synthesize the methods used in [7–11], and try to get the same kinds of intrinsic Harnack inequalities for the most general cases, such as the equation (1.1) with (1.6)–(1.8). In comparison with [7–11], the greatest difficulty comes from the situation in which how the functions φ_0, φ_1 and φ_2 affect the main gradient terms $\mathbf{a}(x,t,u,Du)$ and $\mathbf{b}(x,t,u,Du)$. Now, we introduce the following definition.

Definition 1.1 *A nonnegative function $u(x,t)$ is called the locally weak solution (sub-solution, super-solution) to (1.1) in Ω_T if $u \in C_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(\Omega))$, $u^m \in L^p(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$, and for every compact subset E of Ω, $\forall [t_1,t_2]\subset(0,T]$,*

$$\int_E u\varphi \mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_E(-u\varphi_t + \mathbf{a}(x,t,u,Du)\cdot D\varphi - \mathbf{b}(x,t,u,Du)\varphi)\mathrm{d}x\mathrm{d}t = (\leq,\geq)\,0, \tag{1.10}$$

where $\varphi \in W^{1,2}_{\mathrm{loc}}(0,T;L^2(E)) \cap (0,T;W^{1,p}_0(E)) \cap L^\infty_{\mathrm{loc}}(\Omega)$, $\varphi \geq 0$.

In what follows, let u be a locally bounded and nonnegative local weak solution to (1.1) in Ω_T, $b = 1-m(p-1)>0$ and $\lambda_r = rp - Nb$, $\lambda=\lambda_1$. The main results are as follows.

Theorem 1.1 (L^∞_{loc}-Estimate) *Let $\lambda_r>0$, $r\geq 1$ and $q > 2+\frac{2N}{p}$. Suppose that*

$$\|\varphi_0\|_{L^{\frac{q}{2}}} + \|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q} < \infty \tag{1.11}$$

and

$$(2N+p)C_0 > p(p-1)C_1 + (p-1)C_2. \tag{1.12}$$

Then there exists a constant $\gamma=\gamma(N,m,p,r)$, such that $\forall\,(x_0,t_0)\in\Omega_T$ and $\forall\,\rho>0$ such that $B_{4\rho}(x_0)\subset\Omega$, $\forall\,t>t_0$, we have

$$\sup_{x\in B_\rho(x_0)} u(x) \leq \gamma(t-t_0)^{-\frac{N}{\lambda_r}}\Big(\sup_{t_0\leq\tau\leq t}\int_{B_{2\rho}(x_0)} u^r(x,\tau)\mathrm{d}x\Big)^{\frac{p}{\lambda_r}} + \gamma\Big(\frac{t-t_0}{\rho^p}\Big)^{\frac{1}{b}} + \gamma(\|\varphi_0\|_{L^{\frac{q}{2}}} + \|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q}). \tag{1.13}$$

Theorem 1.2 (Integral Harnack Inequality) *Suppose that*

$$\frac{C_0 b}{p} - (p-1)C_1 - \frac{(p-1)C_2}{p} > 0, \tag{1.14}$$

and that when $m \geq 1$,

$$\varphi_0 \leq t^{\frac{1}{b}} \varphi_{01}, \quad \varphi_{01} \in L^{\frac{q}{2}};$$

when $0 < m < 1$,

$$\varphi_0(x,t) \equiv 0.$$

There exists a constant $\gamma = \gamma(N,m,p,r)$, *such that* $\forall\,(x_0,t_0) \in \Omega_T$ *and* $\forall\,\rho > 0$, *such that* $B_{4\rho}(x_0) \subset \Omega$, $\forall\,t > t_0$, *we have*

$$\sup_{t_0 \leq \tau \leq t} \int_{B_\rho(x_0)} u(x,\tau)\mathrm{d}x \leq \gamma \inf_{t_0 \leq \tau \leq t} \int_{B_{2\rho}(x_0)} u(x,\tau)\mathrm{d}x + \gamma\Big[\|\varphi_{01}\|_{\frac{q}{2}} (t-t_0)^{\frac{q-2}{q}} \rho^{\frac{p}{b}+N\frac{q-2}{q}} + (t-t_0)^{\frac{q-1}{q}} \rho^{N\frac{q-2}{q}} (\|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q}) + \Big(\frac{t-t_0}{\rho^\lambda}\Big)^{\frac{1}{b}}\Big]. \tag{1.15}$$

Theorem 1.3 (Integral Harnack Inequality) *If the constant* $r > 1$ *satisfies*

$$(p-1)C_1 + \frac{(p-1)C_2}{p} < (r-1)C_0 \tag{1.16}$$

and

$$\|\varphi_0\|_{\frac{r}{2}} + \|\varphi_1\|_r + \|\varphi_2\|_r < \infty, \tag{1.17}$$

then

$$\sup_{t_0 \leq \tau \leq t} \int_{B_\rho(x_0)} u^r(x,\tau)\mathrm{d}x \leq \gamma(N,m,p,r,\delta)(\|\varphi_0\|_{\frac{r}{2}} + \|\varphi_1\|_r + \|\varphi_2\|_r)^{\frac{r}{b}} + \gamma\Big\{\int_{B_{2\rho}(x_0)} u^r(x,t_0)\mathrm{d}x + \gamma\Big(\frac{(t-t_0)^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{b}}\Big\}, \quad \forall\,r > 1, \tag{1.18}$$

where $\|f\|_r = \|f\|_{L^r}$.

Comparing Theorem 1.2 with Theorem 1.3, we easily find that there is a gap of the results between 1 and $r_0 = (p-1)C_1 + \frac{(p-1)C_2}{p} + C_0$. In other words, if $r \in (1, r_0]$, the corresponding integral Harnack inequality of $\sup\limits_{t_0 \leq \tau \leq t} \int_{B_\rho(x_0)} u^r(x,\tau)\mathrm{d}x$ is still unknown. However, combining Theorems 1.1 and 1.2 yields the following results.

Corollary 1.1 (L^∞-Estimate at the Same Time Level) *Let* $\lambda > 0$, $\varphi_0 \equiv 0$ *and* (1.14) *be true. There exists a constant* $\gamma = \gamma(N,m,p,r)$, *such that*

(1) $\forall\,0 < t < \infty$, $\forall\,\rho > 0$ *such that* $B_{4\rho} \equiv \{|x| < 4\rho\} \in \Omega$, *and we have*

$$\sup_{x \in B_\rho(x_0)} u(x) \leq \gamma t^{-\frac{N}{\lambda}} \Big(\sup_{t_0 \leq \tau \leq t} \int_{B_{2\rho}(x_0)} u(x,\tau)\mathrm{d}x\Big)^{\frac{p}{\lambda}} + \gamma\Big(\frac{t}{\rho^p}\Big)^{\frac{1}{b}} + \gamma[(\|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q})t\rho^N]^{\frac{p(q-1)}{q\lambda}} + \gamma\|\varphi_0\|_{L^{\frac{q}{2}}} + \|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q}. \tag{1.19}$$

(2) $\forall\,t_0 + 2 \leq \tau \leq t_0 + 4$,

$$\|u(\,\cdot\,,\tau)\|_{\infty,B_\rho} \leq \Big(\sup_{t_0 \leq \tau \leq t_0+4} \int_{B_{2\rho}(x_0)} u(x,\tau)\mathrm{d}x\Big)^{\frac{p}{\lambda}} + \gamma\Big(\frac{1}{\rho^p}\Big)^{\frac{1}{b}} + \gamma[(\|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q})\rho^N]^{\frac{p(q-1)}{q\lambda}} + \gamma\|\varphi_0\|_{L^{\frac{q}{2}}} + \|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q}. \tag{1.20}$$

Theorem 1.4 (Intrinsic Harnack Inequality) *Let $\lambda > 0$, $(x_0,t_0)\in\Omega_T$ and $u(x_0,t_0)>0$. Suppose that*

$$\mathbf{a}(x,t,u,Du) = |Du^m|^{p-2}Du^m, \quad 0\le \mathbf{b}(x,t,u,Du). \tag{1.21}$$

Moreover, suppose that u_t is a regular measure on Ω_T of the locally bounded variation. Then there exist constants $\gamma>1$, $c\in(0,1)$, depending only on N,m,p, such that

$$u(x_0,t_0)\le \gamma \inf_{B_\rho(x_0)} u(\,\cdot\,,t_0+\theta), \tag{1.22}$$

where $\theta = cu^b(x_0,t_0)\rho^p$, provided the cylinder

$$Q_{4\rho}(\theta)\equiv\{|x-x_0|<4\rho\}\times\{t_0-4\theta,t_0+4\theta\}\in\Omega_T. \tag{1.23}$$

2 Proof of Theorem 1.1

Lemma 2.1 (see [9]) *Let $\{Y_n\}$, $n=1,2,\cdots$, be a sequence of equibounded positive numbers satisfying the recursive inequalities*

$$Y_n \le Cb^nY_{n+1}^{1-\alpha},$$

where $C,b>1$ and $\alpha\in(0,1)$ are given constants. Then

$$Y_0\le\Big(\frac{2C}{b^{1-\frac{1}{\alpha}}}\Big)^{\frac{1}{\alpha}}.$$

Without loss of the generality, we can assume that (x_0,t_0) coincides with the origin. Let $\sigma\in(0,1]$ be fixed. Set $\rho_n=\rho(1+\sigma 2^{-n})$, $t_n = t\frac{1-\sigma 2^{-n}}{2}$, $\overline{\rho}_n=\frac{\rho_n+\rho_{n+1}}{2}$, $\overline{t}_n=\frac{t_n+t_{n+1}}{2}$, $B_n = B_{\rho_n}$, $\overline{B}_n = B_{\overline{\rho}_n}$, $Q_n = B_n\times(t_n,t)$, $\overline{Q}_n=\overline{B}_n\times(\overline{t}_n,t)$, $k_n = k(1-\frac{1}{z^{n+1}})$, $\overline{k}_n = (k_n+k_{n+1})$, $n=0,1,2,\cdots$, and $k>0$ is a constant to be chosen later. Let $\zeta_n(x,t)$ ($\overline{\zeta}_n(x,t)$) be a nonnegative piecewise smooth cutoff function in Q_n ($\overline{Q}_n$), which equals 1 in Q_{n+1}, vanishes on the parabolic boundary of $\overline{Q}_n$ respectively, and satisfies $|D\zeta_n|, |D\overline{\zeta}_n|\le \frac{2^{n+2}}{\sigma\rho}$, $0\le\zeta_{n,t},\overline{\zeta}_{n,t}\le\frac{2^{n+3}}{\sigma t}$.

If $m\ge1$, we choose the test function $\varphi(x,t)=(u-\overline{k}_n)_+^{q-1}\overline{\zeta}_n^p$,

$$\iint_{\overline{Q}_n}\frac{\partial u}{\partial t}(u-\overline{k}_n)_+^{q-1}\overline{\zeta}_n^p\mathrm{d}x\mathrm{d}t+\iint_{\overline{Q}_n}\mathbf{a}(x,t,u,Du)D[(u-\overline{k}_n)_+^{q-1}\overline{\zeta}_n^p]\mathrm{d}x\mathrm{d}t$$
$$-\iint_{\overline{Q}_n}\mathbf{b}(x,t,u,Du)(u-\overline{k}_n)_+^{q-1}\overline{\zeta}_n^p\mathrm{d}x\mathrm{d}t=0. \tag{2.1}$$

By the constructive conditions (1.6)–(1.8), integrating by parts, and using the Hölder inequality, we find that there exists a constant $\gamma=\gamma(N,m,p,r)$, such that

$$\sup_{t_n\le\tau\le t}\int_{\overline{B}_n(r)}(u-\overline{k}_n)_+^q\overline{\zeta}_n^p\mathrm{d}x+\iint_{\overline{Q}_n}u^{(m-1)(p-1)}(u-\overline{k}_n)_+^{q-2}|Du|^p\overline{\zeta}_n^p\mathrm{d}x\mathrm{d}\tau$$
$$\le\gamma\frac{2^{n+3}}{\sigma t}\iint_{\overline{Q}_n}(u-\overline{k}_n)_+^q\mathrm{d}x\mathrm{d}\tau+\gamma\Big(\frac{2^{p(n+1)}}{(\sigma\rho)^p}+1\Big)\iint_{\overline{Q}_n}u^{(m-1)(p-1)}(u-\overline{k}_n)_+^{p+q-2}\mathrm{d}x\mathrm{d}\tau$$
$$+\gamma\Big[\Big(\iint_{\overline{Q}_n}\varphi_0^{\frac{q}{2}}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{2}{q}}\Big(\iint_{\overline{Q}_n}(u-\overline{k}_n)_+^q\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{q-2}{q}}$$

$$+\frac{2^{n+2}}{\rho\sigma}\Big(\iint_{\overline{Q}_n}\varphi_1^q\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{q}}\Big(\iint_{\overline{Q}_n}(u-\overline{k}_n)_+^q\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{q-1}{q}}$$
$$+\Big(\iint_{\overline{Q}_n}\varphi_2^q\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{q}}\Big(\iint_{\overline{Q}_n}(u-\overline{k}_n)_+^q\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{q-1}{q}}\Big]. \tag{2.2}$$

Let $w_n=(u-\overline{k}_n)_+^{\frac{q-b}{p}}$ and $\overline{A}_n\equiv\{(x,t)\in Q_n\mid u(x,t)>\overline{k}_n\}$. Clearly,

$$\iint_{\overline{Q}_n}w_n^p\mathrm{d}x\mathrm{d}\tau\geq 2^{-(n+2)(q-b)}k^{q-b}|\overline{A}_n|. \tag{2.3}$$

At the same time, we have

$$\begin{aligned}
&\int_{\overline{B}_n(r)}(u-\overline{k}_n)_+^q\overline{\zeta}_n^p\mathrm{d}x\\
&\geq\int_{\overline{B}_n(\tau)\cap\{u>k_{n+1}\}}(u-\overline{k}_{n+1})_+^{q-b}(u-\overline{k}_n)_+^b\overline{\zeta}_n^p\mathrm{d}x\\
&\geq\Big(\frac{k}{2^{n+3}}\Big)^b\int_{\overline{B}_n(\tau)}(w_{n+1}\overline{\zeta}_n)^p\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
&\iint_{\overline{Q}_n}u^{(m-1)(p-1)}(u-\overline{k}_n)_+^{q-2}|Du|^p\overline{\zeta}_n^p\mathrm{d}x\mathrm{d}\tau\\
&\geq\frac{2^{p-1}(q-b)}{p}\Big[\iint_{\overline{Q}_n}|D(w_{n+1}\overline{\zeta}_n)|^p\mathrm{d}x\mathrm{d}\tau-\frac{2^{(n+2)p}}{(\sigma\rho)^p}\iint_{\overline{Q}_n}w_n^p\mathrm{d}x\mathrm{d}\tau\Big],
\end{aligned}$$

$$\begin{aligned}
&\iint_{\overline{Q}_n}u^{(m-1)(p-1)}(u-\overline{k}_n)_+^{p+q-2}\mathrm{d}x\mathrm{d}\tau\\
&\leq\iint_{\overline{Q}_n}w_n^p\mathrm{d}x\mathrm{d}\tau+k^{(m-1)(p-1)}\Big(\iint_{\overline{Q}_n}w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p+q-2}{q-b}}\overline{A}_n^{\frac{(m-1)(p-1)}{q-b}}\\
&\leq\gamma 2^{(m-1)(p-1)n}\iint_{\overline{Q}_n}w_n^p\mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}
&\iint_{\overline{Q}_n}(u-\overline{k}_n)_+^q\mathrm{d}x\mathrm{d}\tau\\
&\leq\iint_{\overline{Q}_n}(u-\overline{k}_n)_+^{q-b}(u-\overline{k}_n)_+^b\mathrm{d}x\mathrm{d}\tau\\
&\leq\|u\|_{\infty,Q_0}^b\iint_{\overline{Q}_n}u^{(m-1)(p-1)}(u-\overline{k}_n)_+^{p+q-2}\mathrm{d}x\mathrm{d}\tau\\
&\leq\gamma\|u\|_{\infty,Q_0}^b2^{(m-1)(p-1)n}\iint_{\overline{Q}_n}w_n^p\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Hence from (2.2) and (2.3), we have

$$\begin{aligned}
&\Big(\frac{k}{2^n}\Big)^b\sup_{t_n\leq\tau\leq t}\int_{\overline{B}_n(r)}(w_{n+1}\overline{\zeta}_n)^p\mathrm{d}x+\iint_{\overline{Q}_n}|D(w_{n+1}\overline{\zeta}_n)|^p\mathrm{d}x\mathrm{d}\tau\\
&\leq\frac{\gamma 2^{\alpha n}}{\sigma^p t}\Big(\|u\|_{\infty,Q_0}^b+\frac{t}{\rho^p}\Big)\iint_{Q_n}w_n^p\mathrm{d}x\mathrm{d}\tau+\gamma 2^{\alpha n}\Big[\|u\|_{\infty,Q_0}^{b(1-\frac{2}{q})}\|\varphi_0\|_{L^{\frac{q}{2}}}\Big(\iint_{Q_n}w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{2}{q}}\\
&\quad+\|u\|_{\infty,Q_0}^{b(1-\frac{1}{q})}\Big(\frac{1}{\rho}\|\varphi_1\|_{L^q}+\|\varphi_2\|_{L^q}\Big)\Big(\iint_{Q_n}w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\Big],
\end{aligned} \tag{2.4}$$

where $\alpha = \alpha(m, p, b, q)$.

If $\|u\|^b_{\infty,Q_0} \le \frac{t}{\rho^p}$ or $\|u\|^b_{\infty,Q_0} \le 1$, there is nothing to prove. Otherwise, $\|u\|_{\infty,Q_0} > 1$, $\|u\|^b_{\infty,Q_0} > \frac{t}{\rho^p} = \frac{t}{\rho}\rho^{-(p-1)}$, so

$$\frac{t}{\rho} \le \rho^{(p-1)}\|u\|^b_{\infty,Q_0} \le (\mathrm{diam}\Omega)^{(p-1)}\|u\|^b_{\infty,Q_0},$$
$$t \le (\mathrm{diam}\Omega)^p\|u\|^b_{\infty,Q_0}.$$

We can rewrite (2.4) as

$$\begin{aligned}&\Big(\frac{k}{2^n}\Big)^b \sup_{t_n\le\tau\le t}\int_{\overline{B}_n(r)}(w_{n+1}\overline{\zeta}_n)^p\mathrm{d}x + \iint_{\overline{Q}_n}|D(w_{n+1}\overline{\zeta}_n)|^p\mathrm{d}x\mathrm{d}\tau\\ &\le \gamma 2^{\alpha n}\Big[\frac{1}{\sigma^p t}\|u\|^b_{\infty,Q_0}\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau + \|\varphi_0\|_{L^{\frac{q}{2}}}\Big(\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{2}{q}}\\ &\quad + \Big(\frac{1}{\rho}\|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q}\Big)\Big(\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\Big].\end{aligned} \tag{2.5}$$

By the Hölder inequality and the space-time version of the Nirenberg-Gagliardo multiplicative inequality (see [11, p. 74]) and denoting $h = p(1 + \frac{p}{N})$, we get

$$\begin{aligned}&\iint_{\overline{Q}_n}(w_{n+1}\overline{\zeta}_n)^p\mathrm{d}x\mathrm{d}\tau\\ &\le \Big(\iint_{\overline{Q}_n}(w_{n+1}\overline{\zeta}_n)^h\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p}{h}}|\overline{A}_n|^{1-\frac{p}{h}}\\ &\le \gamma\Big\{\iint_{\overline{Q}_n}|D(w_{n+1}\overline{\zeta}_n)|^p\mathrm{d}x\mathrm{d}\tau\Big(\sup_{t_n\le\tau\le t}\int_{\overline{B}_n(r)}(w_{n+1}\overline{\zeta}_n)^p\mathrm{d}x\Big)^{\frac{p}{N}}\Big\}^{\frac{p}{h}}|\overline{A}_n|^{1-\frac{p}{h}}\\ &\le \gamma\Big\{\Big(\frac{k}{2^n}\Big)^b\sup_{t_n\le\tau\le t}\int_{\overline{B}_n(r)}(w_{n+1}\overline{\zeta}_n)^p\mathrm{d}x + \iint_{\overline{Q}_n}|D(w_{n+1}\overline{\zeta}_n)|^p\mathrm{d}x\mathrm{d}\tau\Big\}\Big(\frac{2^n}{k}\Big)^{\frac{pb}{N+p}}|\overline{A}_n|^{1-\frac{p}{h}}\\ &\le \gamma 2^{\alpha n}\|u\|^b_{\infty,Q_0}\Big[\frac{1}{\sigma^p t}\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau + \|\varphi_0\|_{L^{\frac{q}{2}}}\Big(\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{2}{q}}\\ &\quad + \Big(\frac{1}{\rho}\|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q}\Big)\Big(\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\Big]\Big(\frac{2^n}{k}\Big)^{\frac{pb}{N+p}}|\overline{A}_n|^{1-\frac{p}{h}}\\ &\le \gamma 2^{\alpha n}\|u\|^b_{\infty,Q_0}\Big[\frac{1}{\sigma^p t}\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau + \|\varphi_0\|_{L^{\frac{q}{2}}}\Big(\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{2}{q}}\\ &\quad + \Big(\frac{1}{\rho}\|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q}\Big)\Big(\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\Big]\\ &\quad \cdot\Big(\frac{2^n}{k}\Big)^{\frac{pb}{N+p}}2^{(n+2)(q-b)\frac{p}{N+p}}k^{-(q-b)\frac{p}{N+p}}\Big(\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p}{N+p}},\end{aligned} \tag{2.6}$$

where $h = p\big(1 + \frac{p}{N}\big)$.

Because of the assumption $1 - \frac{2}{q} + \frac{p}{N+p} > 1$, by choosing a subsequence, we see that there is an $\varepsilon = \varepsilon(q, p, N) > 0$, such that

$$\iint_{Q_{n+1}} w_{n+1}^p\mathrm{d}x\mathrm{d}\tau \le \gamma d^n M k^{-\frac{pq}{N+p}}\Big(\iint_{Q_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{1+\varepsilon},$$

where

$$M = (\|u\|^b_{\infty,Q_0} + \|\varphi_0\|_{L^{\frac{q}{2}}} + \|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q})\frac{1}{\sigma^p t}$$

and $d = 2^{\alpha+\frac{pq}{N+p}}$. Choosing $q = b + r$ and applying [11, p. 96, Lemma 5.6], we deduce that

$$\lim_{n\to\infty} \iint_{Q_n} w_n^p \mathrm{d}x\mathrm{d}\tau = 0, \tag{2.7}$$

provided that k is chosen to satisfy

$$\iint_{Q_0} u^r \mathrm{d}x\mathrm{d}\tau = C\sigma^{N+p} t^{\frac{N+p}{p}} k^{b+r} (\|u\|^b_{\infty,Q_0} + \|\varphi_0\|_{L^{\frac{q}{2}}} + \|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q})^{\frac{-(N+p)}{p}},$$
$$C = \gamma^{\frac{-N-p}{p}} d^{-(\frac{-N-p}{p})^2}.$$

It follows that

$$\|u\|_{\infty,Q_\infty} \le k \le \Big(\gamma\sigma^{-N-p} t^{-\frac{N+p}{p}} \int_0^t \int_{B_{(1+\sigma)\rho}} u^r \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{b+r}}$$
$$\times (\|u\|^b_{\infty,Q_0} + \|\varphi_0\|_{L^{\frac{q}{2}}} + \|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q})^{\frac{(N+p)}{p(b+r)}}.$$

If

$$\|u\|^b_{\infty,Q_0} \le \|\varphi_0\|_{L^{\frac{q}{2}}} + \|\varphi_1\|_{L^q} + \|\varphi_2\|_{L^q},$$

then Theorem 1.1 is true. Otherwise, we have

$$\|u\|_{\infty,Q_\infty} \le \Big(\gamma\sigma^{-N-p} t^{-\frac{N+p}{p}} \int_0^t \int_{B_{(1+\sigma)\rho}} u^r \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{b+r}} \|u\|^{\frac{b(N+p)}{p(b+r)}}_{\infty,Q_0}. \tag{2.8}$$

Consider the increasing family of radii $\rho_n = \rho \sum_{i=0}^{n} 2^{-i}$ and a family of cylinder $Q_s = B_{\rho_s} \times (t_s, t),\ s = 0, 1, 2, \cdots$. Applying (2.8) to the boxes $Q_s \subset Q_{s+1}$ and by Lemma 2.1, we can easily get the conclusion of Theorem 1.1.

If $0 < m < 1$, let $v = u^m$. The original equation (1.1) becomes

$$(v^{\frac{1}{m}})_t - \mathrm{div}\mathbf{a}(x, t, v^{\frac{1}{m}}, Dv^{\frac{1}{m}}) = \mathbf{b}(x, t, v^{\frac{1}{m}}, Dv^{\frac{1}{m}}), \tag*{(1.1)$'$}$$

and the constructive conditions (1.6)–(1.8) become

$$\mathbf{a}(x, t, v^{\frac{1}{m}}, Dv^{\frac{1}{m}}) \cdot Dv \ge \frac{C_0}{m^{p-1}}|Dv|^p - m\varphi_0 v^{\frac{m-1}{m}}, \tag*{(1.6)$'$}$$

$$|\mathbf{a}(x, t, v^{\frac{1}{m}}, Dv^{\frac{1}{m}})| \le \frac{C_1}{m^{p-1}}|Dv|^{p-1} + \varphi_1, \tag*{(1.7)$'$}$$

$$|\mathbf{b}(x, t, v^{\frac{1}{m}}, Dv^{\frac{1}{m}})| \le \frac{C_2}{m^{p-1}}|Dv|^{p-1} + \varphi_2. \tag*{(1.8)$'$}$$

Now we choose the test function of the weak solutions to (1.1)$'$ as $\varphi = (v - \overline{k}_n)_+^{q-1}\overline{\zeta}_n^p$, $q > 0$ to be chosen later. If we denote

$$G(s) = \int_0^s \Phi'(\tau)(\tau - \overline{k}_n)_+^{q-1}\mathrm{d}\tau, \quad \Phi(s) = s^{\frac{1}{m}}\ (s \ge 0), \quad w_n = (v - \overline{k}_n)_+^{\frac{p+q-2}{p}},$$

and by the constructive conditions (1.6)$'$–(1.8)$'$, integrating by parts, and using the Hölder inequality, we have

$$\begin{aligned}
0 \geq & \int_{\overline{B}_n} G(v)\overline{\zeta}_n^p \mathrm{d}x\Big|_{t_n}^t - \frac{p2^{n+3}}{\sigma t}\int_{t_n}^t\int_{\overline{B}_n} G(v)\overline{\zeta}_n^{p-1}\mathrm{d}x\mathrm{d}\tau \\
& + \Big[\frac{p(q-1)C_0 - p(p-1)C_1 - (p-1)C_2}{pm^{p-1}}\Big]\iint_{\overline{Q}_n}(v-\overline{k}_n)_+^{q-2}\overline{\zeta}_n^p|Dv|^p\mathrm{d}x\mathrm{d}\tau \\
& - \Big[\frac{2^{p(n+2)}C_1}{(\sigma\rho)^p m^{p-1}} + \frac{C_2}{pm^{p-1}}\Big]\iint_{\overline{Q}_n}(v-\overline{k}_n)_+^{p+q-2}\mathrm{d}x\mathrm{d}\tau \\
& - m(q-1)\|v\|_{\infty,Q_0}^{\frac{(q-1-\frac{1}{m})}{(q-1+\frac{1}{m})}}\Big(\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{b(q-1-\frac{1}{m})}{m(q-1+\frac{1}{m})}}\Big(\iint_{\overline{Q}_n}\varphi_0^{\frac{m(q-1)+1}{2}}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{2}{m(q-1)+1}} \\
& - \Big(\frac{2^{n+2}}{\sigma\rho}+1\Big)\Big[\|v\|_{\infty,Q_0}^{\frac{b}{m}}\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big]^{\frac{q-1}{q-1+\frac{1}{m}}}\Big[\sum_{i=1}^2\iint_{\overline{Q}_n}\varphi_i^{m(q-)+1}\mathrm{d}x\mathrm{d}\tau\Big]^{\frac{1}{m(q-)+1}}. \quad (2.9)
\end{aligned}$$

At the same time, we have

$$\begin{aligned}
& \iint_{\overline{Q}_n}(v-\overline{k}_n)_+^{q-2}\overline{\zeta}_n^p|Dv|^p\mathrm{d}x\mathrm{d}\tau \\
= & \frac{p}{p+q-2}\iint_{\overline{Q}_n}|Dw_n|^p\overline{\zeta}_n^p\mathrm{d}x\mathrm{d}\tau \\
\geq & \frac{p}{2^{p-2}(p+q-2)}\Big[\iint_{\overline{Q}_n}|D(w_n\overline{\zeta}_n)|^p\mathrm{d}x\mathrm{d}\tau - \iint_{\overline{Q}_n} w_n^p|D\overline{\zeta}_n|^p\mathrm{d}x\mathrm{d}\tau\Big]. \quad (2.10)
\end{aligned}$$

From (2.9) and (2.10), we get

$$\begin{aligned}
& \int_{\overline{B}_n} G(v)\overline{\zeta}_n^p\mathrm{d}x\Big|_{t_n}^t + \beta_1\iint_{\overline{Q}_n}|D(w_n\overline{\zeta}_n)|^p\mathrm{d}x\mathrm{d}\tau \\
\leq & \frac{p2^{n+3}}{\sigma t}\iint_{\overline{Q}_n} G(v)\overline{\zeta}_n^{p-1}\mathrm{d}x\mathrm{d}\tau + \beta_2\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}\tau \\
& + m(q-1)\|v\|_{\infty,Q_0}^{\frac{(q-1-\frac{1}{m})}{(q-1+\frac{1}{m})}}\Big(\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}t\Big)^{\frac{b(q-1-\frac{1}{m})}{m(q-1+\frac{1}{m})}}\Big(\iint_{\overline{Q}_n}\varphi_0^{\frac{m(q-1)+1}{2}}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{2}{m(q-1)+1}} \\
& + \Big(\frac{2^{n+2}}{\sigma\rho}+1\Big)\Big[\|v\|_{\infty,Q_0}^{\frac{b}{m}}\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big]^{\frac{q-1}{q-1+\frac{1}{m}}}\Big[\sum_{i=1}^2\iint_{\overline{Q}_n}\varphi_i^{m(q-1)+1}\mathrm{d}x\mathrm{d}\tau\Big]^{\frac{1}{m(q-1)+1}}, \quad (2.11)
\end{aligned}$$

where

$$\beta_1 = \frac{p(q-1)C_0 - p(p-1)C_1 - (p-1)C_2}{2^{p-2}(p+q-2)m^{p-1}}, \quad \beta_2 = \frac{\beta_1 2^{(n+2)p}}{(\sigma\rho)^p} + \frac{2^{p(n+2)}C_1}{(\sigma\rho)^p m^{p-1}} + \frac{C_2}{pm^{p-1}}.$$

Now, since

$$\begin{aligned}
(1+mq-m)^{-1}(v-\overline{k}_n)_+^{q-1+\frac{1}{m}} \leq G(v) \leq v^{\frac{1}{m}}(v-\overline{k}_n)_+^{q-1}, \\
\iint_{\overline{Q}_n}(v-\overline{k}_n)_+^{q-1+\frac{1}{m}}\mathrm{d}x\mathrm{d}\tau \geq (k2^{-(n+2)})^{q-1+\frac{1}{m}}|\overline{A}_n|,
\end{aligned}$$

by the Hölder inequality, we have

$$\begin{aligned}\iint_{\overline{Q}_n} G(v)\overline{\zeta}_n^{p-1}\mathrm{d}x\mathrm{d}\tau &\le \iint_{\overline{Q}_n}(v-\overline{k}_n)_+^{q-1+\frac{1}{m}}\mathrm{d}x\mathrm{d}\tau + \overline{k}_n^{\frac{1}{m}}\iint_{\overline{Q}_n}(v-\overline{k}_n)^{q-1}\mathrm{d}x\mathrm{d}\tau\\ &\le \gamma 2^{\frac{n}{m}}\iint_{\overline{Q}_n}(v-\overline{k}_n)_+^{q-1+\frac{1}{m}}\mathrm{d}x\mathrm{d}\tau\\ &\le \gamma 2^{\frac{n}{m}}\|v\|_{\infty,Q_0}^{\frac{b}{m}}\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}\tau, \end{aligned}\tag{2.12}$$

$$\int_{\overline{B}_n} G(v)\overline{\zeta}_n^p\mathrm{d}x \ge (1+mq-m)^{-1}(k2^{-(n+3)})^{\frac{m}{b}}\int_{\overline{B}_n}(w_{n+1}\overline{\zeta}_n)^p\mathrm{d}x. \tag{2.13}$$

From (2.11)–(2.13), we have

$$\begin{aligned}&\Big(\frac{k}{2^n}\Big)^{\frac{b}{m}}\sup_{t_n\le\tau\le t}\int_{\overline{B}_n(\tau)}(w_{n+1}\overline{\zeta}_n)^p\mathrm{d}x + \iint_{\overline{Q}_n}|D(w_{n+1}\overline{\zeta}_n)|^p\mathrm{d}x\mathrm{d}\tau\\ &\le \gamma\Big(\frac{2^{(1+\frac{1}{m})n}}{\sigma^p t}\|v\|_{\infty,Q_0}^{\frac{b}{m}} + \frac{2^{np}}{(\sigma^p\rho)^p}\Big)\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}\tau\\ &\quad + m(q-1)\|v\|_{\infty,Q_0}^{\frac{b(q-1-\frac{1}{m})}{m(q-1+\frac{1}{m})}}\Big(\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}t\Big)^{\frac{q-1-\frac{1}{m}}{q-1+\frac{1}{m}}}\Big(\iint_{\overline{Q}_n}\varphi_0^{\frac{m(q-1)+1}{2}}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{2}{m(q-1)+1}}\\ &\quad + \Big(\frac{2^{n+2}}{\sigma\rho}+1\Big)\Big[\|v\|_{\infty,Q_0}^{\frac{b}{m}}\iint_{\overline{Q}_n} w_n^p\mathrm{d}x\mathrm{d}\tau\Big]^{\frac{q-1}{q-1+\frac{1}{m}}}\Big[\sum_{i=1}^2\iint_{\overline{Q}_n}\varphi_i^{m(q-1)+1}\mathrm{d}x\mathrm{d}\tau\Big]^{\frac{1}{m(q-1)+1}}. \end{aligned}\tag{2.14}$$

Noticing $m<1$, we have

$$\|\varphi_0\|_{L^{\frac{m(q-1)+1}{2}}} \le \gamma\|\varphi\|_{L^{\frac{q}{2}}},$$
$$\|\varphi_i\|_{L^{m(q-1)+1}} \le \gamma\|\varphi\|_{L^q}, \quad i=1,2.$$

So, (2.14) is similar to (2.8), as in the case of $m>1$, and we can get the conclusion in Theorem 1.1.

3 Proof of Theorem 1.2

Lemma 3.1 *Let u be a nonnegative local weak solution to* (1.1) *in* Ω_T. *Suppose that when* $m\ge 1$,

$$\varphi_0 \le t^{\frac{1}{b}}\varphi_{01}, \quad \varphi_{01}\in L^{\frac{q}{2}};$$

when $0<m<1$,

$$\varphi_0(x,t)\equiv 0.$$

There exists a constat $\gamma=\gamma(N,m,p)$, such that $\forall\,(x_0,t_0)\in\Omega_T$, $\forall\,\rho>0$, $B_{4\rho}(x_0)\subset\Omega$, $\forall\,t>t_0$, $\forall\,\sigma\in(0,1)$, we have

$$\begin{aligned}\int_{t_0}^t\int_{B_{\sigma\rho}(x_0)}|Du^m|^{p-1}\mathrm{d}x\mathrm{d}t &\le \gamma K + \frac{\gamma\rho}{(1-\sigma)^{p-1}}\Big\{\Big(\frac{t-t_0}{\rho^\lambda}\Big)^{\frac{1}{p}}\Big[\sup_{t_0\le\tau\le t}\int_{B_\rho(x_0)}u(x,\tau)\mathrm{d}x\Big]^{1-\frac{p}{b}}\\ &\quad + \Big(\frac{t-t_0}{\rho^\lambda}\Big)^{\frac{1}{b}}\Big\}, \end{aligned}\tag{3.1}$$

$$\int_{t_0}^t\int_{B_{\sigma\rho}(x_0)}|Du^m|^{p-1}\mathrm{d}x\mathrm{d}t \le \gamma K + \gamma\sup_{t_0\le\tau\le t}\int_{B_\rho(x_0)}u(x,\tau)\mathrm{d}x + \gamma(\sigma)\Big(\frac{t-t_0}{\rho^\lambda}\Big)^{\frac{1}{b}}, \tag{3.2}$$

where

$$K=\gamma\|\varphi_{01}\|_{\frac{q}{2}}t^{\frac{q-2}{q}}\rho^{\frac{p}{b}+1+N\frac{q-2}{q}}+\frac{\gamma}{1-\sigma}t^{\frac{q-1}{q}}\rho^{N\frac{q-1}{q}+1}\|\varphi_1\|_q+\gamma t^{\frac{q-1}{q}}\rho^{N\frac{q-1}{q}+1}\|\varphi_2\|_q.$$

Clearly, (3.2) is a simple corollary of (3.1).

Proof of (3.1) After a translation, one may assume that $(x_0,t_0)=(0,0)$. Fix $\sigma\in(0,1)$ and let $\zeta(x)$ be a nonnegative piecewise smooth cutoff function in B_ρ that equals 1 on $B_{\sigma\rho}$, and satisfies $\mid D\zeta\mid\leq((1-\sigma)\rho)^{-1}$.

We also separately discuss the two cases $m\geq 1$ and $0<m<1$.

(1) Suppose $m\geq 1$.

In (1.10), take $\varphi=t^{\frac{1}{p}}(u+\varepsilon)^{-\frac{b}{p}}\zeta^p$, $\varepsilon>0$ to be chosen later. By calculation, we have

$$\begin{aligned}
&\frac{1}{m^p}\Big(\frac{C_0b}{p}-C_1(p-1)-\frac{(p-1)C_2}{p}\Big)\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(u+\varepsilon)^{-\frac{b}{p}-1}\zeta^pu^{1-m}|Du^m|^p\mathrm{d}x\mathrm{d}t\\
\leq&\Big(\frac{C_2}{p}+\frac{C_1}{(\rho(1-\sigma))^p}\Big)\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(u+\varepsilon)^{p-\frac{b}{p}-1}u^{(m-1)(p-1)}\mathrm{d}x\mathrm{d}t\\
&+\frac{b}{p}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(u+\varepsilon)^{-\frac{b}{p}-1}\varphi_0\mathrm{d}x\mathrm{d}t+\frac{p}{(1-\sigma)\rho}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(u+\varepsilon)^{-\frac{b}{p}}\varphi_1\mathrm{d}x\mathrm{d}t\\
&+\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(u+\varepsilon)^{-\frac{b}{p}}\varphi_2\mathrm{d}x\mathrm{d}t+\frac{p}{p-b}t^{\frac{1}{p}}\int_{B_\rho}(u+\varepsilon)^{1-\frac{b}{p}}\zeta^p\mathrm{d}x
\end{aligned}\tag{3.3}$$

and

$$\frac{b}{p}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(u+\varepsilon)^{-\frac{b}{p}-1}\varphi_0\mathrm{d}x\mathrm{d}t\leq\frac{b\varepsilon^{-1-\frac{b}{p}}}{p}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}\varphi_0\mathrm{d}x\mathrm{d}t,\tag{3.4}$$

$$\frac{p}{(1-\sigma)\rho}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(u+\varepsilon)^{-\frac{b}{p}}\varphi_1\mathrm{d}x\mathrm{d}t\leq\frac{b\varepsilon^{-\frac{b}{p}}}{(1-\sigma)\rho}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}\varphi_1\mathrm{d}x\mathrm{d}t.\tag{3.5}$$

Let $S=\sup\limits_{0\leq\tau\leq t}\int_{B_\rho}u(x,\tau)\mathrm{d}x$. Then

$$\begin{aligned}
J=&\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(u+\varepsilon)^{-\frac{b}{p}-1}\zeta^pu^{1-m}|Du^m|^p\mathrm{d}x\mathrm{d}t\\
\leq&\gamma\Big(\frac{1}{p}+\frac{1}{(\rho(1-\sigma))^p}\Big)\mathrm{I}_2+\gamma\varepsilon^{-1-\frac{b}{p}}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}\varphi_0\mathrm{d}x\mathrm{d}t\\
&+\frac{\gamma\varepsilon^{-\frac{b}{p}}}{\rho(1-\sigma)}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}\varphi_1\mathrm{d}x\mathrm{d}t+\gamma\mathrm{I}_1+\gamma\int_0^t\int_{B_\rho}t^{\frac{1}{p}}\varphi_2\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where

$$\mathrm{I}_1=t^{\frac{1}{p}}\int_{B_\rho}(u+\varepsilon)^{1-\frac{b}{p}}\zeta^p\mathrm{d}x\leq\gamma(t\rho^{Nb})^{\frac{1}{p}}(S+\varepsilon\rho^N)^{1-\frac{b}{p}}=\gamma\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}\rho(S+\varepsilon\rho^N)^{1-\frac{b}{p}},$$

$$\begin{aligned}
\mathrm{I}_2=&\int_0^t\int_{B_\rho}\tau^{\frac{1}{p}}u^{(m-1)(p-1)}(u+\varepsilon)^{p-\frac{b}{p}-1}\mathrm{d}x\mathrm{d}\tau\\
\leq&\int_0^t\int_{B_\rho}\tau^{\frac{1}{p}}(u+\varepsilon)^{1-b-\frac{b}{p}}\mathrm{d}x\mathrm{d}\tau\leq\varepsilon^{-b}\int_0^t\int_{B_\rho}\tau^{\frac{1}{p}}(u+\varepsilon)^{1-\frac{b}{p}}\mathrm{d}x\mathrm{d}\tau\\
\leq&\gamma\varepsilon^{-b}(t\rho^{Nb})^{\frac{1}{p}}t(S+\varepsilon\rho^N)^{1-\frac{b}{p}}.
\end{aligned}$$

At the same time, by the Hölder inequality,

$$\int_0^t \int_{B_\rho} t^{\frac{1}{p}} \varphi_0 \mathrm{d}x\mathrm{d}t \le \gamma \|\varphi_{01}\|_{\frac{q}{2}} t^{\frac{1}{b}+\frac{1}{p}} \rho^{N\frac{q-2}{q}},$$
$$\int_0^t \int_{B_\rho} t^{\frac{1}{p}} \varphi_i \mathrm{d}x\mathrm{d}t \le \gamma \|\varphi_i\|_q t^{\frac{1}{p}+\frac{q-1}{q}} \rho^{N\frac{q-1}{q}},$$

where $i = 1, 2$. So,

$$J \le \gamma\Big(\frac{1}{p} + \frac{1}{(\rho(1-\sigma))^p}\varepsilon^{-b}(t\rho^{Nb})^{\frac{1}{p}} t(S+\varepsilon\rho^N)^{1-\frac{b}{p}}\Big) + \gamma(t\rho^{Nb})^{\frac{1}{p}}(S+\varepsilon\rho^N)^{1-\frac{b}{p}}$$
$$+ \gamma\varepsilon^{-1-\frac{b}{p}}\|\varphi_{01}\|_{\frac{q}{2}} t^{\frac{1}{b}+\frac{1}{p}}\rho^{N\frac{q-2}{q}} + \gamma\Big[\frac{\varepsilon^{-\frac{b}{p}}}{\rho(1-\sigma)}\|\varphi_1\|_q + \|\varphi_2\|_q\Big] t^{\frac{1}{p}+\frac{q-1}{q}}\rho^{N\frac{q-1}{q}}.$$

Set $\varepsilon^b = \frac{t}{\rho^p}$. By

$$(S + t^{\frac{1}{b}}\rho^{N-\frac{p}{b}})^{1-\frac{b}{p}} = \frac{S + t^{\frac{1}{b}}\rho^{N-\frac{p}{b}}}{(S + t^{\frac{1}{b}}\rho^{N-\frac{p}{b}})^{\frac{b}{p}}} \le \gamma S^{1-\frac{b}{p}} + \gamma t^{\frac{1}{b}(1-\frac{b}{p})}\rho^{(N-\frac{p}{b})(1-\frac{b}{p})},$$

we have

$$J \le \gamma\|\varphi_{01}\|_{\frac{q}{2}} t^{\frac{q-2}{q}}\rho^{\frac{p}{b}+1+N\frac{q-2}{q}} + \frac{\gamma}{1-\sigma} t^{\frac{q-1}{q}}\rho^{N\frac{q-1}{q}+1}\|\varphi_1\|_q + \gamma t^{\frac{q-1}{q}}\rho^{N\frac{q-1}{q}+1}\|\varphi_2\|_q$$
$$+ \frac{\gamma}{(1-\sigma)^p}(t\rho^{Nb})^{\frac{1}{p}}[S^{1-\frac{p}{b}} + t^{\frac{1}{b}(1-\frac{p}{b})}\rho^{(N-\frac{p}{b})(1-\frac{p}{b})}]$$
$$= K + \frac{\gamma}{(1-\sigma)^p}(t\rho^{Nb})^{\frac{1}{p}}[S^{1-\frac{p}{b}} + t^{\frac{1}{b}(1-\frac{p}{b})}\rho^{(N-\frac{p}{b})(1-\frac{p}{b})}].$$

By the Hölder inequality, we get

$$I \equiv \int_0^t \int_{B_{\sigma\rho}} |\, Du^m\, |^{p-1}\, \mathrm{d}x\mathrm{d}\tau$$
$$= \int_0^t \int_{B_{\sigma\rho}} (\tau^{\frac{1}{p}} u^{1-m}(u+\varepsilon)^{-1-\frac{b}{p}}\zeta^p \,|\, Du^m\, |^p)^{\frac{p-1}{p}}$$
$$\times (\tau^{\frac{-p-1}{p^2}} u^{\frac{(m-1)(p-1)}{p}}(u+\varepsilon)^{(1+\frac{b}{p})\frac{p-1}{p}})\mathrm{d}x\mathrm{d}\tau$$
$$\le J^{\frac{p-1}{p}} \mathrm{I}_3{}^{\frac{1}{p}},$$

where

$$\mathrm{I}_3 = \int_0^t \int_{B_{\sigma\rho}} \tau^{\frac{-p-1}{p}} u^{(m-1)(p-1)}(u+\varepsilon)^{(1+\frac{b}{p})(p-1)}\mathrm{d}x\mathrm{d}\tau.$$

Similar to the estimate of I_2, the estimate of I_3 is obtained as follows,

$$\mathrm{I}_3 \le \gamma\rho\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}(S+\varepsilon\rho^N)^{1-\frac{b}{p}} = \gamma\rho\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}(S + t^{\frac{1}{b}}\rho^{N-\frac{p}{b}})^{1-\frac{b}{p}}$$
$$\le \gamma\rho\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}[S^{1-\frac{p}{b}} + t^{\frac{1}{b}(1-\frac{b}{p})}\rho^{(N-\frac{p}{b})(1-\frac{b}{p})}].$$

Thus

$$\begin{aligned}
I &\le J^{\frac{p-1}{p}} \mathrm{I}_3{}^{\frac{1}{p}}\\
&\le \gamma\Big\{K+\frac{1}{(1-\sigma)^p}(t\rho^{Nb})^{\frac{1}{p}}[S^{1-\frac{p}{b}}+t^{\frac{1}{b}(1-\frac{b}{p})}\rho^{(N-\frac{p}{b})(1-\frac{b}{p})}]\Big\}^{\frac{p-1}{p}}\\
&\quad\times\Big\{\rho\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}\Big(S+t^{\frac{1}{b}}\rho^{N-\frac{p}{b}}\Big)^{1-\frac{b}{p}}\Big\}^{\frac{1}{p}}\\
&\le \gamma K+\frac{\gamma\rho}{(1-\sigma)^{p-1}}\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}(S^{1-\frac{b}{p}}+t^{\frac{1}{b}(1-\frac{b}{p})}\rho^{(N-\frac{p}{b})(1-\frac{b}{p})}).
\end{aligned}$$

We get the conclusion easily.

(2) Suppose $0<m<1$.

Let $v=u^m$. Then equation (1.1) becomes

$$(v^{\frac{1}{m}})_t-\mathrm{div}\mathbf{a}(x,t,v^{\frac{1}{m}},Dv^{\frac{1}{m}})=\mathbf{b}(x,t,v^{\frac{1}{m}},Dv^{\frac{1}{m}}), \tag{1.1$'$}$$

and satisfies the constructive conditions (1.6)$'$–(1.8)$'$ with $\varphi_0=0$. Set $\Phi(s)=s^{\frac{1}{m}},\ s\ge 0$ and $G(s)=\int_0^s\Phi'(\tau)(\tau+\varepsilon)^{-\frac{b}{mp}}\mathrm{d}\tau$. Clearly, we have

$$\frac{\partial G(v)}{\partial t}=\frac{\partial\Phi(v)}{\partial t}(v+\varepsilon)^{-\frac{b}{mp}}.$$

Now, we take $\varphi=t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\zeta^p(x)$, where $\zeta(x)$ is the cut function as in the case of $m\ge 1$. Integrating by parts, we can get

$$\begin{aligned}
0\le&-\frac{C_0b}{m^pp}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1}\zeta^p|Dv|^p\mathrm{d}x\mathrm{d}t\\
&+\frac{pC_1}{m^{p-1}(1-\sigma)\rho}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\zeta^{p-1}|Dv|^{p-1}\mathrm{d}x\mathrm{d}t\\
&+\frac{p}{(1-\sigma)\rho}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\zeta^{p-1}\varphi_1\mathrm{d}x\mathrm{d}t\\
&+\frac{C_2}{m^{p-1}}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\zeta^p|Dv|^{p-1}\mathrm{d}x\mathrm{d}t+\int_0^t\int_{B_\rho}\varphi_2t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\zeta^p(x)\mathrm{d}x\mathrm{d}t\\
&-\int_0^t\int_{B_\rho}\Big(\frac{C_2}{m^{p-1}}|Dv|^{p-1}+\varphi_2\Big)\zeta^p(x)(v+\varepsilon)^{-\frac{b}{mp}}\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.6}$$

By the Young's inequality, we have

$$\begin{aligned}
&\frac{pC_1}{m^{p-1}(1-\sigma)\rho}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\zeta^{p-1}|Dv|^{p-1}\mathrm{d}x\mathrm{d}t\\
\le&\frac{(p-1)C_1}{m^{p-1}}\int_{B_\rho}t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1}\zeta^p|Dv|^p\mathrm{d}x\mathrm{d}t\\
&+\frac{C_1}{m^{p-1}(1-\sigma)^p\rho^p}\int_{B_\rho}t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1+p}\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{3.7}$$

$$\frac{C_2}{m^{p-1}}\int_0^t\int_{B_\rho}t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\zeta^p|Dv|^{p-1}\mathrm{d}x\mathrm{d}t$$

$$\leq \frac{C_2(p-1)}{m^{p-1}p}\int_0^t\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1}\zeta^p|Dv|^p\mathrm{d}x\mathrm{d}t$$
$$+\frac{C_2}{m^{p-1}p}\int_0^t\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1+p}\zeta^p\mathrm{d}x\mathrm{d}t. \tag{3.8}$$

By (3.6)–(3.8), we have

$$\Big(\frac{C_0b}{m^pp}-\frac{C_1}{m^{p-1}}-\frac{C_2(p-1)}{m^{p-1}p}\Big)\int_0^t\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1}\zeta^p|Dv|^p\mathrm{d}x\mathrm{d}t$$
$$\leq \frac{C_1}{m^{p-1}(1-\sigma)^p\rho^p}\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1+p}\mathrm{d}x\mathrm{d}t$$
$$+\frac{p}{(1-\sigma)\rho}\int_0^t\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\varphi_1\mathrm{d}x\mathrm{d}t+\frac{C_2}{m^{p-1}p}\int_0^t\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1+p}\mathrm{d}x\mathrm{d}t$$
$$+\int_0^t\int_{B_\rho}\varphi_2 t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\zeta^p(x)\mathrm{d}x\mathrm{d}t+t^{\frac{1}{p}}\int_{B_\rho}G(v(x,t))\zeta^p(x)\mathrm{d}x. \tag{3.9}$$

Set $S=\sup\limits_{0\leq\tau\leq t}\int_{B_\rho}u(x,\tau)\mathrm{d}x$. Because

$$0\leq G(v)\leq p(p-b)^{-1}(v+\varepsilon)^{(1-\frac{b}{p})\frac{1}{m}},$$

by Jesen's inequality, we get

$$t^{\frac{1}{p}}\int_{B_\rho}G(v(x,t))\zeta^p(x)\mathrm{d}x\leq p(p-b)^{-1}t^{\frac{1}{p}}\int_{B_\rho}(v+\varepsilon)^{(1-\frac{b}{p})\frac{1}{m}}\mathrm{d}x\leq\gamma t^{\frac{1}{p}}\int_{B_\rho}(u+\varepsilon)^{1-\frac{b}{p}}\mathrm{d}x$$
$$\leq\gamma t^{\frac{1}{p}}\rho^{\frac{Nb}{p}}(S+\varepsilon^{\frac{1}{m}}\rho^N)^{1-\frac{b}{p}}=\gamma\rho\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}(S+\varepsilon^{\frac{1}{m}}\rho^N)^{1-\frac{b}{p}}. \tag{3.10}$$

Let $\varepsilon=(t\rho^{-p})^{\frac{b}{m}}$. Then

$$\frac{C_1}{m^{p-1}(1-\sigma)^p\rho^p}\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1+p}\mathrm{d}x\mathrm{d}t\leq\frac{\gamma}{(1-\sigma)^p}\rho\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}(S+\varepsilon^{\frac{1}{m}}\rho^N)^{1-\frac{b}{p}}. \tag{3.11}$$

From (3.9), we have

$$\int_0^t\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}-1}\zeta^p|Dv|^p\mathrm{d}x\mathrm{d}t$$
$$\leq\frac{\gamma}{(1-\sigma)^p}\rho\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{p}}(S+\varepsilon^{\frac{1}{m}}\rho^N)^{1-\frac{b}{p}}+\frac{p}{(1-\sigma)\rho}\int_0^t\int_{B_\rho} t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\varphi_1\mathrm{d}x\mathrm{d}t$$
$$+\int_0^t\int_{B_\rho}\varphi_2 t^{\frac{1}{p}}(v+\varepsilon)^{-\frac{b}{mp}}\mathrm{d}x\mathrm{d}t. \tag{3.12}$$

A procedure analogous to that in the case of $m>1$ leads to (3.1) and Lemma 3.1 follows.

Proof of Theorem 1.2 Let us first prove (1.12). Assuming that (x_0,t_0) coincides with the origin and considering the family of expanding concentric balls

$$B_n=\{|x|<\rho_n\},\quad \rho_n=\rho\sum_{i=0}^n 2^{-i},\quad n=0,1,2,\cdots,$$

we have

$$B_0 = B_\rho, \quad B_\infty = B_{2\rho}.$$

Introduce also the "intermediate" spheres

$$\widetilde{B}_n \equiv \Big\{ |x| < \frac{1}{2}(\rho_n + \rho_{n+1}) \Big\}.$$

Let $\zeta_n(x)$ be a piecewise smooth nonnegative cutoff function in $\widetilde{B}_n$ that equals one on B_n such that $|D\zeta_n| \le \frac{2^{n+2}}{\rho}$. In the weak formulation of (1.10), take ζ_n as a test function. Then for any two times levels τ_1, τ_2 in $[0,t]$,

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} \mathbf{a}(x,t,u,Du) \cdot D\zeta_n \mathrm{d}x\mathrm{d}\tau \\
&\le \frac{C_1 2^{n+2}}{\rho} \int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} u^{(m-1)(p-1)} |Du|^{p-1} \mathrm{d}x\mathrm{d}\tau + \int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} \varphi_1 \mathrm{d}x\mathrm{d}\tau, \\
&\int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} \mathbf{b}(x,t,u,Du)\zeta_n(x) \mathrm{d}x\mathrm{d}\tau \\
&\le C_2 \int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} u^{(m-1)(p-1)} |Du|^{p-1} \mathrm{d}x\mathrm{d}\tau + \int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} \varphi_2 \mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

so,

$$\begin{aligned}
\int_{\widetilde{B}_n} u(x,\tau_1)\zeta_n \mathrm{d}x \le& \int_{\widetilde{B}_n} u(x,\tau_2)\zeta_n \mathrm{d}x + \Big(\frac{C_1 2^{n+2}}{m^{p-1}\rho} + \frac{c_2}{m^{p-1}} \Big) \int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} |Du^m|^{p-1} \mathrm{d}x\mathrm{d}\tau \\
&+ \int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} (\varphi_1 + \varphi_2) \mathrm{d}x\mathrm{d}\tau. \qquad (3.13)
\end{aligned}$$

Now, we take τ_2, a time level in $[0,t]$, such that

$$\inf_{0 \le \tau \le t} \int_{B_{2\rho}} u(x,\tau)\mathrm{d}x = \int_{B_{2\rho}} u(x,\tau_2)\mathrm{d}x \equiv I.$$

We also set

$$S_n \equiv \sup_{0 \le \tau \le t} \int_{B_n} u(x,\tau)\mathrm{d}x.$$

Since $\tau_1 \in [0,t]$ is arbitrary, (3.16) implies

$$S_n \le I + \Big(\frac{C_1 2^{n+2}}{m^{p-1}\rho} + \frac{c_2}{m^{p-1}} \Big) \int_0^t \int_{\widetilde{B}_n} |Du^m|^{p-1} \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_{\widetilde{B}_n} (\varphi_1 + \varphi_2) \mathrm{d}x\mathrm{d}\tau. \qquad (3.14)$$

Next, we apply (3.1) over the pair of balls $\widetilde{B}_n$ and B_{n+1} for which $(1-\sigma) \ge 2^{-(n+2)}$, and n is large enough such that $\rho \le C_1 2^{n+2} C_2$. Then

$$\begin{aligned}
&\Big(\frac{C_1 2^{n+2}}{m^{p-1}\rho} + \frac{C_2}{m^{p-1}} \Big) \int_{\tau_1}^{\tau_2} \int_{\widetilde{B}_n} |Du^m|^{p-1} \mathrm{d}x\mathrm{d}\tau \\
&\le \gamma \frac{C_1 2^{n+2}}{m^{p-1}\rho} K + \gamma \frac{C_1 2^{n+2}}{m^{p-1}} \frac{1}{(1-\sigma)^{p-1}} \Big(\frac{t}{\rho^\lambda} \Big)^{\frac{1}{p}} [S_{n+1}^{1-\frac{b}{p}} + t^{\frac{1}{b}(1-\frac{b}{p})} \rho^{(N-\frac{p}{b})(1-\frac{b}{p})}].
\end{aligned}$$

Thus

$$S_n \le \varepsilon S_{n+1} + \gamma \Big[I + \|\varphi_{01}\|_{\frac{q}{2}} t^{\frac{q-2}{q}} \rho^{\frac{p}{b} + N\frac{q-2}{q}} + t^{\frac{q-1}{q}} \rho^{N\frac{q-2}{q}} (\|\varphi_1\|_q + \|\varphi_2\|_q) + \Big(\frac{t}{\rho^\lambda} \Big)^{\frac{1}{b}} \Big] l^n,$$

where $l = 4 \wedge 2^{\frac{p^2}{b}}$. By Lemma 2.1, one gets (1.15) in Theorem 1.2.

4 Proof of Theorem 1.3

Without loss of generality, we also assume $(x_0, t_0) = (0, 0)$. Fix $\sigma \in (0, 1)$ and $\rho > 0$. Divide this problem into two cases.

(1) Suppose $m \geq 1$.

Let $\zeta(x)$ be a standard cutoff function in $B_{(1+\sigma)\rho}$ that equals one in B_ρ and satisfies $|D\zeta| \leq (\sigma\rho)^{-1}$. For $\varepsilon > 0$, taking $\varphi = (u+\varepsilon)^{r-1}\zeta^p(x)$ in (1.10), we have

$$\begin{aligned}&\int_{B_{2\rho}}\int_0^t \frac{\partial(u+\varepsilon)}{\partial t}(u+\varepsilon)^{r-1}\zeta^p(x)\mathrm{d}t\mathrm{d}x\\ &= \frac{1}{r}\int_{B_{2\rho}}(u(x,t)+\varepsilon)^r\zeta^p(x)\mathrm{d}x - \int_{B_{2\rho}}(u(x,0)+\varepsilon)^r\zeta^p(x)\mathrm{d}x.\end{aligned} \tag{4.1}$$

By calculation, we get

$$\begin{aligned}&\frac{1}{r}\sup_{0\leq\tau\leq t}\int_{B_\rho} u^r(x,\tau)\mathrm{d}x\\ &\leq \frac{1}{r}\int_{B_{2\rho}}(u(x,0)+\varepsilon)^r\mathrm{d}x + \Big[\frac{C_2}{p} + \frac{C_1}{(\sigma\rho)^p}\Big]\int_0^t\int_{B_{(1+\sigma)\rho}}(u+\varepsilon)^{r-b}\mathrm{d}x\\ &\quad + (r-1)\int_0^t\int_{B_{2\rho}}(u+\varepsilon)^{r-2}\zeta^p\varphi_0\mathrm{d}x\mathrm{d}t + \frac{p}{\sigma\rho}\int_0^t\int_{B_{2\rho}}(u+\varepsilon)^{r-1}\zeta^{p-1}\varphi_1\mathrm{d}x\mathrm{d}t\\ &\quad + \int_0^t\int_{B_{2\rho}}(u+\varepsilon)^{r-1}\zeta^p\varphi_2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.2}$$

In the last inequality of (4.2), we use the condition

$$\frac{(p-1)C_2}{p} + C_1(p-1) - (r-1)C_0 < 0.$$

Letting $\varepsilon \to 0$ and by the Hölder inquality, we have

$$\begin{aligned}&\frac{1}{r}\sup_{0\leq\tau\leq t}\int_{B_{2\rho}} u^r(x,\tau)\mathrm{d}x\\ &\leq \frac{1}{r}\sup_{0\leq\tau\leq t}\int_{B_\rho} u^r(x,0)\mathrm{d}x + \Big[\frac{C_2}{p} + \frac{C_1}{(\sigma\rho)^p}\Big]t[(1+\sigma)\rho]^{\frac{bN}{r}}\Big(\sup_{0\leq\tau\leq t}\int_{B_{(1+\sigma)\rho}} u^r(x,\tau)\mathrm{d}x\Big)^{\frac{r-b}{r}}\\ &\quad + (r-1)\|\varphi_0\|_{\frac{r}{2}}\Big(\sup_{0\leq\tau\leq t}\int_{B_{(1+\sigma)\rho}} u^r(x,\tau)\mathrm{d}x\Big)^{\frac{r-2}{r}}\\ &\quad + \Big(\frac{p}{\sigma\rho}\|\varphi_1\|_r + \|\varphi_2\|_r\Big)\Big(\sup_{0\leq\tau\leq t}\int_{B_{(1+\sigma)\rho}} u^r(x,\tau)\mathrm{d}x\Big)^{\frac{r-1}{r}},\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\frac{1}{r}\sup_{0\leq\tau\leq t}\int_{B_\rho} u^r(x,\tau)\mathrm{d}x\\ &\leq \gamma\Big[\frac{1}{\sigma^p}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}} + \|\varphi_0\|_{\frac{r}{2}} + \|\varphi_1\|_r + \|\varphi_2\|_r\Big]\Big(\sup_{0\leq\tau\leq t}\int_{B_{(1+\sigma)\rho}} u^r(x,\tau)\mathrm{d}x\Big)^{\frac{r-b}{r}}.\end{aligned}$$

Let $\rho_n = \rho\sum\limits_{i=0}^{n} 2^{-i}$, $B_n = B_{\rho_n}$, $S_n = \sup\limits_{0\leq\tau\leq t}\int_{B_{\rho_n}} u^r(x,\tau)\mathrm{d}x$, $I = \int_{B_{2\rho}} u^r(x,\tau)\mathrm{d}x$. Applying

(3.5) to the pair of ball $B_n \subset B_{n+1}$ for which $\sigma > 2^{-(n+2)}$, we obtain

$$\begin{aligned} S_n &\le I + \gamma\Big[2^{pn}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}} + \|\varphi_0\|_{\frac{r}{2}} + \|\varphi_1\|_r + \|\varphi_2\|_r\Big] S_{n+1}^{\frac{r-b}{r}} \\ &\le \delta S_{n+1} + \gamma(N,m,p,r,\delta)2^{\frac{prn}{b}}\Big(I + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{b}}\Big) \\ &\quad + \gamma(N,m,p,r,\delta)(\|\varphi_0\|_{\frac{r}{2}} + \|\varphi_1\|_r + \|\varphi_2\|_r)^{\frac{r}{b}} \end{aligned}$$

for any given $\delta \in (0,1)$. If for any n,

$$\delta S_{n+1} + \gamma(N,m,p,r,\delta)2^{\frac{prn}{b}}\Big(I + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{b}}\Big) \le \gamma(N,m,p,r,\delta)(\|\varphi_0\|_{\frac{r}{2}} + \|\varphi_1\|_r + \|\varphi_2\|_r)^{\frac{r}{b}},$$

so

$$\sup_{0\le\tau\le t}\int_{B_\rho} u(x,\tau)\mathrm{d}x \le \gamma(N,m,p,r,\delta)(\|\varphi_0\|_{\frac{r}{2}} + \|\varphi_1\|_r + \|\varphi_2\|_r)^{\frac{r}{b}}. \tag{4.4}$$

Then the conclusion is clearly true. Otherwise, by choosing a subsequence, we may assume that

$$S_n \le 2\delta S_{n+1} + \gamma(N,m,p,r,\delta)2^{\frac{prn}{b}}\Big(I + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{b}}\Big).$$

Then, iteration of these inequalities yields

$$S_0 = \sup_{0\le\tau\le t}\int_{B_\rho} u(x,\tau)\mathrm{d}x \le (2\sigma)^n S_\infty + \gamma(N,m,p,r,\delta)\Big(I + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{b}}\Big)\sum_{i=0}^{n}(\sigma 2^{\frac{pr}{b}})^i.$$

Choose $\sigma = 2^{-(1+\frac{pr}{b})}$, so that $\sum_{i=0}^{n}(\sigma 2^{\frac{pr}{b}})^i \le 2$. Let $n \to \infty$. We have

$$\sup_{0\le\tau\le t}\int_{B_\rho} u(x,\tau)\mathrm{d}x \le \gamma\Big(I + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{b}}\Big). \tag{4.5}$$

Combining (4.4) with (4.5), we get (1.18).

(2) Suppose $0 < m < 1$. Let $v = u^m$. Then the equation (1.1) becomes

$$(v^{\frac{1}{m}})_t - \mathrm{div}\mathbf{a}(x,t,v^{\frac{1}{m}},Dv^{\frac{1}{m}}) = \mathbf{b}(x,t,v^{\frac{1}{m}},Dv^{\frac{1}{m}}), \tag{1.1$'$}$$

and satisfies the constructive conditions (1.6)$'$–(1.8)$'$. Set $\Phi(s) = s^{\frac{1}{m}}$, $s \ge 0$ and $G(s) = \int_0^s \Phi'(\tau)(\tau+\varepsilon)^{\frac{r-1}{m}}\mathrm{d}\tau$. Let $\zeta(x)$ be the same cutoff function as above and take the test function $\varphi = (v+\varepsilon)^{\frac{r-1}{m}}\zeta^p(x)$. Also, we have

$$\begin{aligned} &\int_{B_{2\rho}} G(v(x,t))\zeta^p(x)\mathrm{d}x \\ &\le \int_{B_{2\rho}} G(v(x,0))\zeta^p(x)\mathrm{d}x - \Big(\frac{C_0(r-1)}{m^p} - \frac{(p-1)C_1}{m^{p-1}}\Big)\int_0^t\int_{B_{(1+\sigma)\rho}} t^{\frac{1}{p}}(v+\varepsilon)^{\frac{r-1}{m}}\zeta^p|Dv|^p\mathrm{d}x\mathrm{d}t \\ &\quad + \gamma\Big(\frac{C_1}{m^{p-1}(\sigma\rho)^p} + \frac{C_2}{m^{p-1}}\Big)\int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{p+\frac{r-1-m}{m}}\mathrm{d}x\mathrm{d}t \\ &\quad + \frac{p}{\sigma\rho}\int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{\frac{r-1}{m}}\varphi_1\mathrm{d}x\mathrm{d}t + \int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{\frac{r-1}{m}}\varphi_2\mathrm{d}x\mathrm{d}t \\ &\le \int_{B_{2\rho}} G(v(x,0))\mathrm{d}x + \gamma\Big(\frac{C_1}{m^{p-1}(\sigma\rho)^p} + \frac{C_2}{m^{p-1}}\Big)\int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{p+\frac{r-1-m}{m}}\mathrm{d}x\mathrm{d}t \\ &\quad + \frac{p}{\sigma\rho}\int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{\frac{r-1}{m}}\varphi_1\mathrm{d}x\mathrm{d}t + \int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{\frac{r-1}{m}}\varphi_2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.6}$$

Since $\frac{1}{r}s^{\frac{1}{m}} \le G(s) \le \frac{1}{m}(s+\varepsilon)^{\frac{r}{m}}$, from (4.5), we have

$$\begin{aligned}\sup_{0\le\tau\le t}\int_{B_\rho} u^r(x,\tau)\mathrm{d}x \le& \int_{B_{2\rho}}(v(x,0)+\varepsilon)^{\frac{r}{m}}\mathrm{d}x + \gamma\frac{C_1}{(\sigma\rho)^p}\int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{p+\frac{r-1-m}{m}}\mathrm{d}x\mathrm{d}t\\ &+(r-1)\int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{\frac{r-1}{m}-1}\varphi_0^{\frac{m-1}{m}}\mathrm{d}x\mathrm{d}t\\ &+\frac{p}{\sigma\rho}\int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{\frac{r-1}{m}}\varphi_1\mathrm{d}x\mathrm{d}t\\ &+\int_0^t\int_{B_{(1+\sigma)\rho}}(v+\varepsilon)^{\frac{r-1}{m}}\varphi_2\mathrm{d}x\mathrm{d}t. \end{aligned}\tag{4.7}$$

Letting $\varepsilon\to 0$, we have

$$\begin{aligned}\sup_{0\le\tau\le t}\int_{B_\rho} u^r(x,\tau)\mathrm{d}x \le& \int_{B_{2\rho}}u^r(x,0)\mathrm{d}x + \gamma\frac{C_1}{(\sigma\rho)^p}\int_0^t\int_{B_{(1+\sigma)\rho}}u^{r-b}(x,\tau)\mathrm{d}x\mathrm{d}\tau\\ &+(r-1)\int_0^t\int_{B_{(1+\sigma)\rho}}u^{r-2}(x,\tau)\varphi_0\mathrm{d}x\mathrm{d}\tau\\ &+\frac{p}{\sigma\rho}\int_0^t\int_{B_{(1+\sigma)\rho}}u^{r-1}(x,\tau)\varphi_1\mathrm{d}x\mathrm{d}\tau\\ &+\int_0^t\int_{B_{(1+\sigma)\rho}}u^{r-1}(x,\tau)\varphi_2\mathrm{d}x\mathrm{d}\tau. \end{aligned}\tag{4.8}$$

Hence, similar to the case of $m\ge 1$, (1.15) follows.

5 Proof of Theorem 1.4

Let $h,k,\mu,\alpha>0$ and

$$\Psi(x,t)=\frac{k(1-|x|^\alpha)_+^{\frac{p}{m(p-1)}}}{(1+hk^{\frac{b}{p-1}}(\frac{|x|^p}{t})^{\frac{1}{p-1}})^{\frac{p-1}{b}}}.\tag{5.1}$$

Lemma 5.1 *Assume that $\lambda=p-Nb>0$. Then the positive constants h and α can be chosen a priori and are only dependent on N, m and p, so that $\forall\, k>0$, $\forall\,\mu>0$ satisfying $\mu h^{-(p-1)}k^{-b}<1$, $\theta=\min\{\mu((\frac{\lambda}{2p})^{p-1},k^b)\}$,*

$$\Psi_t-\operatorname{div}(|D\Psi^m|^{p-2}D\Psi^m)\le 0,\quad a.e.\ in\ Q(\theta),$$

where

$$Q(\theta)=\{\mu h^{-(p-1)}k^{-b}<|x|^p<1\}\times\{0,\theta\}.\tag{5.2}$$

Let $k,\rho,\xi>0$. Consider the function

$$\Phi(x,t)=\frac{k\rho^{p\xi}}{R^\xi}\Big(1-\Big(\frac{|x|^p}{R(t)}\Big)^{\frac{1}{p-1}}\Big)_+^{1+\frac{1}{m}},\tag{5.3}$$

where $R(t)=k^{-b}t+\rho^p$.

Lemma 5.2 *The number $\xi=\xi(N,m,p)$ can be determined a priori only in terms of N, m and p, so that*

$$L\Phi\equiv\Phi_t-\mathrm{div}(|D\Phi^m|^{p-2}D\Phi^m)\le 0,\quad in\ D_{\{k,\xi\}},$$

where $D_{\{k,\xi\}}\equiv\{|x|^p<R(t)\}\times\{0<t<k^b\rho^p\xi^{-1}\}$.

The proof of the above two lemmas is similar to that of the singular evolution p-Laplacian equation, and only a little modification is needed. For details, we refer to [9].

By virtue of the comparison principle (see [5]), and rescaling of variables, we can easily prove the following two lemmas.

Lemma 5.3 *Let $u(x,t)$ be a nonnegative local weak solution to (1.1) in a domain Q containing*

$$Q^*(\theta)\equiv\{\mu h^{(p-1)}k^{p-2}<|x-\overline{x}|^p l^{-p}<1\}\times\{\overline{t},\overline{t}+l^p\theta\}. \tag{5.4}$$

Suppose that u_t is a regular measure on Q of locally bounded variation, and $u(x,t)\ge k$ for $|x-\overline{x}|^p=\mu l^p h^{(p-1)}k^{p-2}$, $\overline{t}<t<\overline{t}+l^p\theta$. Then

$$u(x,t)\ge\Psi\Big(\frac{x-\overline{x}}{l},\frac{t-\overline{t}}{l^p}\Big),\quad in\ Q^*(\theta).$$

Lemma 5.4 *Let $u(x,t)$ be a nonnegative local weak solution to (1.1) in a domain Q containing*

$$D^*_{\{k,\xi\}}\equiv\{|x-\overline{x}|^p<k^{-b}(t-\overline{t})+\rho^p\}\times\{\overline{t},\overline{t}+k^b\rho^p\xi^{-1}\}. \tag{5.5}$$

Suppose that u_t is a regular measure on Q of locally bounded variation, and $u(x,\overline{t})\ge k$ for $|x-\overline{x}|^p<\rho$. Then

$$u(x,t)\ge\Phi(x-\overline{x},t-\overline{t}),\quad in\ D^*_{\{k,\xi\}}.$$

Proof of Theorem 1.4 Let $(x_0,t_0)\in\Omega_T$, $u(x_0,t_0)>0$ and assume that $\rho>0$ is so small that the cylinder

$$Q_{4\rho}(x_0,t_0)\equiv\{|x-x_0|<4\rho\}\times\{t_0-u^b(x_0,t_0)(4\rho)^p,t_0+u^b(x_0,t_0)(4\rho)^p\}\subset\Omega_T.$$

Without loss of generality, we may assume $(x_0,t_0)=(0,0)$. From (1.21), we easily know that the rescaled function $v(x,t)=\frac{1}{u(0,0)}u(\rho x,u^{\frac{1}{b}}(0,0)t\rho^p)$ satisfies $v(0,0)=1$ and

$$v_t-\mathrm{div}(|Dv^m|^{p-2}Dv^m)=\mathbf{b}(x,t,v,Dv),\quad \text{in } Q, \tag{5.6}$$

where $Q=Q^+\cup Q^-$, $Q^+\equiv B_4\times(-4^p,0]$.

To prove Theorem 1.4, it suffices to determine that constants c and γ_0 in $(0,1)$ depend only on N, m and p, such that

$$\inf_{x\in B_1}v(x,c)\ge\gamma_0.$$

For $\tau\in(0,1)$, consider a family of cylinders $Q_\tau\equiv\overline{B}_r\times[-\delta\tau,0]$, $\delta\in(0,1)$ to be chosen later. Let $M_\tau\equiv\sup_{Q_\tau}v$, $N_r\equiv(1-\tau)^{-\frac{p}{b}}$. Since $v\in L^\infty_{\mathrm{loc}}(\Omega_T)$, $M_0=N_0$ and $N_r\to\infty$ as $\tau\to 1$, the equation $M_\tau=N_\tau$ has the largest root, say τ_0. Thus

$$M_{\tau_0}=(1-\tau_0)^{-\frac{p}{b}},\quad M_{\frac{1+r_0}{2}}\le N_{\frac{1+r_0}{2}}=2^{\frac{p}{b}}(1-\tau_0)^{-\frac{p}{b}}.$$

Since v is Hölder continuous in Q (see [10–12]), there exists a point $(\overline{x},\overline{t}) \in \overline{Q}_{\tau_0}$ such that

$$v(\overline{x},\overline{t}) = M_{\tau_0}, \quad \sup_{|x-\overline{x}|} v(x,\overline{t}) < \frac{1-r_0}{2} v(x,\overline{t})_{\frac{1+r_0}{2}} \le 2^{\frac{p}{b}}(1-\tau_0)^{-\frac{p}{b}}.$$

From this and (1.20), we easily obtain

$$\sup_{|x-\overline{x}|<\frac{1-r_0}{8}} v(x,t) < \gamma_1(1-\tau_0)^{-\frac{p}{b}} \equiv W_0, \quad \forall\, t \in [\overline{t}-1,\overline{t}]. \tag{5.7}$$

Constructing the box

$$Q_{R_0} \equiv \{|x-\overline{x}| < R_0\} \times \{\overline{t} - W_0^b R_0^p\}, \quad R_0 = (8\gamma_1^{\frac{b}{p}})^{-1}(1-\tau_0),$$

we have

$$\sup_{Q_{R_0}} v(x,t) \le W_0.$$

From [10–11], we know that $v(x,t)$ is Hölder continuous and

$$\operatorname{osc}_{B_\rho(\overline{x})} v(x,\overline{t}) \le \gamma W_0 (\rho R_0^{-1})^\alpha, \quad \forall\, 0 < \rho < R_0, \tag{5.8}$$

where $\alpha \in (0,1)$, $\gamma > 1$ is a priori determined in terms of N, m and $4\|v\|_\infty$. Combining (5.7) and (5.8), and taking $\rho = \eta R_0$, we have $\forall |x-\overline{x}| \le \eta R_0$,

$$v(x,\overline{t}) \ge v(\overline{x},\overline{t}) - \operatorname{osc}_{B_\rho(\overline{x})} v(x,\overline{t}).$$

Choose η so small that $1-\gamma\gamma_1\eta^\alpha = \frac{1}{2}$. Let $\varepsilon = \eta(8\gamma^{\frac{b}{p}})^{-1}$. We have

$$v(x,\overline{t}) \ge \frac{1}{2(1-\tau_0)^{\frac{p}{b}}}, \quad \forall |x-\overline{x}| \le \varepsilon(1-\tau_0). \tag{5.9}$$

Next we will employ the local subsolutions to expand the positivity set of v both in the direction of increasing t and sidewise in the space variables.

Consider the comparison function $\Phi(x-\overline{x}, t-\overline{t})$ defined (5.5) with the choice

$$k = \frac{1}{2(1-\tau_0)^{\frac{p}{b}}}, \quad \rho = \varepsilon(1-\tau_0) \quad \text{and} \quad k^b \rho p \xi^{-1} = \varepsilon 2^{-b} \xi^{-1} \equiv 3\delta. \tag{5.10}$$

From (5.9), we have $v(x,\overline{t}) \ge k$ for $|x-\overline{x}| < \rho$. By virtue of Lemma 5.4,

$$v(x,t) \ge \Phi(x-\overline{x}, t-\overline{t}), \quad \text{in } D^*_{k,\xi}.$$

In particular, for $\delta < t-\overline{t} < 3\delta$ and $|x-\overline{x}| \le \rho$, we have

$$v(x,t) \ge \frac{1}{2(1-\tau_0)^{\frac{p}{b}}}\Big(1+\frac{1}{\xi}\Big)^{-\xi}\Big(1-\Big(\frac{3\xi}{1+3\xi}\Big)^{\frac{1}{p-1}}\Big)_+^{1+\frac{1}{m}} \equiv C_0 \frac{1}{(1-\tau_0)^{\frac{p}{b}}}.$$

Since $[\delta, 2\delta] \subset [\overline{t}+\delta, \overline{t}+\delta]$, we have

$$v(x,t) \ge C_0 \frac{1}{(1-\tau_0)^{\frac{p}{b}}}, \quad \forall\, x \in B_\rho(\overline{x}),\ \forall\, t \in [\delta, 2\delta]. \tag{5.11}$$

Consider the comparison function $\Psi(\frac{x-\overline{x}}{3}, \frac{t-\delta}{3^p})$ defined in (5.1) in domain $Q_0 \equiv \{\varepsilon(1-\tau) < |x-\overline{x}| < 3\} \times [\delta, 2\delta]$ with the choice $k = C_0 \frac{1}{(1-\tau_0)^{\frac{p}{b}}}$ and $\mu = \delta(\frac{2p}{\lambda})^{p-1}$. From (5.10), we know that δ can be restricted so small that

$$\mu \leq \min\Big\{\frac{1}{4}, \frac{1}{k^b}, \frac{h^{p-1}\varepsilon^p}{C_0^b 3^p}\Big\}.$$

Thus $Q_0 \subset Q^*(\theta)$ is defined in (5.4).

From (5.11), we obtain by Lemma 5.3 that

$$v(x,t) \geq \Psi\Big(\frac{x-\overline{x}}{3}, \frac{t-\delta}{3^p}\Big).$$

In particular, for $t = 2\delta$ and $|x-\overline{x}| < 2$, we have

$$\begin{aligned} v(x, 2\delta) &\geq \frac{(C_0(1-\tau)^{-\frac{p}{b}}(1-\frac{2}{3})^\alpha)^{\frac{p}{m(p-1)}}}{(1+hC_0(1-\tau)^{-\frac{p}{b}})^{\frac{b}{p-1}}\delta^{-\frac{1}{p-1}}2^{\frac{p}{p-1}})^{\frac{p-1}{b}}} \\ &\geq \inf_{0\leq\tau\leq 1} \frac{(C_0(1-\tau)^{-\frac{p}{b}}(1-\frac{2}{3})^\alpha)^{\frac{p}{m(p-1)}}}{(1+hC_0(1-\tau)^{-\frac{p}{b}})^{\frac{b}{p-1}}\delta^{-\frac{1}{p-1}}2^{\frac{p}{p-1}})^{\frac{p-1}{b}}} \\ &\equiv \inf_{0\leq\tau\leq 1} f(\tau) \equiv \gamma_0(N, m, p). \end{aligned}$$

Since $f(\tau)$ is a positive continuous function in $\tau \in (0,1)$ and

$$\lim_{\tau\to 1} f(\tau) = \Big(C_0\delta\Big(1-\frac{2}{3}\Big)^\alpha\Big)^{\frac{p}{m(p-1)}} h^{-\frac{p-1}{b}} 2^{-\frac{p}{b}} > 0,$$

we have $\gamma_0(N, m, p) > 0$ and Theorem 1.4 follows.

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