Chin. Ann. Math.
32B(3), 2011, 417–434
DOI: 10.1007/s11401-011-0643-6

Chinese Annals of
Mathematics, Series B

Homogenization of a Class of Nonlinear Variational Inequalities with Applications in Fluid Film Flow

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Abstract The authors consider the homogenization of a class of nonlinear variational inequalities, which include rapid oscillations with respect to a parameter. The homogenization of the corresponding class of differential equations is also studied. The results are applied to some models for the pressure in a thin fluid film fluid between two surfaces which are in relative motion. This is an important problem in the lubrication theory. In particular, the analysis includes the effects of surface roughness on both faces and the phenomenon of cavitation. Moreover, the fluid can be modeled as Newtonian or non-Newtonian by using a Rabinowitsch fluid model.

Keywords Homogenization, Two-scale convergence, Lubrication, Variational inequalities
2000 MR Subject Classification 80M40, 76M50, 35J85

1 Introduction

Let $x \in \Omega$ denote the space variable, where Ω is an open bounded subset of $\mathbb{R}^N$, and the time variable, denoted by t, belongs to the interval $(0, T)$ and $\Omega_T = \Omega \times (0, T)$. The N-valued functions $a = a(x, t, y, \tau, \xi)$ and $f = f(x, t, y, \tau)$ and the scalar function $g = g(x, t, y, \tau)$ are defined for $\xi \in \mathbb{R}^N$, $y \in \mathbb{R}^N$ and $\tau \in \mathbb{R}$. Moreover, the function a satisfies certain monotonicity and continuity conditions in ξ. It is also assumed that all of a, f and g are Y-periodic in y and Z-periodic in τ, where $Y = (0, 1)^N$ and $Z = (0, 1)$. We can now use a, f and g to define the following functions:

$$\begin{aligned} a_\varepsilon(x, t, \xi) &= a\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \xi\Big), \\ f_\varepsilon(x, t) &= f\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\Big), \\ g_\varepsilon(x, t) &= g\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\Big), \end{aligned}$$

where $\varepsilon > 0$. Since a, f and g are periodic in y and τ, this means that ε is a parameter which describes the frequency of the oscillations.

Manuscript received October 9, 2009. Revised September 21, 2010. Published online April 19, 2011.
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Let us now study the equation: Find $u_\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ such that

$$\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon)\cdot\nabla_x\phi \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}(f_\varepsilon\cdot\nabla_x\phi + g_\varepsilon\phi)\mathrm{d}x\mathrm{d}t \tag{1.1}$$

holds for all $\phi \in L^p(0,T;W_0^{1,p}(\Omega))$. There are only derivatives with respect to x in (1.1), which implies that t is just a parameter. Corresponding to the equation (1.1), we have the variational inequality: Find $u_\varepsilon \in M$ such that

$$\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon)\cdot\nabla_x(\phi-u_\varepsilon) \,\mathrm{d}x\mathrm{d}t \geq \int_{\Omega_T}(f_\varepsilon\cdot\nabla_x(\phi-u_\varepsilon) + g_\varepsilon(\phi-u_\varepsilon))\mathrm{d}x\mathrm{d}t \tag{1.2}$$

holds for all $\phi \in M$, where

$$M = \{u_\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega)) : u_\varepsilon \geq 0\}.$$

For small values of ε, the functions a_ε, f_ε and g_ε are rapidly oscillating both in space and time. A direct numerical treatment of (1.1) and (1.2) will thus require an enormous number of grid points. Therefore, it is natural to use some type of asymptotic analysis to be able to handle problems of this type. The field of mathematics which has been developed for this purpose is known as homogenization (see, e.g., [17] or [24]). The main contribution of this paper is that we prove homogenization results corresponding to (1.1) and (1.2). This means that we extend the previous results in [8], where the linear problems corresponding to (1.1) and (1.2) were analyzed.

We also show that our general homogenization result can be applied to the analysis of some problems in theory of lubrication. More precisely, we will study the pressure built up in a fluid between two rough surfaces (e.g. in a bearing) which are in relative motion. If the surfaces are rough then the distance between the surfaces will oscillate rapidly both in space and time, which indicates that homogenization can be used priority. A fluid cannot sustain negative pressure. This implies that there may be zones where the lubricant contains air bubbles. This phenomenon is known as cavitation and has a big impact on the hydrodynamic performance. There are several ways to model and analyze this effect. One approach leads to variational inequalities of the type (1.2).

In homogenization both (1.1) and (1.2), we prove that the respective sequence of solutions (u_ε) two-scale converges to $u = u(x,t,\tau)$. In the equation case, u solves a homogenized equation, and in the variational inequality case, u solves a homogenized variational inequality. The homogenized equation is as follows: Find $u \in L^p((0,T)\times Z;W_0^{1,p}(\Omega))$ such that

$$\int_{\Omega_T}\int_Z A(x,t,\tau,\nabla_x u)\cdot\nabla_x\phi \,\mathrm{d}\tau\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Z(F\cdot\nabla_x\phi + G\phi \,\mathrm{d}\tau)\mathrm{d}x\mathrm{d}t \tag{1.3}$$

holds for any $\phi \in L^p((0,T)\times Z;W_0^{1,p}(\Omega))$. The homogenized variational inequality is as follows: Find $u \in V = \{u \in L^p((0,T)\times Z;W_0^{1,p}(\Omega)) : u \geq 0\}$ such that

$$\int_{\Omega_T}\int_Z A(x,t,\tau,\nabla_x u)\cdot\nabla_x(\phi-u) \,\mathrm{d}\tau\mathrm{d}x\mathrm{d}t \geq \int_{\Omega_T}\int_Z(F\cdot\nabla_x(\phi-u) + G(\phi-u) \,\mathrm{d}\tau)\mathrm{d}x\mathrm{d}t \tag{1.4}$$

holds for any $\phi \in V$.

The functions A, F and G in (1.3) and (1.4) are found by solving certain local problems and averaging. The benefit of the homogenized problems for u is that there are no rapid oscillations involved. This means that it is much easier to find the homogenized solution u than the solution u_ε. Roughly speaking, due to the convergence $u_\varepsilon \to u$, the homogenized solution u can be seen as a good approximation of u_ε.

The paper is organized in the following way: Some notations, the precise setting of (1.1) and (1.2) as well as some preliminary results are given in Section 2. In Section 3, we state and develop some results concerning two-scale convergence in order to be able to homogenize (1.1) and (1.2). The variational inequality (1.2) is homogenized in Section 4, and the equation (1.1) is homogenized in Section 5. Several applications in lubrication theory, where the homogenization results can be applied, are discussed in Section 6.

2 Notation and Preliminaries

Let $a : \Omega_T \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^N$ be a function with the following properties: The function $(y, \tau) \to a(x, t, y, \tau, \xi)$ is measurable for every $x \in \Omega$, $t \in (0, T)$ and $\xi \in \mathbb{R}^N$; the function $a(x, t, \cdot, \tau, \xi)$ is Y-periodic; the function $a(x, t, y, \cdot, \xi)$ is Z-periodic. The function $(x, t, \xi) \to a(x, t, y, \tau, \xi)$ is continuous for a.e. y and τ. Let $1 < p < \infty$ and q be the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. We assume that there exist constants $c_1, c_2 > 0$ and two more constants α and β with $0 \leq \alpha \leq \min\{1, p-1\}$ and $\max\{p, 2\} \leq \beta < \infty$ such that a satisfies the following continuity and monotonicity assumptions:

$$a(x, t, y, \tau, 0) = 0, \tag{2.1}$$

$$|a(x, t, y, \tau, \xi_1) - a(x, t, y, \tau, \xi_2)| \leq c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha}|\xi_1 - \xi_2|^\alpha, \tag{2.2}$$

$$(a(x, t, y, \tau, \xi_1) - a(x, t, y, \tau, \xi_2)) \cdot (\xi_1 - \xi_2) \geq c_2(1 + |\xi_1| + |\xi_2|)^{p-\beta}|\xi_1 - \xi_2|^\beta \tag{2.3}$$

for a.e. $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$ and any $\xi_1, \xi_2 \in \mathbb{R}^N$. A direct consequence of (2.1)–(2.3) is that there exist constants $c_3, c_4 > 0$ such that

$$|a(x, t, y, \tau, \xi)| \leq c_3(1 + |\xi|^{p-1}) \quad \text{and} \quad |\xi|^p \leq c_4(1 + a(x, t, y, \tau, \xi) \cdot \xi). \tag{2.4}$$

The functions $f = f(x, t, y, \tau)$ and $g = g(x, t, y, \tau)$ are assumed to be Y-periodic in y and Z-periodic in τ. Moreover, $f \in L^q_{\mathrm{per}}(Y \times Z; C(\overline{\Omega}_T))^N$ and $g \in L^q_{\mathrm{per}}(Y \times Z; C(\overline{\Omega}_T))$.

Now, a, f and g may be used to define the following functions:

$$a_\varepsilon(x, t, \xi) = a\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \xi\Big), \tag{2.5}$$

$$f_\varepsilon(x, t) = f\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\Big), \tag{2.6}$$

$$g_\varepsilon(x, t) = g\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\Big). \tag{2.7}$$

Without loss of generality, we shall from now on assume that $|Y| = |Z| = 1$ unless something else is stated.

The functions A, F and G, in the homogenized equation (1.3) and the homogenized variational inequality (1.4), are defined as

$$A(x, t, \tau, \xi) = \int_Y a(x, t, y, \tau, \xi + \nabla_y w_\xi)\, \mathrm{d}y, \tag{2.8}$$

where w_ξ is the solution to the local problem: Find $w_\xi \in W^{1,p}_{\text{per}}(Y)$ such that

$$\int_Y [a(x,t,y,\tau,\xi+\nabla w_\xi(y)) - f(x,t,y,\tau)]\cdot\nabla\phi \, \mathrm{d}y = 0$$

for every $\phi \in W^{1,p}_{\text{per}}(Y)$,

$$F(x,t,\tau) = \int_Y f(x,t,y,\tau)\, \mathrm{d}y \quad \text{and} \quad G(x,t,\tau) = \int_Y g(x,t,y,\tau)\, \mathrm{d}y. \tag{2.9}$$

By using similar arguments as in [32], it can be shown that the properties of a imply that the homogenized operator A defined in (2.8) has the following continuity and monotonicity properties: There exist constants $\widetilde{c}_1$, $\widetilde{c}_2 > 0$ such that

$$|A(x,t,\tau,\xi_1) - A(x,t,\tau,\xi_2)| \le \widetilde{c}_1(1+|\xi_1|+|\xi_2|)^{p-1-\frac{\alpha}{\beta-\alpha}}|\xi_1-\xi_2|^{\frac{\alpha}{\beta-\alpha}}, \tag{2.10}$$

$$(A(x,t,\tau,\xi_1) - A(x,t,\tau,\xi_2))\cdot(\xi_1-\xi_2) \ge \widetilde{c}_2(1+|\xi_1|+|\xi_2|)^{p-\beta}|\xi_1-\xi_2|^{\beta} \tag{2.11}$$

for any $\xi_1, \xi_2 \in \mathbb{R}^N$.

The continuity and monotonicity conditions of a and A guarantee that there exists a unique solution to equation (1.1), variational inequality (1.2), corresponding homogenized equation (1.3) and homogenized variational inequality (1.4). The proofs of these facts are based on standard existence and uniqueness results (see, e.g., [37, Theorem 26.A, p. 557 and Theorem 32.C, p. 875].

3 Two-Scale Convergence

The concept of two-scale convergence was developed in the study of different types of homogenization problems. For more information concerning the general theory of two-scale convergence, we refer to [1, 31, 34]. In this section, we present and develop the ideas such that they can be used to homogenize (1.1) and (1.2). Let us start with the following definition.

Definition 3.1 *Let (u_ε) be a bounded sequence in $L^p(\Omega_T)$ and $u \in L^p(\Omega_T \times Y \times Z)$. Then we say that (u_ε) two-scale converges to u if*

$$\int_0^T\int_\Omega u_\varepsilon(x,t)\phi\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\,\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega\int_Y\int_Z u(x,t,y,\tau)\phi(x,t,y,\tau)\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \tag{3.1}$$

as $\varepsilon \to 0$, for every test function $\phi \in C_0^\infty(\Omega_T; C^\infty_{\text{per}}(Y\times Z))$.

We remark that if the measure of Y and Z is not one, then we have to divide the right-hand side of (3.1) by $|Y||Z|$. This fact should be kept in mind throughout this paper. We also note that two-scale convergence implies that u_ε converges weakly to $\overline{u} = \int_Y\int_Z u\,\mathrm{d}\tau\mathrm{d}y$ in $L^p(\Omega_T)$.

One of the main results concerning two-scale convergence is the following compactness result (the proof of this result may be found in e.g. [1, 31]):

Theorem 3.1 *If (u_ε) is a bounded sequence in $L^p(\Omega_T)$, then there exist a subsequence and a $u \in L^p(\Omega_T\times Y\times Z)$ such that the subsequence two-scale converges to u.*

In addition to two-scale convergence, one can also define the concept of strong two-scale convergence.

Definition 3.2 *Let (u_ε) be a bounded sequence in $L^p(\Omega_T)$ and $u \in L^p(\Omega_T \times Y \times Z)$. Then we say (u_ε) two-scale converges strongly to u if for any bounded sequence (v_ε) in $L^q(\Omega_T)$ which two-scale converges to $v \in L^q(\Omega_T \times Y \times Z)$, we have that*

$$\int_{\Omega_T} u_\varepsilon v_\varepsilon \, \mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} \int_Y \int_Z uv \, \mathrm{d}\tau \mathrm{d}x\mathrm{d}t.$$

The following theorem relates two-scale convergence and strong two-scale convergence (see, e.g., [31]).

Theorem 3.2 *Two-scale convergence of the sequence (u_ε) in $L^p(\Omega_T)$ to $u \in L^p(\Omega_T \times Y \times Z)$ together with*

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} |u_\varepsilon|^p \, \mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \int_Y \int_Z |u|^p \, \mathrm{d}\tau \mathrm{d}y\mathrm{d}x\mathrm{d}t \tag{3.2}$$

is equivalent to strong two-scale convergence of (u_ε) to u.

Let $\mathcal{A}_q$ be the set of functions $u \in L^q(\Omega_T \times Y \times Z)$ such that $u_\varepsilon(x,t) = u(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ is measurable and satisfies the following two convergence properties:

(1) The sequence (u_ε) two-scale converges to u.

(2) The sequence (u_ε) satisfies $\lim\limits_{\varepsilon \to 0} \|u_\varepsilon\|_{L^q(\Omega_T)} = \|u\|_{L^q(\Omega_T \times Y \times Z)}$.

For example, functions in $L^q_{\mathrm{per}}(Y \times Z; C(\overline{\Omega}_T)) \cup L^q(\Omega_T; C_{\mathrm{per}}(Y \times Z))$ belong to $\mathcal{A}_q$ (for details, see e.g. [1, 31, 32]). By Theorem 3.2, we have that if $u \in \mathcal{A}_q$, then (u_ε) two-scale converges strongly to $u \in L^q(\Omega_T \times Y \times Z)$. In applications of two-scale convergence, it is often important to enlarge the class of test functions ϕ for which the convergence (3.1) holds. Indeed, we have the following corollary of Theorem 3.2.

Corollary 3.1 *If a sequence (u_ε) in $L^p(\Omega_T)$ two-scale converges to $u \in L^p(\Omega_T \times Y \times Z)$, then*

$$\int_\Omega u_\varepsilon \phi\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big) \, \mathrm{d}\tau \mathrm{d}y\mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} \int_Y \int_Z u\phi \, \mathrm{d}\tau \mathrm{d}y\mathrm{d}x\mathrm{d}t$$

for any $\phi \in \mathcal{A}_q$.

Because of this, we will call the functions $\mathcal{A}_q$ admissible test functions. Another consequence of Theorem 3.2 is the following corollary.

Corollary 3.2 *Let (u_ε) be a bounded sequence in $L^p(\Omega_T)$ which two-scale converges to $u \in L^p(\Omega_T \times Y \times Z)$ and also satisfies that*

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} |u_\varepsilon|^p \, \mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \int_Y \int_Z |u|^p \, \mathrm{d}\tau \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

If the limit u belongs to $\mathcal{A}_p$, then

$$\int_{\Omega_T} \Big|u_\varepsilon(x,t) - u\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big|^p \mathrm{d}x\mathrm{d}t \to 0. \tag{3.3}$$

This is realized by the following argumentation: By Theorem 3.2 u_ε two-scale converges strongly to $u \in L^p(\Omega_T \times Y \times Z)$. Since $u \in \mathcal{A}_p$, we also have that $u(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ two-scale converges strongly to $u \in L^p(\Omega_T \times Y \times Z)$. Hence, it follows that $u_\varepsilon(x,t) - u(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ two-scale converges strongly to 0 and (3.3) holds by taking Theorem 3.2 into account again.

The following compactness result for sequences in $L^p(0,T;W_0^{1,p}(\Omega))$ was proved in [30].

Theorem 3.3 *Assume that (u_ε) is a sequence in $L^p(0,T;W_0^{1,p}(\Omega))$ such that $u_\varepsilon(x,t)$ two-scale converges to $u(x,t,y,\tau)$ and $\nabla_x u_\varepsilon(x,t)$ two-scale converges to $z(x,t,y,\tau)$. Then the two-scale limit u is independent of y and $u \in L^p((0,T)\times Z;W_0^{1,p}(\Omega))$. Moreover, $z(x,t,y,\tau) = \nabla_x u(x,t,\tau) + \nabla_y u_1(x,t,y,\tau)$, where $u_1 \in L^p(\Omega_T \times Z, W^{1,p}_{\mathrm{per}}(Y))$.*

We will now derive two results regarding two-scale convergence and monotonicity. As we will see later that both of them are fundamental in the homogenization of (1.1) and (1.2). The first theorem is as follows.

Theorem 3.4 *Let a satisfy the conditions (2.1)–(2.3). Moreover, let (v_ε) be a bounded sequence in $L^p(\Omega_T)^N$ which two-scale converges to $v \in L^p(\Omega_T \times Y \times Z)^N$ and assume that $a_\varepsilon(x,t,v_\varepsilon(x,t))$ two-scale converges to $a_0(x,t,y,\tau) \in L^q(\Omega_T \times Y \times Z)^N$. Then*

$$\liminf_{\varepsilon\to 0} \int_{\Omega_T} a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon},v_\varepsilon(x,t)\Big)\cdot v_\varepsilon(x,t)\,\mathrm{d}x\mathrm{d}t$$
$$\geq \int_{\Omega_T}\int_Y\int_Z a_0(x,t,y,\tau)\cdot v(x,t,y,\tau)\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t, \tag{3.4}$$

and if (3.4) holds as equality, then $a_0(x,t,y,\tau) = a(x,t,y,\tau,v(x,t,y,\tau))$.

Proof We prove (3.4). Assume that (3.4) does not hold, i.e.,

$$\liminf_{\varepsilon\to 0} \int_{\Omega_T} a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon},v_\varepsilon(x,t)\Big)\cdot v_\varepsilon(x,t)\,\mathrm{d}x\mathrm{d}t$$
$$< \int_{\Omega_T}\int_Y\int_Z a_0(x,t,y,\tau)\cdot v(x,t,y,\tau)\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

Then there exists a positive constant $k > 0$ such that

$$\liminf_{\varepsilon\to 0} \int_{\Omega_T} a_\varepsilon(x,t,v_\varepsilon)\cdot v_\varepsilon\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Y\int_Z a_0\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t - k. \tag{3.5}$$

Let $\phi \in C_0^\infty(\Omega_T;C^\infty_{\mathrm{per}}(Y\times Z)^N)$ and define $\phi_\varepsilon(x,t) = \phi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$. From the monotonicity assumption (2.3) on a, we can deduce

$$\int_{\Omega_T} [a_\varepsilon(x,t,v_\varepsilon) - a_\varepsilon(x,t,\phi_\varepsilon)]\cdot[v_\varepsilon - \phi_\varepsilon]\,\mathrm{d}x\mathrm{d}t \geq 0. \tag{3.6}$$

Since it is assumed that the function $(x,t,\xi) \to a(x,t,y,\tau,\xi)$ is continuous and $\phi \in C_0^\infty(\Omega_T;C^\infty_{\mathrm{per}}(Y\times Z)^N)$, we have that $a_\varepsilon(x,t,\phi_\varepsilon)$ is an admissible test function, but in fact $a_\varepsilon(x,t,\phi_\varepsilon)$ belongs to $L^q_{\mathrm{per}}(Y\times Z;C(\overline{\Omega}_T)) \subset \mathcal{A}_q$. This together with Corollary 3.1 implies

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} a_\varepsilon(x,t,\phi_\varepsilon)\cdot v_\varepsilon\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Y\int_Z a(x,t,y,\tau,\phi)\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t, \tag{3.7}$$
$$\lim_{\varepsilon\to 0}\int_{\Omega_T} a_\varepsilon(x,t,\phi_\varepsilon)\cdot \phi_\varepsilon\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Y\int_Z a(x,t,y,\tau,\phi)\cdot \phi\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \tag{3.8}$$

Let us now apply $\liminf_{\varepsilon\to 0}$ to both sides in the inequality (3.6) (actually, the limit exists for all three terms in the right-hand side). With the help of relations (3.5), (3.7) and (3.8), we obtain that

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - a(x,t,y,\tau,\phi)]\cdot[v-\phi]\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \geq k. \tag{3.9}$$

By density and continuity (see [18, p. 77]), this inequality also holds for any $\phi \in L^p(\Omega_T \times Y \times Z)^N$. Let $sw(x,t,y,\tau) = v(x,t,y,\tau) - \phi(x,t,y,\tau)$, $s > 0$. Then after dividing it by s, we get

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - a(x,t,y,\tau,v-sw)]\cdot w\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \geq \frac{k}{s}.$$

Letting $s \to 0$, we have

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - a(x,t,y,\tau,v)]\cdot w\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \geq \infty.$$

By repeating the procedure for $s < 0$, it implies that

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - a(x,t,y,\tau,v)]\cdot w\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \leq -\infty.$$

This is a contradiction, so that (3.4) must hold.

If (3.4) holds as equality, we can repeat the procedure above (with $k = 0$), and we get that

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - a(x,t,y,\tau,v)]\cdot w\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t = 0$$

for any $w \in L^p(\Omega_T \times Y \times Z)^N$. Hence $a_0(x,t,y,\tau) = a(x,t,y,\tau,v(x,t,y,\tau))$.

The second theorem concerning two-scale convergence and monotonicity which is important in the homogenization of (1.1) and (1.2) is the theorem below.

Theorem 3.5 *Let a satisfy the conditions* (2.1)–(2.3). *Moreover, let (v_ε) be a bounded sequence in $L^p(\Omega_T)^N$ such that v_ε two-scale converges to $v \in L^p(\Omega_T \times Y \times Z)^N$, $|v_\varepsilon|^{p-2}v_\varepsilon$ two-scale converges to $w \in L^q(\Omega_T \times Y \times Z)^N$ and $a_\varepsilon(x,t,v_\varepsilon(x,t))$ two-scale converges to $a_0(x,t,y,\tau) \in L^q(\Omega_T \times Y \times Z)^N$. If*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} a_\varepsilon(x,t,v_\varepsilon)\cdot v_\varepsilon\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Y\int_Z a_0\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t, \tag{3.10}$$

then v_ε two-scale converges strongly to $v \in L^p(\Omega_T \times Y \times Z)^N$.

Proof The left-hand side of (3.10) can be rewritten as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T} a_\varepsilon(x,t,v_\varepsilon)\cdot v_\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&= \limsup_{\varepsilon\to 0}\int_{\Omega_T} k|v_\varepsilon|^p\,\mathrm{d}x\mathrm{d}t - \limsup_{\varepsilon\to 0}\int_{\Omega_T} k|v_\varepsilon|^p\,\mathrm{d}x\mathrm{d}t + \lim_{\varepsilon\to 0}\int_{\Omega_T} a_\varepsilon(x,t,v_\varepsilon)\cdot v_\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&= \limsup_{\varepsilon\to 0}\int_{\Omega_T} k|v_\varepsilon|^p\,\mathrm{d}x\mathrm{d}t + \liminf_{\varepsilon\to 0}\int_{\Omega_T} [a_\varepsilon(x,t,v_\varepsilon) - k|v_\varepsilon|^{p-2}v_\varepsilon]\cdot v_\varepsilon\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

This together with (3.10) gives

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_{\Omega_T} k|v_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t+\liminf_{\varepsilon\to 0}\int_{\Omega_T}[a_\varepsilon(x,t,v_\varepsilon)-k|v_\varepsilon|^{p-2}v_\varepsilon]\cdot v_\varepsilon \,\mathrm{d}x\mathrm{d}t\\ &=\int_{\Omega_T}\int_Y\int_Z kw\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t+\int_{\Omega_T}\int_Y\int_Z (a_0-kw)\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \qquad (3.11)\end{aligned}$$

For $k>0$ sufficiently small, the function $a(x,t,y,\tau,\xi)-k|\xi|^{p-2}\xi$ satisfies the conditions in Theorem 3.4, therefore

$$\liminf_{\varepsilon\to 0}\int_{\Omega_T}[a_\varepsilon(x,t,v_\varepsilon)-k|v_\varepsilon|^{p-2}v_\varepsilon]\cdot v_\varepsilon \,\mathrm{d}x\mathrm{d}t\geq\int_{\Omega_T}\int_Y\int_Z (a_0-kw)\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

This and (3.11) imply

$$\limsup_{\varepsilon\to 0}\int_{\Omega_T}|v_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t\leq\int_{\Omega_T}\int_Y\int_Z w\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \qquad (3.12)$$

The function $|\xi|^{p-2}\xi$ satisfies the conditions in Theorem 3.4 which implies that

$$\liminf_{\varepsilon\to 0}\int_{\Omega_T}|v_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t=\liminf_{\varepsilon\to 0}\int_{\Omega_T}|v_\varepsilon|^{p-2}v_\varepsilon\cdot v_\varepsilon \,\mathrm{d}x\mathrm{d}t\geq\int_{\Omega_T}\int_Y\int_Z w\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \qquad (3.13)$$

By (3.12) and (3.13), it follows that

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}|v_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t=\lim_{\varepsilon\to 0}\int_{\Omega_T}|v_\varepsilon|^{p-2}v_\varepsilon\cdot v_\varepsilon \,\mathrm{d}x\mathrm{d}t=\int_{\Omega_T}\int_Y\int_Z w\cdot v\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

Hence it follows by Theorem 3.4 that $w=|v|^{p-2}v$, and Theorem 3.2 gives that v_ε two-scale converges strongly to $v\in L^p(\Omega_T\times Y\times Z)^N$.

4 Homogenization of the Variational Inequality

In this section, we prove a homogenization result for the class of variational inequalities (1.2).

Let $M=\{u\in L^p(0,T;W_0^{1,p}(\Omega)):u\geq 0\}$ and consider the problem: Find $u_\varepsilon\in M$ such that

$$\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon)\cdot\nabla_x(\phi-u_\varepsilon)\,\mathrm{d}x\mathrm{d}t\geq\int_{\Omega_T}(f_\varepsilon\cdot\nabla_x(\phi-u_\varepsilon)+g_\varepsilon(\phi-u_\varepsilon))\mathrm{d}x\mathrm{d}t \qquad (4.1)$$

holds for all $\phi\in M$. Then the sequence of solutions, (u_ε), two-scale converges to $u\in V$, where

$$V=\{u\in L^p((0,T)\times Z;W_0^{1,p}(\Omega)):u\geq 0\}.$$

Moreover, u solves the homogenized variational inequality: Find $u\in V$, such that

$$\int_{\Omega_T}\int_Z A(x,t,\tau,\nabla_x u)\cdot\nabla_x(\phi-u)\,\mathrm{d}\tau\mathrm{d}x\mathrm{d}t\geq\int_{\Omega_T}\int_Z(F\cdot\nabla_x(\phi-u)+G(\phi-u))\mathrm{d}\tau\mathrm{d}x\mathrm{d}t \qquad (4.2)$$

holds for any $\phi\in V$ (recall that A, F and G are defined in (2.8) and (2.9), respectively). More precisely, we have the following result.

Theorem 4.1 *The sequence of solutions u_ε to* (4.1) *two-scale converges to $u \in L^p((0,T) \times Z; W_0^{1,p}(\Omega))$, where u is the unique solution to the homogenized variational inequality* (4.2). *Moreover, $\nabla_x u_\varepsilon$ two-scale converges strongly to $\nabla_x u(x,t,\tau) + \nabla_y u_1(x,t,y,z)$, where $u_1 \in L^p(\Omega_T \times Z; W_{\rm per}^{1,p}(Y))$ and*

$$\int_Y [a(x,t,y,\tau,\nabla_x u(x,t,\tau) + \nabla_y u_1(x,t,y,z)) - f(x,t,y,\tau)] \cdot \nabla \phi \ {\rm d}y = 0,$$

for every $\phi \in W_{\rm per}^{1,p}(Y)$.

Proof We divide the proof into several steps.

Step 1 Let us start by proving that $a_\varepsilon(x,t,\nabla_x u_\varepsilon)$ is bounded in $L^q(\Omega_T)^N$ and that $\nabla_x u_\varepsilon$ is bounded in $L^p(\Omega_T)^N$. By choosing $\phi = 0$ in (4.1), we obtain the following inequality:

$$\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon) \cdot \nabla_x u_\varepsilon \ {\rm d}x{\rm d}t \le \int_{\Omega_T} (f_\varepsilon \cdot \nabla_x u_\varepsilon + g_\varepsilon u_\varepsilon){\rm d}x{\rm d}t. \tag{4.3}$$

The estimate (2.4) and the inequality (4.3) imply that

$$\begin{aligned}\int_{\Omega_T} |\nabla_x u_\varepsilon|^p {\rm d}x{\rm d}t &\le c_4 \int_{\Omega_T} (1 + a_\varepsilon(x,\nabla_x u_\varepsilon) \cdot \nabla_x u_\varepsilon){\rm d}x{\rm d}t \\ &\le c_4(|\Omega_T| + \|f_\varepsilon\|_{L^q(\Omega_T)^N} \|\nabla_x u_\varepsilon\|_{L^p(\Omega_T)^N} + \|g_\varepsilon\|_{L^q(\Omega_T)} \|u_\varepsilon\|_{L^p(\Omega_T)}).\end{aligned}$$

The Poincaré inequality gives that there exists a $c > 0$, such that

$$\|\nabla_x u_\varepsilon\|^p_{L^p(\Omega_T)^N} \le c(1 + \|\nabla_x u_\varepsilon\|_{L^p(\Omega_T)^N}).$$

If $\|\nabla_x u_\varepsilon\|_{L^p(\Omega_T)^N} \le 1$, we could do so to assume the opposite. Then it holds that

$$\|\nabla_x u_\varepsilon\|^p_{L^p(\Omega_T)^N} \le 2c\|\nabla_x u_\varepsilon\|_{L^p(\Omega_T)^N}.$$

From this, it is clear that the sequence (u_ε) is bounded in $L^p(0,T; W_0^{1,p}(\Omega))$. The continuity assumption (2.2) gives that

$$|a_\varepsilon(x,t,\nabla_x u_\varepsilon)| \le c_1(1 + |\nabla_x u_\varepsilon|)^{p-1-\alpha} |\nabla_x u_\varepsilon|^\alpha \le c_1(1 + |\nabla_x u_\varepsilon|)^{p-1},$$

which gives that $a_\varepsilon(x,t,\nabla_x u_\varepsilon)$ is bounded in $L^q(\Omega_T)^N$, since $\nabla_x u_\varepsilon$ is bounded in $L^p(\Omega_T)^N$.

By Theorem 3.1, there exists a subsequence (still denoted by ε), such that

(1) There exists a $u \in L^p(\Omega_T \times Y \times Z)$, such that u_ε two-scale converges to u,

(2) There exists an $\eta \in L^p(\Omega_T \times Y \times Z)^N$, such that $\nabla_x u_\varepsilon$ two-scale converges to η,

(3) There exists an $a_0 \in L^q(\Omega_T \times Y \times Z)^N$, such that $a_\varepsilon(x,t,\nabla_x u_\varepsilon)$ two-scale converges to a_0.

In addition, it follows from Theorem 3.3 that $u \in L^p((0,T) \times Z; W_0^{1,p}(\Omega))$ (in particular u does not depend on y) and $\eta(x,t,y,\tau) = \nabla_x u(x,t,\tau) + \nabla_y u_1(x,t,y,\tau)$, where $u_1 \in L^p(\Omega_T \times Z, W_{\rm per}^{1,p}(Y))$.

Step 2 Let us now show that

$$\int_{\Omega_T} \int_Y \int_Z [a_0 - f] \cdot \nabla_y \phi_1 \ {\rm d}\tau{\rm d}y{\rm d}x{\rm d}t = 0 \tag{4.4}$$

for any $\phi_1 \in C_0^\infty(\Omega_T; C_{\mathrm{per}}^\infty(Y \times Z))$. Fix $\phi_1 \in C_0^\infty(\Omega_T; C_{\mathrm{per}}^\infty(Y \times Z))$. Choose $\phi_0 \in C_0^\infty(\Omega_T)$ such that

$$\phi_0(x,t) \geq \Big|\phi_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big|.$$

We can then choose

$$\phi(x,t) = \varepsilon\Big[\phi_0(x,t) + \phi_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big] + u_\varepsilon(x,t)$$

as a test function in (4.1). Indeed,

$$\int_{\Omega_T} ([a_\varepsilon(x,t,\nabla_x u_\varepsilon) - f_\varepsilon]\cdot[\varepsilon\nabla_x\phi_0 + \varepsilon\nabla_x\phi_1 + \nabla_y\phi_1] - g_\varepsilon[\varepsilon\phi_0 + \varepsilon\phi_1])\mathrm{d}x\mathrm{d}t \geq 0.$$

Passing to the limit gives

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - f]\cdot\nabla_y\phi_1 \,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \geq 0. \tag{4.5}$$

By choosing ϕ as

$$\phi(x,t) = \varepsilon\Big[\phi_0(x,t) - \phi_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big] + u_\varepsilon(x,t),$$

we obtain the reversed inequality, i.e.,

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - f]\cdot\nabla_y\phi_1 \,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \leq 0. \tag{4.6}$$

From (4.5) and (4.6) it is clear that (4.4) holds.

Step 3 Let $\phi \in C_0^\infty(\Omega_T; C_{\mathrm{per}}^\infty(Z))$ and assume $\phi \geq 0$. Define the function ϕ_ε as $\phi_\varepsilon(x,t) = \phi(x,t,\frac{t}{\varepsilon})$. From the variational inequality (4.1), we have

$$\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon)\cdot\nabla_x(\phi_\varepsilon - u_\varepsilon)\mathrm{d}x\mathrm{d}t \geq \int_{\Omega_T}(f_\varepsilon\cdot\nabla_x(\phi_\varepsilon - u_\varepsilon) + g_\varepsilon(\phi_\varepsilon - u_\varepsilon))\mathrm{d}x\mathrm{d}t.$$

In view of Theorem 3.4, we obtain that in the limit

$$\begin{aligned}&\int_{\Omega_T}\int_Y\int_Z a_0\cdot[\nabla_x\phi - \nabla_x u - \nabla_y u_1]\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t\\ \geq &\int_{\Omega_T}\int_Y\int_Z (f\cdot[\nabla_x\phi - \nabla_x u - \nabla_y u_1] + g(\phi - u))\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.7}$$

By density, it is possible to choose $\phi_1 = u_1$ in (4.4). This together with (4.7) gives that

$$\int_{\Omega_T}\int_Y\int_Z a_0\cdot\nabla_x(\phi - u)\,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \geq \int_{\Omega_T}\int_Y\int_Z (f\cdot\nabla_x(\phi - u) + g(\phi - u))\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \tag{4.8}$$

for any $\phi \in C_0^\infty(\Omega_T; C_{\mathrm{per}}^\infty(Z))$ such that $\phi \geq 0$.

Step 4 Let us now show that

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon)\cdot\nabla_x u_\varepsilon \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Y\int_Z (f\cdot[\nabla_x u + \nabla_y u_1] + gu)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \tag{4.9}$$

First, choosing $\phi = 0$ in (4.1), we have

$$\int_{\Omega_T} a_\varepsilon(x, t, \nabla_x u_\varepsilon) \cdot \nabla_x u_\varepsilon \ \mathrm{d}x\mathrm{d}t \leq \int_{\Omega_T} (f_\varepsilon \cdot \nabla_x u_\varepsilon + g_\varepsilon u_\varepsilon)\mathrm{d}x\mathrm{d}t.$$

In the limit, we get

$$\limsup_{\varepsilon \to 0} \int_{\Omega_T} a_\varepsilon(x, \nabla_x u_\varepsilon) \cdot \nabla_x u_\varepsilon \ \mathrm{d}x\mathrm{d}t \leq \int_{\Omega_T} \int_Y \int_Z (f \cdot [\nabla_x u + \nabla_y u_1] + gu)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \quad (4.10)$$

Next, choosing $\phi = 2u_\varepsilon$ in (4.1), we get

$$\int_{\Omega_T} a_\varepsilon(x, t, \nabla_x u_\varepsilon) \cdot \nabla_x u_\varepsilon \ \mathrm{d}x\mathrm{d}t \geq \int_{\Omega_T} (f_\varepsilon \cdot \nabla_x u_\varepsilon + g_\varepsilon u_\varepsilon)\mathrm{d}x\mathrm{d}t.$$

In the limit, we get

$$\liminf_{\varepsilon \to 0} \int_{\Omega_T} a_\varepsilon(x, \nabla_x u_\varepsilon) \cdot \nabla_x u_\varepsilon \ \mathrm{d}x\mathrm{d}t \geq \int_{\Omega_T} \int_Y \int_Z (f \cdot [\nabla_x u + \nabla_y u_1] + gu)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \quad (4.11)$$

The relation (4.9) is now followed by (4.10) and (4.11).

Step 5 Prove that

$$\int_{\Omega_T} \int_Y \int_Z a_0 \cdot [\nabla_x u + \nabla_y u_1] \ \mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \int_Y \int_Z (f \cdot [\nabla_x u + \nabla_y u_1] + gu)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \quad (4.12)$$

By choosing $\phi = 0$ in (4.8), we get

$$\int_{\Omega_T} \int_Y \int_Z a_0 \cdot \nabla_x u \ \mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \leq \int_{\Omega_T} \int_Y \int_Z (f \cdot \nabla_x u + gu)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

The reversed inequality is obtained by choosing $\phi = 2u$ in (4.8), which is possible after a density argument. Hence,

$$\int_{\Omega_T} \int_Y \int_Z a_0 \cdot \nabla_x u \ \mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \int_Y \int_Z (f \cdot \nabla_x u + gu)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

This together with (4.4) and a density argument implies (4.12).

Step 6 The relations (4.9) and (4.12) give that

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} a_\varepsilon(x, t, \nabla_x u_\varepsilon) \cdot \nabla_x u_\varepsilon \ \mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \int_Y \int_Z a_0 \cdot [\nabla_x u + \nabla_y u_1] \ \mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

Theorem 3.4 about two-scale convergence and monotonicity gives that

$$a_0(x, t, y, \tau) = a(x, t, y, \tau, \nabla_x u(x, t, \tau) + \nabla_y u_1(x, t, y, \tau)). \quad (4.13)$$

From Theorem 3.5 about strong two-scale convergence and monotonicity, it follows that $\nabla_x u_\varepsilon$ two-scale converges strongly to $\nabla_x u + \nabla_y u_1$.

Step 7 Finally, the relations (4.13) and (4.8) imply the homogenized variational inequality

$$\begin{aligned} &\int_{\Omega_T} \int_Y \int_Z a(x, t, y, \tau, \nabla_x u + \nabla_y u_1) \cdot \nabla_x(\phi - u) \ \mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ \geq &\int_{\Omega_T} \int_Y \int_Z (f \cdot \nabla_x(\phi - u) + g(\phi - u))\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \end{aligned}$$

for any $\phi \in V$. Or in other words,

$$\int_{\Omega_T}\int_Z A(x,t,\tau,\nabla_x u)\cdot\nabla_x(\phi-u)\mathrm{d}\tau\mathrm{d}x\mathrm{d}t \geq \int_{\Omega_T}\int_Z (F\cdot\nabla_x(\phi-u)+G(\phi-u))\mathrm{d}\tau\mathrm{d}x\mathrm{d}t \quad (4.14)$$

for any $\phi \in V$. So far we have only proved that the theorem holds for a subsequence. However, by the continuity and monotonicity properties (2.10) and (2.11) of A, it follows that the homogenized variational inequality (4.14) has a unique solution, which implies that the theorem holds for the whole sequence.

If $\nabla_x u$ and $\nabla_y u_1$ are admissible (each component belongs to $\mathcal{A}_p$, see Section 3), then Corollary 3.2 implies that

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big|\nabla_x u_\varepsilon(x,t)-\nabla_x u\Big(x,t,\frac{t}{\varepsilon}\Big)-\nabla_y u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big|^p\mathrm{d}x\mathrm{d}t=0. \quad (4.15)$$

If in addition, u and u_1 are admissible, then (4.15) together with the Poincaré inequality gives

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big|u_\varepsilon(x,t)-u\Big(x,t,\frac{t}{\varepsilon}\Big)\Big|^p\mathrm{d}x\mathrm{d}t\\
&=\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big|u_\varepsilon(x,t)-u\Big(x,t,\frac{t}{\varepsilon}\Big)-\varepsilon u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)+\varepsilon u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big|^p\mathrm{d}x\mathrm{d}t\\
&\leq c\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big|\nabla_x u_\varepsilon(x,t)-\nabla_x u\Big(x,t,\frac{t}{\varepsilon}\Big)-\nabla_y u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big|^p\mathrm{d}x\mathrm{d}t\\
&\quad+\lim_{\varepsilon\to 0}c\varepsilon^p\int_{\Omega_T}\Big|\nabla_x u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big|^p+\Big|u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big|^p\mathrm{d}x\mathrm{d}t=0. \qquad (4.16)
\end{aligned}$$

5 Homogenization of the Equation

In this section, we consider homogenization of the differential equations (1.1), which are related to the variational inequalities in the previous section. Indeed, consider the problem: Find $u_\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))$, such that

$$\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon)\cdot\nabla_x\phi\ \mathrm{d}x\mathrm{d}t=\int_{\Omega_T}(f_\varepsilon\cdot\nabla_x\phi+g_\varepsilon\phi)\mathrm{d}x\mathrm{d}t \quad (5.1)$$

holds for all $\phi\in L^p(0,T;W_0^{1,p}(\Omega))$. The sequence of solutions (5.1) then two-scale converges to u, where u solves the homogenized equation: Find $u\in L^p((0,T)\times Z;W_0^{1,p}(\Omega))$, such that

$$\int_{\Omega_T}\int_Z A(x,t,\tau,\nabla_x u)\cdot\nabla_x\phi\ \mathrm{d}\tau\mathrm{d}x\mathrm{d}t=\int_{\Omega_T}\int_Z(F\cdot\nabla_x\phi+G\phi)\mathrm{d}\tau\mathrm{d}x\mathrm{d}t \quad (5.2)$$

holds for any $\phi\in L^p((0,T)\times Z;W_0^{1,p}(\Omega))$. More precisely, we have the following result.

Theorem 5.1 *The sequence of solutions u_ε to* (5.1) *two-scale converges to $u\in L^p((0,T)\times Z;W_0^{1,p}(\Omega))$, where u is the unique solution to the homogenized equation* (5.2). *Moreover, $\nabla_x u_\varepsilon$ two-scale converges strongly to $\nabla_x u(x,t,\tau)+\nabla_y u_1(x,t,y,z)$, where $u_1\in L^p(\Omega_T\times Z;W^{1,p}_{\mathrm{per}}(Y))$ and*

$$\int_Y[a(x,t,y,\tau,\nabla_x u(x,t,\tau)+\nabla_y u_1(x,t,y,z))-f(x,t,y,\tau)]\cdot\nabla\phi\ \mathrm{d}y=0$$

for every $\phi\in W^{1,p}_{\mathrm{per}}(Y)$.

Proof By choosing $\phi = u_\varepsilon$ in (5.1), we get that

$$\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon)\cdot\nabla_x u_\varepsilon \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} (f_\varepsilon\cdot\nabla_x u_\varepsilon + g_\varepsilon u_\varepsilon)\mathrm{d}x\mathrm{d}t. \tag{5.3}$$

We can now use similar arguments as in the proof of Theorem 4.1 to deduce that u_ε, $\nabla_x u_\varepsilon$ and $a_\varepsilon(x,t,\nabla_x u_\varepsilon)$ are bounded in $L^p(\Omega_T)$, $L^p(\Omega_T)^N$ and $L^q(\Omega_T)^N$, respectively. According to Theorems 3.1 and 3.3, we can then extract a subsequence (still denoted by ε), such that

(1) There exists a $u \in L^p((0,T)\times Z; W_0^{1,p}(\Omega))$, such that u_ε two-scale converges,

(2) There exists a $u_1 \in L^p(\Omega_T\times Z, W^{1,p}_{\mathrm{per}}(Y))$, such that $\nabla_x u_\varepsilon$ two-scale converges to $\nabla_x u(x,t,\tau)+\nabla_y u_1(x,t,y,\tau)$,

(3) There exists an $a_0 \in L^q(\Omega_T\times Y\times Z)^N$, such that $a_\varepsilon(x,t,\nabla_x u_\varepsilon)$ two-scale converges to a_0.

Let $\phi\in C_0^\infty(\Omega_T; C^\infty_{\mathrm{per}}(Z))$. Define the function $\phi_\varepsilon(x,t)=\phi(x,t,\frac{t}{\varepsilon})$. Then ϕ_ε may be used as a test function in (5.1). Indeed

$$\int_{\Omega_T}[a_\varepsilon(x,t,\nabla_x u_\varepsilon)-f_\varepsilon]\cdot\nabla_x\phi_\varepsilon \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} g_\varepsilon\phi_\varepsilon \,\mathrm{d}x\mathrm{d}t.$$

Passing to the limit gives that

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - f]\cdot\nabla_x\phi \,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Y\int_Z g\phi \,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t \tag{5.4}$$

for any $\phi\in C_0^\infty(\Omega_T; C^\infty_{\mathrm{per}}(Z))$. We can also choose $\phi_1\in C_0^\infty(\Omega_T; C^\infty_{\mathrm{per}}(Y\times Z))$ and use $\varepsilon\phi_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ as a test function in (5.1).

$$\begin{aligned}&\int_{\Omega_T}[a_\varepsilon(x,t,\nabla_x u_\varepsilon)-f_\varepsilon]\cdot\Big[\varepsilon\nabla_x\phi_1\Big(x,t,\frac{t}{\varepsilon},\frac{\tau}{\varepsilon}\Big)+\nabla_y\phi_1\Big(x,t,\frac{t}{\varepsilon},\frac{\tau}{\varepsilon}\Big)\Big]\mathrm{d}x\mathrm{d}t\\ &=\varepsilon\int_{\Omega_T} g_\varepsilon\phi_1\Big(x,t,\frac{x}{\varepsilon},\frac{\tau}{\varepsilon}\Big)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Passing to the limit gives that

$$\int_{\Omega_T}\int_Y\int_Z [a_0 - f]\cdot\nabla_y\phi_1 \,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t = 0. \tag{5.5}$$

From (5.4)–(5.5) and density argument, it follows that

$$\int_{\Omega_T}\int_Y\int_Z a_0\cdot[\nabla_x u+\nabla_y u_1] \,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Y\int_Z (f\cdot[\nabla_x u+\nabla_y u_1]+gu)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t. \tag{5.6}$$

By passing to the limit in (5.3) and taking the equality (5.6) into account, we obtain that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} a_\varepsilon(x,t,\nabla_x u_\varepsilon)\cdot\nabla_x u_\varepsilon \,\mathrm{d}x\mathrm{d}t &= \int_{\Omega_T}\int_Y\int_Z (f\cdot[\nabla_x u+\nabla_y u_1]+gu)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t\\ &= \int_{\Omega_T}\int_Y\int_Z a_0\cdot[\nabla_x u+\nabla_y u_1] \,\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}t.\end{aligned}$$

This together with the Theorem 3.4 implies that

$$a_0(x,t,y,\tau) = a(x,t,y,\tau,\nabla_x u(x,t,\tau)+\nabla_y u_1(x,t,y,\tau)) \tag{5.7}$$

and from the Theorem 3.5 about strong two-scale convergence and monotonicity it follows that $\nabla_x u_\varepsilon$ two-scale converges strongly to $\nabla_x u + \nabla_y u_1$. By inserting (5.7) into the relation (5.4), we get the homogenized equation: Find $u \in L^p((0,T)\times Z; W_0^{1,p}(\Omega))$, such that

$$\int_{\Omega_T}\int_Z A(x,t,\tau,\nabla_x u)\cdot\nabla_x\phi_0 \,\mathrm{d}\tau\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\int_Z (F\cdot\nabla_x\phi_0 + G\phi_0)\mathrm{d}\tau\mathrm{d}x\mathrm{d}t \tag{5.8}$$

holds for any $\phi \in C_0^\infty(\Omega_T; C_{\mathrm{per}}^\infty(Z))$. By density, (5.8) also holds for any $\phi \in L^p((0,T)\times Z; W_0^{1,p}(\Omega))$. The theorem is now proved for a subsequence. However, the continuity and monotonicity conditions (2.10) and (2.11) imply that the homogenized equation (5.8) has a unique solution. Hence, the desired result holds for the whole sequence.

As for the variational inequality, we have that if u, u_1, $\nabla_x u_1$ and $\nabla_y u_1$ are admissible, then Corollary 3.2 implies that

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big|\nabla_x u_\varepsilon(x,t) - \nabla_x u\Big(x,t,\frac{t}{\varepsilon}\Big) - \nabla_y u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\Big|^p \mathrm{d}x\mathrm{d}t = 0, \tag{5.9}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big|u_\varepsilon(x,t) - u\Big(x,t,\frac{t}{\varepsilon}\Big)\Big|^p \mathrm{d}x\mathrm{d}t = 0. \tag{5.10}$$

6 Some Applications in the Theory of Lubrication

We will now discuss some applications in lubrication theory, where we can apply the homogenization results to the equation (1.1) and the variational inequality (1.2), see Theorem 5.1 and Theorem 4.1. More precisely, we will study the pressure distribution in an incompressible fluid film between two surfaces which are in relative motion. For example, this type of flow takes place in different types of bearings.

If the surfaces are rough, then the distance between the surfaces will oscillate rapidly both in space and time. This is the motivation of using homogenization in the analysis. When the pressure is known, it can be used to find other relevant quantities as a load carrying capacity and friction.

For simplicity, assume that both surfaces are moving in the x_1-direction. The velocity of the upper surface is $V_u = (v_u, 0)$ and for the lower surface it is $V_l = (v_l, 0)$. To express the film thickness, we introduce the following auxiliary function:

$$h(x,t,y,\tau) = h^0(x,t) + h^u(y - \tau V_u) - h^l(y - \tau V_l),$$

where h^0, h^u and h^l are continuously differentiable functions. Moreover, h^u and h^l are assumed to be periodic. Without loss of generality, it can also be assumed that the cell of periodicity is $Y = (0,1)\times(0,1)$ for both h^u and h^l, i.e., the unit cube in $\mathbb{R}^2$. We also assume that v_u and v_l are such that h is periodic in τ and we denote the cell of periodicity by Z. Let the bearing domain be an open bounded subset of $\mathbb{R}^2$ denoted by Ω, the space variable $x \in \Omega$ and $t \in (0,T) \subset \mathbb{R}$ represents the time. By using the auxiliary function h, we can model the film thickness h_ε by

$$h_\varepsilon(x,t) = h\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big), \quad \varepsilon > 0.$$

This means that $\varepsilon > 0$ is a parameter which describes the roughness wavelength; h^0 describes the global film thickness; the periodic functions h^u and h^l represent the roughness contribution of the upper and lower surfaces, respectively. We note that if $h^0 = h^0(x)$, one surface is stationary and rough and the other is smooth and moving, then $h_\varepsilon = h_\varepsilon(x)$, i.e., the distance between the surfaces does not depend on t. We will refer to the stationary case.

In order to apply our homogenization results, it is crucial to observe that the special form of h_ε implies that there is a relation between differentiation with respect to the time variable t and the spatial variable x. Indeed, define h_ε^u and h_ε^l as

$$h_\varepsilon^u(x,t) = h^u\Big(\frac{x}{\varepsilon} - \frac{t}{\varepsilon}V_u\Big) \quad \text{and} \quad h_\varepsilon^l(x,t) = h^l\Big(\frac{x}{\varepsilon} - \frac{t}{\varepsilon}V_l\Big)$$

and observe that

$$\frac{\partial h_\varepsilon^u}{\partial t} = -v_u\frac{\partial h_\varepsilon^u}{\partial x_1} \quad \text{and} \quad \frac{\partial h_\varepsilon^l}{\partial t} = -v_l\frac{\partial h_\varepsilon^l}{\partial x_1}.$$

This means that

$$\frac{\partial h_\varepsilon}{\partial t} = \frac{\partial h^0}{\partial t} + \frac{\partial h_\varepsilon^u}{\partial t} - \frac{\partial h_\varepsilon^l}{\partial t} = \frac{\partial h^0}{\partial t} - v_u\frac{\partial h_\varepsilon^u}{\partial x_1} + v_l\frac{\partial h_\varepsilon^l}{\partial x_1}.$$

Below, we present some Reynolds type equations which are frequently used to describe the pressure distribution, p_ε, in the lubricant film.

(1) Assume that the fluid is Newtonian, with viscosity μ. Then the pressure is modeled by the Reynolds equation (see, e.g., [35] or [20]):

$$\int_{\Omega_T} h_\varepsilon^3\nabla_x p_\varepsilon\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\Big(\lambda h_\varepsilon\frac{\partial\phi}{\partial x_1} - \gamma\frac{\partial h_\varepsilon}{\partial t}\phi\Big)\mathrm{d}x\mathrm{d}t, \tag{6.1}$$

where $\lambda = 6\mu(v_u + v_l)$ and $\gamma = 12\mu$. The general homogenization result in Theorem 5.1 can be applied by choosing $a_\varepsilon(x,t,\xi) = h_\varepsilon^3\xi$, $g_\varepsilon(x,t) = -\gamma\frac{\partial h^0}{\partial t}$ and $f_\varepsilon(x,t) = (f_\varepsilon^1(x,t),0)$, where

$$f_\varepsilon^1(x,t) = \lambda h_\varepsilon - \gamma v_u h_\varepsilon^u + \gamma v_l h_\varepsilon^l.$$

(2) The above example in polar coordinates, with x_1 as the angular coordinate and x_2 as the radial coordinate, reads

$$\int_{\Omega_T} A_\varepsilon(x,t)\nabla_x p_\varepsilon\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\Big(\lambda x_2 h_\varepsilon\frac{\partial\phi}{\partial x_1} - \gamma x_2\frac{\partial h_\varepsilon}{\partial t}\phi\Big)\mathrm{d}x\mathrm{d}t, \tag{6.2}$$

where $\lambda = 6\mu(v_u + v_l)$, $\gamma = 12\mu$ and

$$A_\varepsilon(x,t) = \begin{pmatrix} \frac{h_\varepsilon^3}{x_2} & 0 \\ 0 & x_2 h_\varepsilon^3 \end{pmatrix}.$$

In this case, we choose $a_\varepsilon(x,t,\xi) = A_\varepsilon(x,t)\xi$, $g_\varepsilon(x,t) = -\gamma x_2\frac{\partial h^0}{\partial t}$ and $f_\varepsilon(x,t) = (f_\varepsilon^1(x,t),0)$, where

$$f_\varepsilon^1(x,t) = \lambda x_2 h_\varepsilon - \gamma v_u x_2 h_\varepsilon^u + \gamma v_l x_2 h_\varepsilon^l.$$

Note that now v_u and v_l are angular speeds.

(3) Assume that the fluid is non-Newtonian, obeying a cubic Rabinowitsch fluid model with the viscosity μ and the constant κ accounting for the non-Newtonian effects. For this type of fluid the following governing equation for the pressure was presented in [28] (see also [21]):

$$\begin{aligned}&\int_{\Omega_T}\Big\{h_\varepsilon^3\nabla_x p_\varepsilon\cdot\nabla\phi+\frac{3\kappa h_\varepsilon^5}{20}\Big[\Big(\frac{\partial p}{\partial x_1}\Big)^3\frac{\partial\phi}{\partial x_1}+\Big(\frac{\partial p}{\partial x_2}\Big)^3\frac{\partial\phi}{\partial x_2}\Big]\Big\}\mathrm{d}x\mathrm{d}t\\ &=\int_{\Omega_T}\Big(\lambda h_\varepsilon\frac{\partial\phi}{\partial x_1}-\gamma\frac{\partial h_\varepsilon}{\partial t}\phi\Big)\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{6.3}$$

where $\lambda=6\mu(v_u+v_l)$ and $\gamma=12\mu$. In this case the homogenization result can be applied by choosing g_ε and f_ε as in the first case (Newtonian and Cartesian coordinates) but with

$$a_\varepsilon(x,t,\xi)=h_\varepsilon^3\xi+\frac{3\kappa h_\varepsilon^5}{20}\begin{pmatrix}\xi_1^3\\ \xi_2^3\end{pmatrix}.$$

Depending on h_ε, the pressure solution to the equations (6.1), (6.2) and (6.3), for example, might very well be negative in some regions. However, a fluid cannot sustain negative pressure. In such areas there will be zones where the lubricant contains air bubbles. This phenomenon is known as cavitation and has a big impact on the hydrodynamic performance. There are several ways to model and analyze cavitation. One way is to use variational inequalities (see, e.g., the books [27] and [8]), another is to apply the Elrod-Adams model, which introduces a new unknown saturation function which leads to systems of differential equations (see [19]). For example, the first mentioned approach leads to that the pressure instead of satisfying the equation (6.1) satisfies the variational inequality: Find $p_\varepsilon\geq 0$, such that

$$\int_{\Omega_T}h_\varepsilon^3\nabla p_\varepsilon\cdot\nabla_x(\phi-u_\varepsilon)\ \mathrm{d}x\mathrm{d}t\geq\int_{\Omega_T}\Big[\lambda h_\varepsilon\frac{\partial}{\partial x_1}(\phi-u_\varepsilon)-\gamma\frac{\partial h_\varepsilon}{\partial t}(\phi-u_\varepsilon)\Big]\mathrm{d}x\mathrm{d}t\tag{6.4}$$

holds for any testfunction $\phi\geq 0$. In the same way, we can motivate variational inequalities corresponding to (6.2) and (6.3).

When the pressure is known, it can be used to find two fundamental quantities, load carrying capacity and friction force. The load carrying capacity is found by integrating the pressure over Ω (bearing domain). The friction force, F_ε, is found by integrating the shear stress at the surface. The convergence of the friction force F_ε is strongly related to the two-scale convergence of ∇p_ε to $\nabla_x p_0+\nabla_y p_1$ (see [3, 13]).

We conclude by giving guidance to some of the literature, where homogenization was applied to lubrication problems. In the stationary case and with Newtonian fluid, the equation (6.1) was homogenized in [36] by using two-scale convergence, by H-convergence in [16] and by the formal method of multiple scale expansions in [4, 9, 25]. By introducing two parameters, one for the film thickness and one for the fineness of the roughness, the authors of [7, 13] studied homogenization of the Stokes flow and its relation to Reynolds flow in the stationary and Newtonian case. These works rigorously verify that homogenization of Reynolds equations (6.1) and (6.2) may be used when the filmthickness is small compared with the wavelength of the roughness. Nonlinear minimization problems corresponding to the equation problem studied in this work were studied in [30]. The stationary case with different types of non-Newtonian fluids was analyzed by multiple scale expansions in [26]. Roughness effects in stationary Newtonian lubrication taking cavitation into account by the Elrod-Adams model in [19] were studied by two-scale convergence in [10, 33] and by asymptotic expansions in [11, 12]. In addition to cavitation, several local scales are taken into account in [33] and elsto hydrodynamic phenomenon in [12]. Cavitation in the unstationary case with Newtonian fluid was modeled by variational inequalities in [8], and the effects of surface roughness were analyzed by two-scale convergence. The present work generalizes the results in [8] to include non-linear problems. From the viewpoint of application, this means that we can study Non-Newtonian fluids (see (6.3)). Recently, the idea of using bounds related to the homogenized equation has been used successfully for Newtonian fluids (see [2, 6, 29]). We also want to mention that there are several papers in which the homogenization process is illustrated by numerical investigations, examples being [5, 8, 25, 26, 33]. In the above guidance to the literature, all the references concern fluid flow. Some interesting related works considering air flow are [14, 15, 22, 23].

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