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A Note on Heegaard Splittings of Amalgamated 3-Manifolds*

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Abstract Let M be a compact orientable irreducible 3-manifold, and F be an essential connected closed surface in M which cuts M into two manifolds M_1 and M_2. If M_i has a minimal Heegaard splitting $M_i = V_i \cup_{H_i} W_i$ with $d(H_1) + d(H_2) \geq 2(g(M_1) + g(M_2) - g(F)) + 1$, then $g(M) = g(M_1) + g(M_2) - g(F)$.

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1 Introduction

All 3-manifolds in this paper are assumed to be compact and orientable.

Let M be a 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $S = \partial_+ W = \partial_+ V$, then we say that M has a Heegaard splitting, denoted by $M = V \cup_S W$, and S is called a Heegaard surface of M. Moreover, if the genus $g(S)$ of S is minimal among all Heegaard surfaces of M, then $g(S)$ is called the genus of M, denoted by $g(M)$. If there are essential disks $B \subset V$ and $D \subset W$ such that $\partial B = \partial D$ (resp. $\partial B \cap \partial D = \emptyset$), then $V \cup_S W$ is said to be reducible (resp. weakly reducible). Otherwise, it is said to be irreducible (resp. strongly irreducible).

Let $M = V \cup_S W$ be a Heegaard splitting. The distance between two essential simple closed curves α and β on S, denoted by $d(\alpha, \beta)$, is the smallest integer $n \geq 0$, so there is a sequence of essential simple closed curves $\alpha_0 = \alpha, \cdots, \alpha_n = \beta$ on S such that α_{i-1} is disjoint from α_i for $1 \leq i \leq n$. The distance of the Heegaard splitting $W \cup_S V$ is $d(S) = \min\{d(\alpha, \beta)\}$, where α bounds a disk in V and β bounds a disk in W (see [7]).

Let M be a 3-manifold, and A be an incompressible annulus on ∂M. Let $M = V \cup W$ be a Heegaard splitting with $A \subset \partial_- W$. Recalling that a spine annulus in W is an essential annulus of which one boundary component lies in $\partial_- W$, the other lies in $\partial_+ W$. A spine annulus A_s of W is called an A-spine annulus if one component of ∂A_s lies in A. $V \cup W$ is said to be A-primitive if there is an essential disk in V which intersects an A-spine annulus of W in one

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point. A 3-manifold M is said to be A-primitive if one of the minimal Heegaard splittings of M is A-primitive.

If a surface F in a 3-manifold M is incompressible and not parallel to ∂M, then F is said to be essential.

Let M be an irreducible 3-manifold, F be an essential connected closed surface in M which cuts M into two 3-manifolds M_1 and M_2. If $M_i = V_i \cup_{H_i} W_i$ is a Heegaard splitting of M_i $(i = 1, 2)$, then M has a natural Heegaard splitting called the amalgamation of $V_1 \cup_{H_1} W_1$ and $V_2 \cup_{H_2} W_2$ (see [23]). From this construction, we have $g(M) \leq g(M_1)+g(M_2)-g(F)$. Now an interesting problem is: When $g(M) = g(M_1)+g(M_2)-g(F)$ or $g(M) < g(M_1)+g(M_2)-g(F)$?

If $g(F) = 0$, any natural Heegaard splitting of the amalgamated 3-manifold M along unstablized Heegaard splittings of M_1 and M_2 is unstablized. It is proved in [1] and [18] respectively.

There are some examples for $g(M) < g(M_1) + g(M_2) - g(F)$ (see [9, 25]).

A sufficient condition for $g(M) = g(M_1) + g(M_2) - g(F)$ was first given in [8] by using Hemple distance in [7]. Then there are some results about this (see [3, 5, 11, 17, 19, 26]).

The amalgamated 3-manifold M can be viewed as gluing M_1 to M_2 via a homeomorphism $f : F_1 \to F_2$, where $F_i \subset \partial M_i$ $(i = 1, 2)$. So, from the viewpoint of homeomorphic maps of surfaces, there are some results about $g(M) = g(M_1) + g(M_2) - g(F)$ (see [10, 12, 14, 24]).

In this paper, we give a bound of the sum of the distance of two Heegaard splittings of M_1 and M_2, such that for any minimal Heegaard splitting $V \cup_S W$ of M, $g(S) = g(M_1)+g(M_2)-g(F)$. The main result is as follows.

Theorem 1.1 *Let M be an irreducible 3-manifold, F be an essential connected closed surface in M which cuts M into two 3-manifolds M_1 and M_2. If M_i has a minimal Heegaard splitting $M_i = V_i \cup_{H_i} W_i$ $(i = 1, 2)$ with $d(H_1) + d(H_2) \geq 2(g(M_1) + g(M_2) - g(F)) + 1$, then $g(M) = g(M_1) + g(M_2) - g(F)$.*

Corollary 1.1 *If $d(H_i) \geq 2g(M_i) - g(F)$ for $i = 1, 2$, then $g(M) = g(M_1) + g(M_2) - g(F)$.*

Question 1.1 Wether the bound of the sum of this two distances is the best?

2 Preliminary

There are some lemmas which can be used to prove the main theorem.

Lemma 2.1 (see [4, 19]) *Let $M = V \cup_S W$ be a Heegaard splitting, and F be an incompressible surface in M. Then either F can be isotoped to be disjoint from S or $d(S) \leq 2-\chi(F)$.*

Lemma 2.2 *Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting, and F be an essential connected closed surface in M which cuts M into two manifolds M_1 and M_2. Then S can be isotoped so that*

(1) *one of $S \cap M_1$ and $S \cap M_2$, say $S \cap M_1$, is incompressible while all components of $S \cap M_2$ are incompressible except one bicompressible component, or*

(2) *one of $S \cap M_1$ and $S \cap M_2$, say $S \cap M_1$, is incompressible while $S \cap M_2$ is compressible only in one side, say $M_2 \cap V$, and there is a Heegaard surface S' isotopic to S such that*

(i) *$S' \cap M_1$ is only compressible in $M_1 \cap W$ or incompressible while $S' \cap M_2$ is incompressible,*

(ii) *S' is obtained by ∂-compressing S in M_2 only one time.*

This is a stronger version of [8, 17]. The arguments in [8, 17] contain this result. We give an outline of the proof.

Proof of Lemma 2.2 By [8, Proposition 2.6], S can be isotoped such that at least one of $S \cap M_1$ and $S \cap M_2$ is incompressible, and $S \cap F$ is a collection of essential simple closed curves on both S and F. We may assume $S \cap M_1$ is incompressible. Then, there are three cases for $S \cap M_2$:

Case 1 $S \cap M_2$ is incompressible.

Case 2 $S \cap M_2$ is bicompressible.

Case 3 $S \cap M_2$ is only compressible in $M_2 \cap V$, incompressible in $M_2 \cap W$.

If Case 2 holds, then there is nothing to prove. Now we consider Case 1 and Case 3.

Case 1 Suppose that $S \cap M_1$ is incompressible, and $S \cap M_2$ is incompressible.

Since F is essential, $V \cup_S W$ is a non-trivial Heegaard splitting of M. So, there are essential disks in both V and W. We may assume that among all the isotopies of S, satisfying

(1) $S \cap F$ is a collection of essential simple closed curves on both S and F,

(2) $S \cap M_1$ and $S \cap M_2$ are incompressible,

there is an essential disk D in V or W such that $|D \cap F|$ is minimal. Since F is essential, $D \cap F$ is a collection of arcs in D. Let a be an outermost arc in D. There is an arc $b \in \partial D$ such that $a \cup b$ bounds a subdisk $D' \subset D$ with $\mathrm{int}D' \cap F = \emptyset$. We now prove that D' is a ∂-compressing disk for S_1 or S_2. Suppose that D' lies in M_1. If b is inessential on S_1, then b can be isotoped to b' which lies in F such that $a \cup b'$ bounds a disk in M_1. Since F is essential, $a \cup b'$ is an inessential simple closed curve on F. Since M_1 is irreducible, it is easy to see that D' can be pushed into M_2 decreasing $|D \cap F|$, a contradiction. So, b is an essential arc on S_1 or S_2.

We may assume that D' is a ∂-compressing disk for S_1. ∂-compressing S_1 along D', we get a surface S' isotopic to S. Note that $S' \cap M_1$ is incompressible, except for possibly a ∂-parallel disk. If there is a ∂-parallel disk, then we push it into M_2. We still denote it by S'. Note that $S' \cap F$ is also a collection of essential simple closed curves on both S' and F. By the minimality of $|D \cap F|$, $S' \cap M_2$ is bicompressible or only compressible in $M_2 \cap V$ (resp. $M_2 \cap W$), incompressible in $M_2 \cap W$ (resp. $M_2 \cap V$). So, we have Case 2 or Case 3.

Case 3 Suppose that $S \cap M_1$ is incompressible, $S \cap M_2$ is only compressible in $M_2 \cap V$, incompressible in $M_2 \cap W$.

As in Case 1, we may assume that among all the isotopies of S, satisfying

(1) $S \cap F$ is a collection of essential simple closed curves on both S and F,

(2) one of $S \cap M_1$ and $S \cap M_2$ is incompressible, the other is only compressible in one side,

there is an essential disk D in W or V with $|D \cap F|$ being minimal. (If one of $S \cap M_1$ and $S \cap M_2$ is compressible in V (resp. W), we say that D lies in W (resp. V).)

Suppose that $S \cap M_1$ is incompressible, $S \cap M_2$ is compressible in $M_2 \cap V$ and incompressible in $M_2 \cap W$. Then, there is an essential disk D in W such that $|D \cap F|$ is minimal.

By the proof as in Case 1, we choose a subdisk D' of D with $\mathrm{int}D' \cap F = \emptyset$. Note that D' is a ∂-compressing disk for S_1 or S_2. If D' lies in S_1, then ∂-compressing S_1 along D', we get a surface S' isotopic to S such that $S' \cap M_1$ is still incompressible, except for possibly a ∂-parallel disk. By pushing the ∂-parallel into M_2, we still denote this surface by S'. Note that $S' \cap F$ is also a collection of essential simple closed curves on both S' and F, and the isotopy decreases $|D \cap F|$. Since F is essential, S' cannot be disjoint from F. Then, $S' \cap M_1$ is also incompressible, and $S' \cap M_2$ is also compressible in $M_2 \cap V$. By the minimality of $|D \cap F|$, $S' \cap M_2$ is bicompressible. This is Case 2.

Now suppose that D' lies in S_2. ∂-compressing S_2 along D', we obtain a surface S' isotopic to S. Note that $S' \cap M_2$ is still incompressible in $M_2 \cap W$, except for possibly a ∂-parallel disk. If there is a ∂-parallel disk, then we push it into M_1. We still denote it by S'. Note that $S' \cap F$ is a collection of essential simple closed curves on both S' and F. If $S' \cap M_2$ is still compressible in $M_2 \cap V$, by the minimality of $|D \cap F|$, $S' \cap M_1$ is compressible in $M_1 \cap V$. We also denote the disk isotopic to D by D. Then $D \cap F \neq \emptyset$. We isotope S' via ∂-compressing along subdisk of D as above, by the minimality of $|D \cap F|$ and the finiteness of $D \cap F$, we have Case 2.

If $S' \cap M_2$ is incompressible in M_2, by the minimality of $|D \cap F|$, Lemma 2.2 holds.

Lemma 2.3 *Let M be an irreducible 3-manifold, $V \cup_P W$ be a Heegaard splitting of M, Q be a properly embedded separating connected bicompressible bounded surface in M which cuts M into two 3-manifolds X and Y. Suppose ∂Q lies in one component of $\partial_- V$. If $d(P) \geq 5$, then $d(P) \leq 2 - \chi(Q)$ or Q lies in $\partial_- V \times I$.*

This is the stronger version of the proof in [13]. There are some differences from [13], because in [13], Q is closed. In fact, the condition $d(P) \geq 5$ can be deleted (see [4]). The argument in [13] contains this result. We do not prove it here.

3 The Proof of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1 and Corollary 1.1 Let $V \cup_S W$ be a minimal Heegaard splitting of M, and $k_i = d(H_i)$ for $i = 1, 2$. Since F is essential, S cannot be isotoped to be disjoint from F.

Case 1 $V \cup_S W$ is strongly irreducible.

Let $S_1 = S \cap M_1$, $S_2 = S \cap M_2$. By Lemma 2.2, there are the following two cases.

Case 1.1 Suppose that S_1 is incompressible, S_2 is compressible in $M_2 \cap V$ and incompressible in $M_2 \cap W$.

By Lemma 2.1, we have $\chi(S_1) \leq 2 - k_1$. By Lemma 2.2, we get an incompressible surface S_2' after ∂-compressing S_2 in M_2 only one time. By Lemma 2.1, we have $\chi(S_2') \leq 2 - k_2$. Since $\chi(S_2) = \chi(S_2') - 1$, $\chi(S) = \chi(S_1) + \chi(S_2) \leq 2 - k_1 + 1 - k_2 = 3 - (k_1 + k_2) \leq 3 - 2(g(M_1) + g(M_2) - g(F)) - 1$, i.e., $g(S) \geq g(M_1) + g(M_2) - g(F)$.

Case 1.2 Suppose that S_1 is incompressible, S_2 is bicompressible.

By Lemma 2.1, we have $\chi(S_1) \leq 2 - k_1$. If S_2 is not connected, let H be an incompressible component. Since S is strongly irreducible, the bicompressible component of S_2 is not annulus.

So $\chi(H) \geq \chi(S_2) + 2$. By Lemma 2.1, $\chi(H) \leq 2 - k_2$. So $\chi(S_2) \leq -k_2$. Then $\chi(S) = \chi(S_1) + \chi(S_2) \leq 2 - k_1 - k_2 \leq 2 - 2(g(M_1) + g(M_2) - g(F)) - 1$, i.e., $g(S) \geq g(M_1) + g(M_2) - g(F) + 1$, a contradiction.

Hence, S_2 is connected. After maximally compressing S_2 in $M_2 \cap V$ (resp. $M_2 \cap W$), we denote it by S_V (resp. S_W). By the no nested lemma in [21], S_V and S_W are incompressible. If there is a bounded component H of S_V (resp. S_W) which is not ∂-parallel, by Lemma 2.1, $\chi(S_2) + 2 \leq \chi(H) \leq 2 - k_2$. Then $\chi(S) = \chi(S_1) + \chi(S_2) \leq 2 - k_1 - k_2 \leq 2 - 2(g(M_1) + g(M_2) - g(F)) - 1$, i.e., $g(S) \geq g(M_1) + g(M_2) - g(F) + 1$, which is a contradiction.

Hence, each bounded component of S_V and S_W is ∂-parallel. If the bounded components of S_V (resp. S_W) are nested, since S_2 is connected, they are as in Figure 1. Let H be the outermost component of S_V, F_H be a subsurface of F parallel to H, and $F \times I$ be a small regular neighborhood of F in M_2, where $F = F \times \{0\}$. Then $F' = \overline{H - F \times I} \cup (\overline{F - F_H} \times \{1\})$ is parallel to F. We can push F' slightly such that $F' \cap H = \emptyset$. Since H is outermost, F' is disjoint from S_2. So, F' lies in V or W, a contradiction. Hence, each bounded component of S_V is ∂-parallel and non-nested. So does S_W.

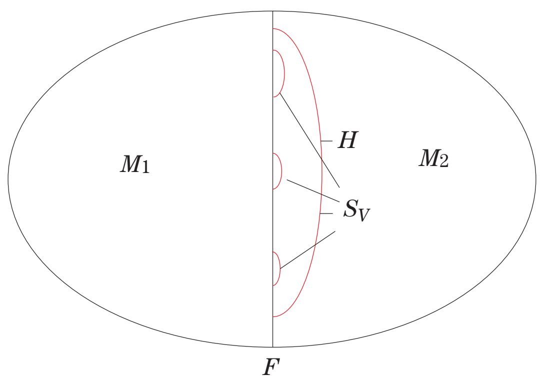

Figure 1 The bounded component of S_V

Let $F_V = F \cap (M_2 \cap V)$, $F_W = F \cap (M_2 \cap W)$. It is easy to see that $H' = \overline{S_2 - F \times I} \cup (F_V \times \{1\})$ and $H'' = \overline{S_2 - F \times I} \cup (F_W \times \{1\})$ are Heegaard surfaces of M_2. We have $\chi(S_2) + \chi(F_V) = \chi(H')$, $\chi(S_2) + \chi(F_W) = \chi(H'')$. Note that $\chi(H') \leq \chi(H_2)$, $\chi(H'') \leq \chi(H_2)$, so $2\chi(S_2) + \chi(F) = \chi(H') + \chi(H'') \leq 2\chi(H_2)$, i.e., $\chi(S_2) \leq \chi(H_2) - \frac{1}{2}\chi(F)$. Since $\chi(S_1) \leq 2 - k_1$, $\chi(S) = \chi(S_1) + \chi(S_2) \leq 2 - k_1 + \chi(H_2) - \frac{1}{2}\chi(F)$. Since $g(S) \leq g(M_1) + g(M_2) - g(F)$, $k_1 \leq 2g(M_1) - g(F) + 1$. Since $k_1 + k_2 \geq 2(g(M_1) + g(M_2) - g(F)) + 1$, $k_2 \geq 2g(M_2) - g(F) \geq 2(g(F) + 1) - g(F) \geq 3$. But H' and H'' are A-primitive, by [14], $d(H'), d(H'') \leq 2$, so $g(H'), g(H'') \geq g(H_2) + 1$, i.e., $\chi(H'), \chi(H'') \leq \chi(H_2) - 2$. Then, by the proof as above, $\chi(S_2) \leq \chi(H_2) - 2 - \frac{1}{2}\chi(F)$. Thus, $\chi(S) = \chi(S_1) + \chi(S_2) \leq 2 - k_1 + \chi(H_2) - 2 - \frac{1}{2}\chi(F)$. We have $k_1 \leq 2g(M_1) - g(F) - 1$. Then $k_2 \geq 2g(M_2) - g(F) + 2 \geq 5$. Thus, by Lemma 2.3, we have $\chi(S_2) \leq 2 - k_2$. Hence, $\chi(S) = \chi(S_1) + \chi(S_2) \leq 2 - k_1 + 2 - k_2 \leq 4 - 2(g(M_1) + g(M_2) - g(F)) - 1$, i.e., $g(S) \geq g(M_1) + g(M_2) - g(F) - \frac{1}{2}$.

Note that $k_1 \leq 2g(M_1) - g(F) - 1$. If we suppose $k_i \geq 2g(M_i) - g(F)$ for each i, then this case does not happen. Corollary 1.1 holds.

Case 2 $V \cup_S W$ is weakly reducible.

Since $V \cup_S W$ is weakly reducible, by [2, 16, 22], we have

$$M = V \cup_S W = (V_1 \cup_{P_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{P_n} W_n),$$

where each $V_i \cup_{P_i} W_i$ is strongly irreducible, each F_i is incompressible.

If there is F_i for some i, such that $F_i \cap F \neq \emptyset$ after isotopies, we may assume that $F_i \cap F$ is a collection of essential simple closed curves on both F_i and F, and $|F_i \cap F|$ is minimal. Let $F_i^1 = F_i \cap M_1$, $F_i^2 = F_i \cap M_2$. By Lemma 2.1, $\chi(F_i^1) \leq 2 - k_1$, $\chi(F_i^2) \leq 2 - k_2$. Then, we have $\chi(S)+4 \leq \chi(F_i) = \chi(F_i^1)+\chi(F_i^2) \leq 2-k_1+2-k_2$. Since $k_1+k_2 \geq 2(g(M_1)+g(M_2)-g(F))+1$, $g(S) > g(M_1) + g(M_2) - g(F) + \frac{3}{2}$, which is a contradiction.

Hence, for each i, $F_i \cap F = \emptyset$. Then, F lies in $V_i \cup_{P_i} W_i$ for some i. If $P_i \cap F = \emptyset$, then F is parallel to F_{i-1} or F_i, we have $g(S) = g(M_1) + g(M_2) - g(F)$. Next, we suppose $P_i \cap F \neq \emptyset$. Note that P_i is strongly irreducible. Let $P_i^1 = P_i \cap M_1$, $P_i^2 = P_i \cap M_2$, and $M_i = V_i \cup_{P_i} W_i$. By Lemma 2.2, there are two cases.

Case 2.1 Suppose that P_i^1 is incompressible in $M_i \cap M_1$, P_i^2 is compressible in $V_i \cap M_2$, and incompressible in $W_i \cap M_2$.

Since each F_i is incompressible, P_i^1 is incompressible in M_1. By Lemma 2.1, $\chi(P_i^1) \leq 2-k_1$. Then we get $P_i^{2'}$ obtained by ∂-compressing P_i^2, such that $P_i^{2'}$ is incompressible in $M_2 \cap M_i$. So, $P_i^{2'}$ is incompressible in M_2. Hence, again by Lemma 2.1, $\chi(P_i^{2'}) = \chi(P_i^2)+1 \leq 2-k_2$, i.e., $\chi(P_i^2) \leq 1 - k_2$. So, $\chi(P_i) = \chi(P_i^1) + \chi(P_i^2) \leq 2 - k_1 + 1 - k_2 \leq 2 - 2(g(M_1) + g(M_2) - g(F))$. Since $\chi(S) + 2 \leq \chi(P_i)$, we have $g(S) > g(M_1) + g(M_2) - g(F)$, which is a contradiction.

Case 2.2 Suppose that P_i^1 is incompressible in $M_i \cap M_1$, P_i^2 is bicompressible in $M_i \cap M_2$.

Since each F_i is incompressible, P_i^1 is incompressible in M_1, all compressing disks of P_i^2 in M_2 lie in $M_2 \cap M_i$. By Lemma 2.1, $\chi(P_i^1) \leq 2 - k_1$. If P_i^2 is not connected, let H be an incompressible component of P_i^2, then $\chi(H) \leq 2 - k_2$. Since $\chi(P_i^2) + 2 \leq \chi(H)$, we have $\chi(P_i^2) \leq -k_2$. So $\chi(P_i) = \chi(P_i^1) + \chi(P_i^2) \leq 2 - k_1 - k_2$. Since $\chi(S) \leq \chi(P_i) - 2$, we have $g(S) > g(M_1) + g(M_2) - g(F)$, which is a contradiction.

Hence, P_i^2 is connected. Maximally compressing P_i^2 in $M_2 \cap V_i$ (resp. $M_2 \cap W_i$), we denote it by P_V (resp. P_W). By the no nested lemma in [21], P_V and P_W are incompressible. If there is a bounded component H of P_V (resp. P_W) which is not ∂-parallel, by Lemma 2.1, $\chi(H) \leq 2 - k_2$. Since $\chi(P_i^2) + 2 \leq \chi(H)$, we have $\chi(P_i^2) \leq -k_2$. Since $\chi(S) \leq \chi(P_i) - 2$, we have $g(S) > g(M_1) + g(M_2) - g(F)$, which is a contradiction.

Hence, each bounded component of P_V and P_W is ∂-parallel. If the bounded components of P_V (resp. P_W) are nested, since P_i^2 is connected, they are also as in Figure 1. Let H be the outermost component of P_V, F_H be a subsurface of F parallel to H, and $F \times I$ be a small regular neighborhood of F in M_2, where $F = F \times \{0\}$. Then $F' = \overline{H - F \times I} \cup (\overline{F - F_H} \times \{1\})$ is parallel to F. We can push F' slightly such that $F' \cap H = \emptyset$. Since H is outermost, F' is disjoint from P_i^2. So, F' lies in V_i or W_i, which is a contradiction. Hence, each bounded component of P_V is ∂-parallel and non-nested. So does P_W.

Let $F_V = F \cap (M_2 \cap V_i)$, $F_W = F \cap (M_2 \cap W_i)$, and $M_i' = M_2 \cap M_i$. It is easy to see that $H' = \overline{P_i^2 - F \times I} \cup (F_V \times \{1\})$ and $H'' = \overline{P_i^2 - F \times I} \cup (F_W \times \{1\})$ are Heegaard surfaces of M_i'. We have $\chi(P_i^2)+\chi(F_V)=\chi(H')$, $\chi(P_i^2)+\chi(F_W)=\chi(H'')$. So $2\chi(P_i^2)+\chi(F)=\chi(H')+\chi(H'')\leq 4-4g(M_i')$, i.e., $\chi(P_i^2)\leq 2-2g(M_i')-\frac{1}{2}\chi(F)$. We have $\chi(S)\leq\chi(P_i)-2\leq 2-k_1+2-2g(M_i')-\frac{1}{2}\chi(F)-2$, i.e., $k_1\leq 2g(M_1)+2g(M_2)-2g(M_i')-g(F)-1$. Since $k_1+k_2\geq 2(g(M_1)+g(M_2)-g(F))+1$, we have $k_2\geq 2g(M_i')-g(F)+2\geq 5$. By Lemma 2.3, we get $\chi(P_i^2)\leq 2-k_2$. Hence, $\chi(S)\leq\chi(P_i)-2\leq 2-k_1+2-k_2-2$, i.e., $g(S)\geq g(M_1)+g(M_2)-g(F)+1$, which is a contradiction. Theorem 1.1 is proved.

Note that if $k_i\geq 2g(M_i)-g(F)$ for each i, then the proof of Case 2.1 is the same. In Case 2.2, $2g(M_1)-g(F)\leq k_1\leq 2g(M_1)+2g(M_2)-2g(M_i')-g(F)-1$, we have $g(M_2)\geq g(M_i')+1$. Note that $k_2\geq 2g(M_2)-g(F)\geq 5$. By Lemma 2.3, $\chi(P_i^2)\leq 2-k_2$. By Lemma 2.1, $\chi(P_i^1)\leq 2-k_1$. Thus, $\chi(S)\leq\chi(P_i)-2\leq 2-(k_1+k_2)$. We have $g(S)\geq g(M_1)+g(M_2)-g(F)$. Corollary 1.1 holds.

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References

[1] Bachman, D., Connected sums of unstabilized Heegaard splittings are unstabilized, *Geom. Topol.*, **12**, 2008, 2327–2378.

[2] Casson, A. and McA Gordon, C., Reducing Heegaard splittings, *Topology Appl.*, **27**, 1987, 275–283.

[3] Du, K., Lei, F. and Ma, J., Distance and self-amalgamation of Heegaard splittings, preprint.

[4] Du, K. and Qiu, R., The self-amalgamation of high distance Heegaard splittings is always efficient, *Topology Appl.*, **157**(7), 2010, 1136–1141.

[5] Du, K., Qiu, R., Ma, J. and Zhang, M., Distance and Heegaard genera of annular 3-manifold, *J. Knot Theory Ramifications*, to appear.

[6] Hartshorn, K., Heegaard splittings of Haken manifolds have bounded distance, *Pacific J. Math.*, **204**, 2002, 61–75.

[7] Hempel, J., 3-Manifolds as viewed from the curve complex, *Topology*, **40**, 2001, 631–657.

[8] Kobayashi, T. and Qiu, R., The amalgamation of high distance Heegaard splittings is unstabilized, *Math. Ann.*, **341**(3), 2008, 707–715.

[9] Kobayashi, T., Qiu, R., Rieck, Y. and Wang S., Separating incompressible surfaces and stabilizations of Heegaard splittings, *Math. Proc. Cambridge Philos. Soc.*, **137**(3), 2004, 633–643.

[10] Lackenby, M., The Heegaard genus of amalgamated 3-manifolds, *Geom. Dedicata*, **109**, 2004, 139–145.

[11] Lei, F. and Yang, G., A lower bound of genus of amalgamations of Heegaard splittings, *Math. Proc. Camb. Phil. Soc.*, **146**, 2009, 615–623.

[12] Li, T., Heegaard surfaces and the distance of amalgamation. arXiv: math.GT/0807.2869

[13] Li, T., Saddle tangencies and the distance of Heegaard splittings, *Albegr. Geom. Topol.*, **7**, 2007, 1119–1134.

[14] Li, T., On the Heegaard splittings of amalgamated 3-manifolds, Workshop on Heegaard Splittings, Geom. Topol. Monogr., **12**, Geom. Topol. Publ., Coventry, 2007, 157–190.

[15] Moriah, Y., Heegaard splittings of knot exteriors, Workshop on Heegaard Splittings, Geom. Topol. Monogr., **12**, Geom. Topol. Publ., Coventry, 2007, 191–232.

[16] Qiu, R., Untelescopings of Heegaard splittings, *Northeast. Math. J.*, **16**(4), 2000, 484–490.

[17] Qiu, R. and Lei, F., On the Heegaard genera of 3-manifolds containing non-separating surfaces, Topology and Physics, Nankai Tracts Math., **12**, World Sci. Publ., Hackensack, New Jersey, 2008, 341–347.

[18] Qiu, R. and Scharlemann, M., A proof of the Gordon conjecture, *Adv. Math.*, **222**(6), 2009, 2085–2106.

[19] Qiu, R., Wang, S. and Zhang, M., Additivity of Heegaard genera of bounded surface sums, *Topology and Its Appl.*, **157**, 2010, 1593–1601.

[20] Scharlemann, M., Proximity in the curve complex: boundary reduction and bicompressible surfaces, *Pacific J. Math.*, **228**(2), 2006, 325–348.

[21] Scharlemann, M., Local detection of strongly irreducible Heegaard splittings, *Topology and Its Appl.*, **90**, 1998, 135–147.

[22] Scharlemann, M. and Thompson, A., Thin position for 3-manifolds, Geometric Topology (Haifa, 1992), Contemp. Math., Vol. 164, A. M. S., Providence, RI, 1994, 231–238.

[23] Schultens, J., The classification of Heegaard splittings for (compact orientable surface) $\times S^1$, *Proc. London Math. Soc.*, **67**(3), 1993, 425–448.

[24] Souto, J., The Heegaard genus and distance in curve complex, preprint.

[25] Schultens, J. and Weidmann, R., Destabilizing amalgamated Heegaard splittings, Workshop on Heegaard Splittings, Geom. Topol. Monogr., **12**, Geom. Topol. Publ., Coventry, 2007, 319–334.

[26] Yang, G. and Lei, F., On amalgamations of Heegaard splittings with high distance, *Proc. Amer. Math. Soc.*, **137**(2), 2009, 723–731.