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Laplacian on Complex Finsler Manifolds*

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Abstract In this paper, the Laplacian on the holomorphic tangent bundle $T^{1,0}M$ of a complex manifold M endowed with a strongly pseudoconvex complex Finsler metric is defined and its explicit expression is obtained by using the Chern Finsler connection associated with (M, F). Utilizing the initiated "Bochner technique", a vanishing theorem for vector fields on the holomorphic tangent bundle $T^{1,0}M$ is obtained.

Keywords Laplacian, Strongly pseudoconvex complex Finsler metric, Chern Finsler connection
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1 Introduction

It is well-known that the Laplacian plays an important role in the theory of harmonic integral and Bochner technique in both Riemannian and Kähler manifolds (see [1–6]). The Laplacian also makes sense in Finsler cases (see [7–14]). Let M be a complex manifold endowed with a strongly pseudoconvex Finsler metric in the sense of [15]. In [16–17], by using the complex Rund connection, Zhong defined the horizontal and vertical Laplacians in an invariant way on a strongly pseudoconvex Finsler manifold and used the horizontal Laplacian to obtain a vanishing theorem of p-forms on the base manifold M under the assumption that F is a Kähelr Finsler metric on M.

As an application of the horizontal Laplacian associated with a complex Finsler manifold (M, F), the Bochner technique (see [1–3]) or Bochner Kodaira technique (see [18–20]) has also been studied (see [21–22]). In this paper, the authors derive the Laplacian on the holomorphic tangent bundle $T^{1,0}M$ for a strongly pseudoconvex Finsler manifold in terms of the Chern-Finsler connection associated with (M, F). Furthermore, by using the Chern-Finsler connection, the authors obtain the so-called Weitzenböck formula for the Laplacian. Finally, as an application, the authors obtain a Bochner type vanishing theorem for vector fields on the holomorphic tangent bundle $T^{1,0}M$.

2 Preliminaries

Let M be a complex manifold of complex dimension m. Denote $\pi : T^{1,0}M \longrightarrow M$ the holomorphic tangent bundle of M. Note that $T^{1,0}M$ is a non-compact complex manifold, even if M is compact. For a local complex coordinate system $z = (z^1, \cdots, z^m)$ on M, a holomorphic

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tangent vector v of M is written as

$$v = v^\mu \partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial z^\mu}, \quad \dot{\partial}_\alpha = \frac{\partial}{\partial v^\alpha},$$

and we take $(z, v) = (z^1, \cdots, z^m, v^1, \cdots, v^m)$ as local holomorphic coordinate neighborhood of $T^{1,0}M$. Let $\widetilde{M} = T^{1,0}M \backslash \{0\}$ denote $T^{1,0}M$ without the zero section. $\{\partial_\mu, \dot{\partial}_\alpha\}$ gives a local holomorphic frame field of the holomorphic tangent bundle $T^{1,0}\widetilde{M}$ of $\widetilde{M}$. A complex Finsler metric on a complex manifold M is a continuous function $F : T^{1,0}M \longrightarrow R^+$ with the following properties (see [15]):

(i) $G = F^2$ is smooth on $\widetilde{M}$;
(ii) $F(v) > 0$ for all $v \in \widetilde{M}$ and $F(v) = 0$ for all $v = 0$;
(iii) $F(\zeta v) = |\zeta| F(v)$ for all $v \in T^{1,0}M$ and $\zeta \in C$.

The pair (M, F) is called a complex Finsler manifold. A complex Finsler metric F is said to be strongly pseudoconvex if the Levi matrix $(G_{\alpha\overline{\beta}})$ is positive definite on $\widetilde{M}$, where $G_{\alpha\overline{\beta}} = \dot{\partial}_\alpha \dot{\partial}_{\overline{\beta}} G$, and the pair (M, F) is called a strongly pseudoconvex Finsler manifold.

Let $\widetilde{\pi} : T^{1,0}\widetilde{M} \longrightarrow \widetilde{M}$ denote the natural projection. The differential $\mathrm{d}\pi : T^{\mathbb{C}}\widetilde{M} \longrightarrow T^{\mathbb{C}}M$ of $\pi : \widetilde{M} \longrightarrow M$ defines the vertical bundle $\mathcal{V}$ over $\widetilde{M}$ by

$$\mathcal{V} = \mathrm{Ker}\, \mathrm{d}\pi \cap T^{1,0}\widetilde{M}, \tag{2.1}$$

which is the holomorphic vector bundle of rank m over $\widetilde{M}$, and $\{\dot{\partial}_\alpha\}$ gives a local frame for $\mathcal{V}$. As is defined in [15], there is a horizontal bundle $\mathcal{H}$ over $\widetilde{M}$ such that $T^{1,0}\widetilde{M} = \mathcal{V} \oplus \mathcal{H}$, and the local frame for $\mathcal{H}$ is given by

$$\delta_\mu = \partial_\mu - \Gamma^\alpha_\mu \dot{\partial}_\alpha, \qquad \Gamma^\alpha_\mu = G^{\overline{\tau}\alpha} G_{\overline{\tau};\mu}, \tag{2.2}$$

where $(G^{\overline{\tau}\alpha}) = (G_{\alpha\overline{\tau}})^{-1}$, $G_{\overline{\tau};\mu} = \dot{\partial}_{\overline{\tau}} \partial_\mu G$. Thus $\{\delta_\mu, \dot{\partial}_\alpha\}$ gives a local frame for $T^{1,0}\widetilde{M}$. Let $\{\mathrm{d}z^\mu, \delta v^\alpha\}$ be the dual frame for $T^{1,0*}\widetilde{M}$, where $\delta v^\alpha = \mathrm{d}v^\alpha + \Gamma^\alpha_\mu \mathrm{d}z^\mu$. The frames $\{\delta_\mu, \dot{\partial}_\alpha\}$ and $\{\mathrm{d}z^\mu, \delta v^\alpha\}$ are called the adapted frames for $T^{1,0}\widetilde{M}$ and $T^{1,0*}\widetilde{M}$ respectively, and the following Lie brackets hold (see [15]):

$$\begin{aligned} &[\delta_\mu, \delta_\nu] = 0, && [\delta_\mu, \dot{\partial}_\alpha] = \Gamma^\sigma_{\alpha;\mu} \dot{\partial}_\sigma, \quad [\dot{\partial}_\alpha, \dot{\partial}_\beta] = 0, \\ &[\delta_\mu, \delta_{\overline{\nu}}] = \delta_{\overline{\nu}}(\Gamma^\alpha_\mu) \dot{\partial}_\alpha - \delta_\mu(\overline{\Gamma^\tau_\nu}) \dot{\partial}_{\overline{\tau}}, && [\delta_\mu, \dot{\partial}_{\overline{\alpha}}] = \Gamma^\sigma_{\overline{\alpha};\mu} \dot{\partial}_\sigma, \quad [\dot{\partial}_\alpha, \dot{\partial}_{\overline{\beta}}] = 0. \end{aligned} \tag{2.3}$$

For a strongly pseudoconvex Finsler metric, there is a unique Chern-Finsler connection D on $T^{1,0}\widetilde{M}$. Its connection form ω^α_β is given by

$$\omega^\alpha_\beta = G^{\overline{\tau}\alpha} \partial G_{\beta\overline{\tau}} = \Gamma^\alpha_{\beta;\mu} \mathrm{d}z^\mu + \Gamma^\alpha_{\beta\gamma} \delta v^\gamma,$$

where $\Gamma^\alpha_{\beta;\mu} = G^{\overline{\tau}\alpha} \delta_\mu G_{\beta\overline{\tau}}$, $\Gamma^\alpha_{\beta\gamma} = G^{\overline{\tau}\alpha} \dot{\partial}_\gamma G_{\beta\overline{\tau}}$.

By defining $D(\overline{X}) = \overline{DX}$ and complex linearity, the Chern-Finsler connection D can be extended to the whole complex vector bundle $T^{\mathbb{C}}\widetilde{M}$ and its dual complex vector bundle $T^{\mathbb{C}*}\widetilde{M}$ by requiring $D\varphi(X) + \varphi(DX) = \mathrm{d}\varphi(X)$ for every $\varphi \in \chi(T^{\mathbb{C}*}\widetilde{M})$ and $X \in \chi(T^{\mathbb{C}}\widetilde{M})$. Thus the Chern-Finsler connection can also be extended to the complex linear connection $D : \chi(T^{r,s}_{\mathbb{C}}\widetilde{M}) \longrightarrow \chi(T^{r,s}_{\mathbb{C}}\widetilde{M} \otimes T^{r,s}_{\mathbb{C}}\widetilde{M})$ in the usual way. All the extended connections are still called the Chern-Finsler connection with the conjugation preserving the type. Let $\bigtriangledown$ be the covariant differentiation defined by D. Then, according to the adapted local frame $\{\delta_\alpha, \dot{\partial}_\alpha, \delta_{\overline{\alpha}}, \dot{\partial}_{\overline{\alpha}}\}$

of $T^{\mathbb{C}}(\widetilde{M})$ and the local dual frame $\{\mathrm{d}z^\alpha, \delta v^\alpha, \mathrm{d}\overline{z}^\alpha, \delta\overline{v}^\alpha\}$ for $T^{\mathbb{C}*}\widetilde{M}$, we have

$$\begin{aligned} &\nabla_{\delta_\alpha}\mathrm{d}z^\beta = -\Gamma^\beta_{\gamma;\alpha}\mathrm{d}z^\gamma, \quad \nabla_{\delta_\alpha}\mathrm{d}\overline{z}^\beta = 0, \quad \nabla_{\delta_\alpha}\delta v^\beta = -\Gamma^\beta_{\gamma;\alpha}\delta v^\gamma, \quad \nabla_{\delta_\alpha}\delta\overline{v}^\beta = 0, \\ &\nabla_{\dot{\partial}_\alpha}\mathrm{d}z^\beta = -\Gamma^\beta_{\gamma\alpha}\mathrm{d}z^\gamma, \quad \nabla_{\dot{\partial}_\alpha}\mathrm{d}\overline{z}^\beta = 0, \quad \nabla_{\dot{\partial}_\alpha}\delta v^\beta = -\Gamma^\beta_{\gamma\alpha}\delta v^\gamma, \quad \nabla_{\dot{\partial}_\alpha}\delta\overline{v}^\beta = 0. \end{aligned} \tag{2.4}$$

Locally, the $(2,0)$-torsion θ of the connection is given by

$$\theta = \theta^\sigma \otimes \delta_\sigma, \tag{2.5}$$

where $\theta^\sigma = \frac{1}{2}[\Gamma^\sigma_{\mu;\nu} - \Gamma^\sigma_{\nu;\mu}]\mathrm{d}z^\mu \wedge \mathrm{d}z^\nu + \Gamma^\sigma_{\gamma\nu}\delta v^\gamma \wedge \mathrm{d}z^\nu$. The $(1,1)$-torsion τ for the connection is given by

$$\tau = \tau^\alpha \otimes \dot{\partial}_\alpha, \tag{2.6}$$

where $\tau^\alpha = -\delta_{\overline{\nu}}(\Gamma^\alpha_\mu)\mathrm{d}z^\mu \wedge \mathrm{d}\overline{z}^\nu - \Gamma^\alpha_{\overline{\beta};\mu}\mathrm{d}z^\mu \wedge \delta\overline{v}^\beta$. The curvature operator $\Omega = \Omega^\alpha_\beta \otimes [\mathrm{d}z^\beta \otimes \delta_\alpha + \delta v^\beta \otimes \dot{\partial}_\alpha]$, where $\Omega^\alpha_\beta = \overline{\partial}\omega^\alpha_\beta$. Then for any $X, Y \in \mathcal{X}(T^{1,0}\widetilde{M})$, we have

$$\nabla_X\nabla_Y - \nabla_Y\nabla_X = \nabla_{[X,Y]}, \tag{2.7}$$

$$\nabla_X\nabla_{\overline{Y}} - \nabla_{\overline{Y}}\nabla_X = \nabla_{[X,\overline{Y}]} + \Omega(X, \overline{Y}). \tag{2.8}$$

3 Decomposition of the Exterior Derivative on $\widetilde{M}$

Now we consider the space

$$\mathcal{A} = \bigoplus_{p,q,r,s} \mathcal{A}^{p,q;r,s}$$

of C^∞ complex-valued forms with compact support on $\widetilde{M}$, which is defined by [17, 23, 24], where $\mathcal{A}^{p,q;r,s}$ is the non-zero set of $(p,q;r,s)$-forms only when they act on p vector fields of h-type, on q-vector fields of $\overline{h}$-type, on r vector fields of v-type, and on s vector fields of $\overline{v}$-type. Let $\{\mathrm{d}z^\alpha, \delta v^\alpha, \mathrm{d}\overline{z}^\alpha, \delta\overline{v}^\alpha\}$ be the adapted local frame of $T^{\mathbb{C}*}\widetilde{M}$. For any $\varphi \in \mathcal{A}^{p,q;r,s}$, it can be represented by

$$\varphi = \frac{1}{p!q!r!s!}\varphi_{A_p\overline{B}_qC_r\overline{D}_s}\mathrm{d}z^{A_p} \wedge \mathrm{d}\overline{z}^{B_q} \wedge \delta v^{C_r} \wedge \delta\overline{v}^{D_s},$$

where $A_p = (\mu_1, \cdots, \mu_p)$, $B_q = (\nu_1, \cdots, \nu_q)$, $C_r = (\alpha_1, \cdots, \alpha_r)$, $D_s = (\beta_1, \cdots, \beta_s)$, and $\mathrm{d}z^{A_p} = \mathrm{d}z^{\mu_1} \wedge \cdots \wedge \mathrm{d}z^{\mu_p}$, $\mathrm{d}\overline{z}^{B_q} = \mathrm{d}\overline{z}^{\nu_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{\nu_q}$, $\delta v^{C_r} = \delta v^{\alpha_1} \wedge \cdots \wedge \delta v^{\alpha_r}$, $\delta\overline{v}^{D_s} = \delta\overline{v}^{\beta_1} \wedge \cdots \wedge \delta\overline{v}^{\beta_s}$.

For the adapted local frame $\{\delta v^\alpha, \delta\overline{v}^\alpha\}$, we have

$$\begin{aligned} \partial(\delta v^\alpha) &= \Gamma^\alpha_{\beta;\mu}\delta v^\beta \wedge \mathrm{d}z^\mu, \\ \overline{\partial}(\delta v^\alpha) &= \delta_{\overline{\beta}}(\Gamma^\alpha_\mu)\mathrm{d}\overline{z}^\beta \wedge \mathrm{d}z^\mu + \dot{\partial}_{\overline{\beta}}(\Gamma^\alpha_\mu)\delta\overline{v}^\beta \wedge \mathrm{d}z^\mu, \\ \partial(\delta\overline{v}^\alpha) &= \delta_\beta(\overline{\Gamma^\alpha_\mu})\mathrm{d}z^\beta \wedge \mathrm{d}\overline{z}^\mu + \dot{\partial}_\beta(\overline{\Gamma^\alpha_\mu})\delta v^\beta \wedge \mathrm{d}\overline{z}^\mu, \\ \overline{\partial}(\delta\overline{v}^\alpha) &= \overline{\Gamma^\alpha_{\beta;\mu}}\delta\overline{v}^\beta \wedge \mathrm{d}\overline{z}^\mu, \end{aligned} \tag{3.1}$$

where in the first equation we have used the identities (see [15])

$$\delta_\beta(\Gamma^\alpha_\mu) - \delta_\mu(\Gamma^\alpha_\beta) = 0, \quad \dot{\partial}_\beta(\Gamma^\alpha_\mu) = \Gamma^\alpha_{\beta;\mu}.$$

Consequently for $\varphi \in \mathcal{A}^{p,q;r,s}$, we get

$$\begin{aligned} \partial\varphi &\in \mathcal{A}^{p+1,q;r,s} \oplus \mathcal{A}^{p,q;r+1,s} \oplus \mathcal{A}^{p+1,q+1;r,s-1} \oplus \mathcal{A}^{p,q+1;r+1,s-1}, \\ \overline{\partial}\varphi &\in \mathcal{A}^{p,q+1;r,s} \oplus \mathcal{A}^{p,q;r,s+1} \oplus \mathcal{A}^{p+1,q+1;r-1,s} \oplus \mathcal{A}^{p+1,q;r-1,s+1}. \end{aligned} \tag{3.2}$$

In the following, we shall rewrite the exterior derivative operators on $\widetilde{M}$ by using the Chern Finsler connection $\bigtriangledown$. For simplicity, we need to introduce some notations: denote $i(X)$ the usual interior product induced by $X \in \mathcal{X}(T^{\mathbb{C}}(T^{1,0}M))$; and denote $e(\varphi)$ the exterior product induced by φ, for $\psi \in \mathcal{A}^{p,q;r,s}$, $e(\varphi)\psi = \varphi \wedge \psi$. Then for any $X, Y \in \mathcal{X}(T^{\mathbb{C}}(T^{1,0}M))$, and $\varphi \in \mathcal{A}^{p,q;r,s}$, we have

$$i(X)i(Y)\varphi = -i(Y)i(X)\varphi, \quad i(\overline{X})\overline{\varphi} = \overline{i(X)\varphi}, \tag{3.3}$$

and for any $\varphi, \psi, \phi \in \mathcal{A}^{p,q;r,s}$, we have

$$\begin{aligned} &e(\varphi)e(\psi)\phi = (-1)^{p+q+r+s} e(\psi)e(\varphi)\phi, \\ &i(X)e(\psi)\phi = [i(X)\psi] \wedge \phi + (-1)^{p+q+r+s} e(\psi)i(X)\phi. \end{aligned} \tag{3.4}$$

Under the adapted local frame,

$$\begin{aligned} &i(\delta_\alpha)\mathrm{d}z^\beta = \delta^\beta_\alpha, \quad i(\delta_\alpha)\mathrm{d}\overline{z}^\beta = 0, \quad i(\delta_\alpha)\delta v^\beta = 0, \quad i(\delta_\alpha)\delta\overline{v}^\beta = 0, \\ &i(\dot{\partial}_\alpha)\mathrm{d}z^\beta = 0, \quad i(\dot{\partial}_\alpha)\mathrm{d}\overline{z}^\beta = 0, \quad i(\dot{\partial}_\alpha)\delta v^\beta = \delta^\beta_\alpha, \quad i(\dot{\partial}_\alpha)\delta\overline{v}^\beta = 0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &i(\delta_{\overline{\nu}})e(\delta\overline{v}^\sigma) = -e(\delta\overline{v}^\sigma)i(\delta_{\overline{\nu}}), \quad i(\dot{\partial}_\gamma)e(\delta v^\alpha) = \delta^\alpha_\gamma - e(\delta v^\alpha)i(\dot{\partial}_\gamma), \\ &i(\delta_\mu)e(\mathrm{d}z^\nu) = \delta^\nu_\mu - e(\mathrm{d}z^\nu)i(\delta_\mu). \end{aligned} \tag{3.6}$$

It is easy to check the following lemma.

Lemma 3.1 *Let $\varphi \in \mathcal{A}^{p,q;r,s}$. Then we have*

$$\begin{aligned}
\bigtriangledown_{\delta_\alpha} i(\delta_\beta)\varphi &= i(\delta_\beta) \bigtriangledown_{\delta_\alpha} \varphi + \Gamma^\gamma_{\beta;\alpha} i(\delta_\gamma)\varphi, \\
\bigtriangledown_{\delta_\alpha} i(\delta_{\overline{\beta}})\varphi &= i(\delta_{\overline{\beta}}) \bigtriangledown_{\delta_\alpha} \varphi, \\
\bigtriangledown_{\delta_\alpha} i(\dot{\partial}_\beta)\varphi &= i(\dot{\partial}_\beta) \bigtriangledown_{\delta_\alpha} \varphi + \Gamma^\gamma_{\beta;\alpha} i(\dot{\partial}_\gamma)\varphi, \\
\bigtriangledown_{\delta_\alpha} i(\dot{\partial}_{\overline{\beta}})\varphi &= i(\dot{\partial}_{\overline{\beta}}) \bigtriangledown_{\delta_\alpha} \varphi, \\
\bigtriangledown_{\dot{\partial}_\alpha} i(\dot{\partial}_\beta)\varphi &= i(\dot{\partial}_\beta) \bigtriangledown_{\dot{\partial}_\alpha} \varphi + \Gamma^\gamma_{\beta\alpha} i(\dot{\partial}_\gamma)\varphi, \\
\bigtriangledown_{\dot{\partial}_\alpha} i(\dot{\partial}_{\overline{\beta}})\varphi &= i(\dot{\partial}_{\overline{\beta}}) \bigtriangledown_{\dot{\partial}_\alpha} \varphi, \\
\bigtriangledown_{\dot{\partial}_\alpha} i(\delta_\beta)\varphi &= i(\delta_\beta) \bigtriangledown_{\dot{\partial}_\alpha} \varphi + \Gamma^\gamma_{\beta\alpha} i(\delta_\gamma)\varphi, \\
\bigtriangledown_{\dot{\partial}_{\overline{\alpha}}} i(\delta_\beta)\varphi &= i(\delta_\beta) \bigtriangledown_{\dot{\partial}_{\overline{\alpha}}} \varphi, \\
\bigtriangledown_{\delta_\alpha} e(\mathrm{d}z^\beta)\varphi &= -e(\mathrm{d}z^\beta) \bigtriangledown_{\delta_\alpha} \varphi - \Gamma^\beta_{\mu;\alpha} e(\mathrm{d}z^\mu)\varphi, \\
\bigtriangledown_{\delta_\alpha} e(\mathrm{d}\overline{z}^\beta)\varphi &= -e(\mathrm{d}\overline{z}^\beta) \bigtriangledown_{\delta_\alpha} \varphi, \\
\bigtriangledown_{\delta_\alpha} e(\delta v^\beta)\varphi &= -e(\delta v^\beta) \bigtriangledown_{\delta_\alpha} \varphi - \Gamma^\beta_{\gamma;\alpha} e(\delta v^\gamma)\varphi, \\
\bigtriangledown_{\delta_\alpha} e(\delta\overline{v}^\beta)\varphi &= -e(\delta\overline{v}^\beta) \bigtriangledown_{\delta_\alpha} \varphi, \\
\bigtriangledown_{\dot{\partial}_\alpha} e(\delta v^\beta)\varphi &= -e(\delta v^\beta) \bigtriangledown_{\dot{\partial}_\alpha} \varphi - \Gamma^\beta_{\gamma\alpha} e(\delta v^\gamma)\varphi, \\
\bigtriangledown_{\dot{\partial}_\alpha} e(\delta\overline{v}^\beta)\varphi &= -e(\delta\overline{v}^\beta) \bigtriangledown_{\dot{\partial}_\alpha} \varphi, \\
\bigtriangledown_{\dot{\partial}_\alpha} e(\mathrm{d}z^\beta)\varphi &= -e(\mathrm{d}z^\beta) \bigtriangledown_{\dot{\partial}_\alpha} \varphi - \Gamma^\beta_{\gamma\alpha} e(\mathrm{d}z^\gamma)\varphi, \\
\bigtriangledown_{\dot{\partial}_{\overline{\alpha}}} e(\mathrm{d}z^\beta)\varphi &= -e(\mathrm{d}z^\beta) \bigtriangledown_{\dot{\partial}_{\overline{\alpha}}} \varphi.
\end{aligned}$$

Combined with (2.4), the exterior derivative operator $\mathrm{d} = \partial + \overline{\partial}$ can be represented by using the Chern Finsler connection

$$\partial = e(\mathrm{d}z^\alpha) \bigtriangledown_{\delta_\alpha} + \Gamma^\gamma_{\beta;\alpha} e(\mathrm{d}z^\alpha)e(\mathrm{d}z^\beta)i(\delta_\gamma) + e(\delta v^\alpha) \bigtriangledown_{\dot{\partial}_\alpha} + \Gamma^\gamma_{\alpha\beta} e(\delta v^\alpha)e(\mathrm{d}z^\beta)i(\delta_\gamma)$$

$$+\,\delta_\beta(\overline{\Gamma^\alpha_\mu})e(\mathrm{d}z^\beta)e(\mathrm{d}\overline{z}^\mu)i(\dot{\partial}_{\overline{\alpha}})+\dot{\partial}_\beta(\overline{\Gamma^\alpha_\mu})e(\delta v^\beta)e(\mathrm{d}\overline{z}^\mu)i(\dot{\partial}_{\overline{\alpha}}), \tag{3.7}$$

$$\overline{\partial}=e(\mathrm{d}\overline{z}^\alpha)\bigtriangledown_{\delta_{\overline{\alpha}}}+\overline{\Gamma^\gamma_{\beta;\alpha}}e(\mathrm{d}\overline{z}^\alpha)e(\mathrm{d}\overline{z}^\beta)i(\delta_{\overline{\gamma}})+e(\delta\overline{v}^\alpha)\bigtriangledown_{\dot{\partial}_{\overline{\alpha}}}+\overline{\Gamma^\gamma_{\alpha\beta}}e(\delta\overline{v}^\alpha)e(\mathrm{d}\overline{z}^\beta)i(\delta_{\overline{\gamma}})$$
$$+\,\delta_{\overline{\beta}}(\Gamma^\alpha_\mu)e(\mathrm{d}\overline{z}^\beta)e(\mathrm{d}z^\mu)i(\dot{\partial}_\alpha)+\dot{\partial}_{\overline{\beta}}(\Gamma^\alpha_\mu)e(\delta\overline{v}^\beta)e(\mathrm{d}z^\mu)i(\dot{\partial}_\alpha). \tag{3.8}$$

In accordance with the classification, denote the operators (see [17])

$$D_{\mathcal{H}}=e(\mathrm{d}z^\alpha)\bigtriangledown_{\delta_\alpha}+\Gamma^\gamma_{\beta;\alpha}e(\mathrm{d}z^\alpha)e(\mathrm{d}z^\beta)i(\delta_\gamma):\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p+1,q;r,s}, \tag{3.9}$$

$$\overline{D}_{\mathcal{H}}=e(\mathrm{d}\overline{z}^\alpha)\bigtriangledown_{\delta_{\overline{\alpha}}}+\overline{\Gamma^\gamma_{\beta;\alpha}}e(\mathrm{d}\overline{z}^\alpha)e(\mathrm{d}\overline{z}^\beta)i(\delta_{\overline{\gamma}}):\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p,q+1;r,s}, \tag{3.10}$$

$$D_{\mathcal{V}}=e(\delta v^\alpha)\bigtriangledown_{\dot{\partial}_\alpha}+\Gamma^\gamma_{\alpha\beta}e(\delta v^\alpha)e(\mathrm{d}z^\beta)i(\delta_\gamma):\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p,q;r+1,s}, \tag{3.11}$$

$$\overline{D}_{\mathcal{V}}=e(\delta\overline{v}^\alpha)\bigtriangledown_{\dot{\partial}_{\overline{\alpha}}}+\overline{\Gamma^\gamma_{\alpha\beta}}e(\delta\overline{v}^\alpha)e(\mathrm{d}\overline{z}^\beta)i(\delta_{\overline{\gamma}}):\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p,q;r,s+1}, \tag{3.12}$$

$$D_3=\delta_\beta(\overline{\Gamma^\alpha_\mu})e(\mathrm{d}z^\beta)e(\mathrm{d}\overline{z}^\mu)i(\dot{\partial}_{\overline{\alpha}}):\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p+1,q+1;r,s-1}, \tag{3.13}$$

$$\overline{D}_3=\delta_{\overline{\beta}}(\Gamma^\alpha_\mu)e(\mathrm{d}\overline{z}^\beta)e(\mathrm{d}z^\mu)i(\dot{\partial}_\alpha):\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p+1,q+1;r-1,s}, \tag{3.14}$$

$$D_4=\dot{\partial}_\beta(\overline{\Gamma^\alpha_\mu})e(\delta v^\beta)e(\mathrm{d}\overline{z}^\mu)i(\dot{\partial}_{\overline{\alpha}}):\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p,q+1;r+1,s-1}, \tag{3.15}$$

$$\overline{D}_4=\dot{\partial}_{\overline{\beta}}(\Gamma^\alpha_\mu)e(\delta\overline{v}^\beta)e(\mathrm{d}z^\mu)i(\dot{\partial}_\alpha):\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p+1,q;r-1,s+1}. \tag{3.16}$$

Then the exterior derivative operator $\mathrm{d}=\partial+\overline{\partial}$ can be rewritten as

$$\mathrm{d}=D_{\mathcal{H}}+D_{\mathcal{V}}+D_3+D_4+\overline{D}_{\mathcal{H}}+\overline{D}_{\mathcal{V}}+\overline{D}_3+\overline{D}_4. \tag{3.17}$$

4 The Hodge-Laplace Operator and Weitzenböck Formulas on $\widetilde{M}$

Let (M,F) be an m-dimensional strongly pseudoconvex complex Finsler manifold, where F is the strongly pseudoconvex complex Finsler metric. Then, F induces a Hermitian metric

$$\widetilde{G}=G_{\alpha\overline{\beta}}\mathrm{d}z^\alpha\otimes\mathrm{d}\overline{z}^\beta+G_{\alpha\overline{\beta}}\delta v^\alpha\otimes\delta\overline{v}^\beta, \tag{4.1}$$

on $T^{\mathbb{C}}\widetilde{M}$, denoted by $\langle\,,\,\rangle$. Associated to this metric, we have the Kaehler form $\omega=\mathrm{i}G_{\alpha\overline{\beta}}\mathrm{d}z^\alpha\wedge\mathrm{d}\overline{z}^\beta+\mathrm{i}G_{\alpha\overline{\beta}}\delta v^\alpha\wedge\delta\overline{v}^\beta$ and $\omega^{2m}=(-1)^m(2m)!\mathcal{G}^2\mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^m\wedge d\overline{z}^1\wedge\cdots\wedge\mathrm{d}\overline{z}^m\wedge\delta v^1\wedge\cdots\wedge\delta v^m\wedge\delta\overline{v}^1\wedge\cdots\wedge\delta\overline{v}^m$, where $\mathcal{G}=\det(G_{\alpha\overline{\beta}})$. It is easy to check that $\mathrm{d}V_{\widetilde{M}}=\frac{\omega^{2m}}{(2m)!}$ defines a global invariant volume form of $\widetilde{M}$. Associated with the decomposition $T^{1,0}\widetilde{M}=\mathcal{H}\oplus\mathcal{V}$, and according to the adapted local frame $\{\delta_\alpha,\dot{\partial}_\alpha,\delta_{\overline{\alpha}},\dot{\partial}_{\overline{\alpha}}\}$ of $T^{\mathbb{C}}(\widetilde{M})$ and $\{\mathrm{d}z^\alpha,\delta v^\alpha,\mathrm{d}\overline{z}^\alpha,\delta\overline{v}^\alpha\}$ for $T^{\mathbb{C}*}\widetilde{M}$, we have

$$\begin{aligned}&\langle\delta_\alpha,\delta_\beta\rangle=G_{\alpha\overline{\beta}},\qquad\langle\delta_\alpha,\dot{\partial}_\beta\rangle=0,\qquad\langle\dot{\partial}_\alpha,\dot{\partial}_\beta\rangle=G_{\alpha\overline{\beta}},\\&\langle\mathrm{d}z^\alpha,\mathrm{d}z^\beta\rangle=G^{\overline{\beta}\alpha},\quad\langle\mathrm{d}z^\alpha,\delta v^\beta\rangle=0,\quad\langle\delta v^\alpha,\delta v^\beta\rangle=G^{\overline{\beta}\alpha}.\end{aligned} \tag{4.2}$$

Now, if we denote $G^{\overline{A}_pB_p}=G^{\overline{\alpha}_1\beta_1}\cdots G^{\overline{\alpha}_2\beta_2}\cdots G^{\overline{\alpha}_p\beta_p}$ on $\widetilde{M}$, then for any $\varphi,\psi\in\mathcal{A}^{p,q;r,s}$, at each point $v\in T^{1,0}M$, the inner product can be defined by

$$\langle\varphi,\psi\rangle=\frac{1}{p!q!r!s!}\varphi_{A_p\overline{B}_qC_r\overline{D}_s}\overline{\psi^{\overline{A}_pB_q\overline{C}_rD_s}}, \tag{4.3}$$

where $\psi^{\overline{A}_pB_q\overline{C}_rD_s}=\psi_{E_p\overline{F}_qH_r\overline{L}_s}G^{\overline{A}_pE_p}G^{\overline{F}_qB_q}G^{\overline{C}_rH_r}G^{\overline{L}_sD_s}$.

Definition 4.1 *Let (M,F) be an m-dimensional strongly pseudoconvex compact complex Finsler manifold. The global inner product of two forms $\varphi,\psi\in\mathcal{A}^{p,q;r,s}$ is defined by*

$$(\varphi,\psi)=\int_{\widetilde{M}}\langle\varphi,\psi\rangle\frac{\omega^{2m}}{(2m)!}. \tag{4.4}$$

It is easy to check that $(\ ,\ )$ *satisfies the following properties:*

(1) $(\varphi,\varphi)\geq 0$ *and* $(\varphi,\varphi)=0$ *if and only if* $\varphi=0$;

(2) $(a\varphi+b\phi,\psi)=a(\varphi,\psi)+b(\phi,\psi)$ *for any* $\phi,\varphi,\psi\in\mathcal{A}^{p,q;r,s}$ *and* $a,b\in\mathbb{C}$;

(3) $\overline{(\varphi,\psi)}=(\psi,\varphi)$.

We define $\|\varphi\|=\sqrt{(\varphi,\varphi)}$ as usual.

As in Hermitian geometry (see [6]), we define the Hodge star operator

$$*:\ \mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{m-q,m-p;m-s,m-r},$$

such that for any $\psi\in\mathcal{A}^{p,q;r,s}$,

$$\begin{aligned}*\psi=&(-1)^{m(p+r+1)+(r+s)(p+q)}G_{A_qA_{m-q}\overline{B}_p\overline{B}_{m-p}}G_{C_sC_{m-s}\overline{D}_r\overline{D}_{m-r}}\\&\cdot\psi^{\overline{B}_pA_q\overline{D}_rC_s}\mathrm{d}z^{A_{m-q}}\wedge\mathrm{d}\overline{z}^{B_{m-p}}\wedge\delta v^{C_{m-s}}\wedge\delta\overline{v}^{D_{m-r}}.\end{aligned}\tag{4.5}$$

Then we have the result as follows.

Theorem 4.1 *Let* (M,F) *be an* m*-dimensional strongly pseudoconvex compact complex Finsler manifold. Then the complex linear map* $*:\ \mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{m-q,m-p;m-s,m-r}$ *satisfies*

(i) $\langle\varphi,\psi\rangle\frac{\omega^{2m}}{(2m)!}=\varphi\wedge*\overline{\psi}$;

(ii) $\overline{*\psi}=*\overline{\psi}$ (*i.e.,* $*$ *is a real operator*);

(iii) $**\psi=(-1)^{p+q+r+s}\psi$.

Now we define the adjoint operators of the operators (3.9)–(3.16) as follows.

Definition 4.2 $D_{\mathcal{H}}^*=-*\overline{D}_{\mathcal{H}}*$, $D_{\mathcal{V}}^*=-*\overline{D}_{\mathcal{V}}*$, $D_3^*=-*\overline{D}_3*$, $D_4^*=-*\overline{D}_4*$. *By conjugation, we have*

$$\overline{D}_{\mathcal{H}}^*=-*D_{\mathcal{H}}*,\quad\overline{D}_{\mathcal{V}}^*=-*D_{\mathcal{V}}*,\quad\overline{D}_3^*=-*D_3*,\quad\overline{D}_4^*=-*D_4*.$$

Then we can define the adjoint of d *as* $\mathrm{d}^*=-*\mathrm{d}*$. *It follows from* (3.17) *that*

$$\mathrm{d}^*=D_{\mathcal{H}}^*+D_{\mathcal{V}}^*+D_3^*+D_4^*+\overline{D}_{\mathcal{H}}^*+\overline{D}_{\mathcal{V}}^*+\overline{D}_3^*+\overline{D}_4^*.\tag{4.6}$$

It is easy to see that

$$\begin{aligned}D_{\mathcal{H}}^*&:\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p,q-1;r,s},\\\overline{D}_{\mathcal{H}}^*&:\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p-1,q;r,s},\\D_{\mathcal{V}}^*&:\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p,q;r,s-1},\\\overline{D}_{\mathcal{V}}^*&:\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p,q;r-1,s},\\D_3^*&:\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p-1,q-1;r+1,s},\\\overline{D}_3^*&:\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p-1,q-1;r,s+1},\\D_4^*&:\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p-1,q;r+1,s-1},\\\overline{D}_4^*&:\mathcal{A}^{p,q;r,s}\longrightarrow\mathcal{A}^{p,q-1;r-1,s+1}.\end{aligned}$$

Taking $\varphi\in\mathcal{A}^{p-1,q;r,s}$, $\psi\in\mathcal{A}^{p,q;r,s}$, then by type reason we have

$$\begin{aligned}(D_{\mathcal{H}}\varphi,\psi)=(\mathrm{d}\varphi,\psi)&=\int_{\widetilde{M}}\mathrm{d}\varphi\wedge*\overline{\psi}\\&=(-1)^{p+q+r+s+1}\int_{\widetilde{M}}\varphi\wedge\mathrm{d}*\overline{\psi}+\int_{\widetilde{M}}\mathrm{d}(\varphi\wedge*\overline{\psi})\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{p+q+r+s+1} \int_{\widetilde{M}} \varphi \wedge \mathrm{d} * \overline{\psi} \\
&= - \int_{\widetilde{M}} \varphi \wedge * \overline{(*\mathrm{d}*)\psi} = - \int_{\widetilde{M}} \varphi \wedge * \overline{(*D_{\mathcal{H}}*)\psi} \\
&= (\varphi, \overline{D}^*_{\mathcal{H}} \psi).
\end{aligned}
$$

Similarly,

$$
(D_{\mathcal{V}}\varphi, \psi) = (\varphi, \overline{D}^*_{\mathcal{V}}\psi), \quad (D_3\varphi, \psi) = (\varphi, \overline{D}^*_3\psi), \quad (D_4\varphi, \psi) = (\varphi, \overline{D}^*_4\psi).
$$

By conjugation, we have

$$
\begin{aligned}
(\overline{D}_{\mathcal{H}}\varphi, \psi) &= (\varphi, D^*_{\mathcal{H}}\psi), \quad & (\overline{D}_{\mathcal{V}}\varphi, \psi) &= (\varphi, D^*_{\mathcal{V}}\psi), \\
(\overline{D}_3\varphi, \psi) &= (\varphi, D^*_3\psi), \quad & (\overline{D}_4\varphi, \psi) &= (\varphi, D^*_4\psi).
\end{aligned}
$$

According to (3.9)–(3.16), we get the explicit expressions of the adjoint operators by solving the above formulas.

Theorem 4.2 *Let (M, F) be an m-dimensional strongly pseudoconvex compact complex Finsler manifold. Then we have*

$$
\begin{aligned}
D^*_{\mathcal{H}} &= -G^{\overline{\beta}\alpha} \bigtriangledown_{\delta_\alpha} i(\delta_{\overline{\beta}}) + G^{\overline{\beta}\alpha}[\Gamma^\gamma_{\alpha;\gamma} - 2\Gamma^\gamma_{\gamma;\alpha}] i(\delta_{\overline{\beta}}) \\
&\quad + \Gamma^\gamma_{\beta;\alpha} G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}}), \qquad (4.7) \\
\overline{D}^*_{\mathcal{H}} &= -G^{\overline{\beta}\alpha} \bigtriangledown_{\delta_{\overline{\beta}}} i(\delta_\alpha) + G^{\overline{\beta}\alpha}[\overline{\Gamma^\gamma_{\beta;\gamma}} - 2\overline{\Gamma^\gamma_{\gamma;\beta}}] i(\delta_\alpha) \\
&\quad + \overline{\Gamma^\gamma_{\beta;\alpha}} G^{\overline{\beta}\sigma} G^{\overline{\alpha}\tau} G_{\varepsilon\overline{\gamma}} e(\mathrm{d}z^\varepsilon) i(\delta_\sigma) i(\delta_\tau), \qquad (4.8) \\
D^*_{\mathcal{V}} &= -G^{\overline{\beta}\alpha} \bigtriangledown_{\dot{\partial}_\alpha} i(\dot{\partial}_{\overline{\beta}}) - G^{\overline{\beta}\alpha} \Gamma^\gamma_{\gamma\alpha} i(\dot{\partial}_{\overline{\beta}}) \\
&\quad + \Gamma^\gamma_{\alpha\beta} G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) i(\delta_{\overline{\tau}}) i(\dot{\partial}_{\overline{\sigma}}), \qquad (4.9) \\
\overline{D}^*_{\mathcal{V}} &= -G^{\overline{\beta}\alpha} \bigtriangledown_{\dot{\partial}_{\overline{\beta}}} i(\dot{\partial}_\alpha) - G^{\overline{\beta}\alpha} \overline{\Gamma^\gamma_{\gamma\beta}} i(\dot{\partial}_\alpha) \\
&\quad + \overline{\Gamma^\gamma_{\alpha\beta}} G^{\overline{\alpha}\sigma} G^{\overline{\beta}\tau} G_{\varepsilon\overline{\gamma}} e(\mathrm{d}z^\varepsilon) i(\delta_\tau) i(\dot{\partial}_\sigma), \qquad (4.10) \\
D^*_3 &= \delta_\beta(\overline{\Gamma^\alpha_\mu}) G^{\overline{\gamma}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(\delta v^\tau) i(\delta_\nu) i(\delta_{\overline{\gamma}}), \qquad (4.11) \\
\overline{D}^*_3 &= -\delta_{\overline{\beta}}(\Gamma^\alpha_\mu) G^{\overline{\beta}\gamma} G^{\overline{\nu}\mu} G_{\alpha\overline{\sigma}} e(\delta\overline{v}^\sigma) i(\delta_\gamma) i(\delta_{\overline{\nu}}), \qquad (4.12) \\
D^*_4 &= \dot{\partial}_\beta(\overline{\Gamma^\alpha_\mu}) G^{\overline{\gamma}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(\delta v^\tau) i(\delta_\nu) i(\dot{\partial}_{\overline{\gamma}}), \qquad (4.13) \\
\overline{D}^*_4 &= -\dot{\partial}_{\overline{\beta}}(\Gamma^\alpha_\mu) G^{\overline{\lambda}\mu} G^{\overline{\beta}\varepsilon} G_{\alpha\overline{\sigma}} e(\delta\overline{v}^\sigma) i(\dot{\partial}_\varepsilon) i(\delta_{\overline{\lambda}}). \qquad (4.14)
\end{aligned}
$$

Proof We only need to prove (4.7), and the others are similar to obtain. For any $\varphi, \psi \in \mathcal{A}^{p,q;r,s}$, according to (3.10), we have

$$
\begin{aligned}
(\varphi, D^*_{\mathcal{H}}\psi) &= (\overline{D}_{\mathcal{H}}\varphi, \psi) = (e(\mathrm{d}\overline{z}^\alpha) \bigtriangledown_{\delta_{\overline{\alpha}}} \varphi + \overline{\Gamma^\gamma_{\beta;\alpha}} e(\mathrm{d}\overline{z}^\alpha) e(\mathrm{d}\overline{z}^\beta) i(\delta_{\overline{\gamma}})\varphi, \psi) \\
&= \int_{\widetilde{M}} \langle e(\mathrm{d}\overline{z}^\alpha) \bigtriangledown_{\delta_{\overline{\alpha}}} \varphi, \psi\rangle \mathrm{d}V + \int_{\widetilde{M}} \langle \overline{\Gamma^\gamma_{\beta;\alpha}} e(\mathrm{d}\overline{z}^\alpha) e(\mathrm{d}\overline{z}^\beta) i(\delta_{\overline{\gamma}})\varphi, \psi\rangle \mathrm{d}V,
\end{aligned}
$$

while

$$
\begin{aligned}
&\int_{\widetilde{M}} \langle e(\mathrm{d}\overline{z}^\alpha) \bigtriangledown_{\delta_{\overline{\alpha}}} \varphi, \psi\rangle \mathrm{d}V \\
= &\int_{\widetilde{M}} G^{\overline{\alpha}\beta} \langle \bigtriangledown_{\delta_{\overline{\alpha}}} \varphi, i(\delta_{\overline{\beta}})\psi\rangle \mathrm{d}V
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\widetilde{M}} G^{\overline{\alpha}\beta}\delta_{\overline{\alpha}}(\langle\varphi, i(\delta_{\overline{\beta}})\psi\rangle)\mathcal{G}^2(-1)^m \mathrm{d}z\wedge \mathrm{d}\overline{z}\wedge\delta v\wedge\delta\overline{v} - \int_{\widetilde{M}} G^{\overline{\alpha}\beta}\langle\varphi, \bigtriangledown_{\delta_\alpha} i(\delta_{\overline{\beta}})\psi\rangle \mathrm{d}V \\
&= \int_{\widetilde{M}} \{\delta_{\overline{\alpha}}[G^{\overline{\alpha}\beta}\langle\varphi, i(\delta_{\overline{\beta}})\psi\rangle\mathcal{G}^2] - \delta_{\overline{\alpha}}(G^{\overline{\alpha}\beta})\langle\varphi, i(\delta_{\overline{\beta}})\psi\rangle\mathcal{G}^2 - G^{\overline{\alpha}\beta}\langle\varphi, i(\delta_{\overline{\beta}})\psi\rangle\delta_{\overline{\alpha}}(\mathcal{G}^2)\} \\
&\quad \cdot (-1)^m \mathrm{d}z\wedge \mathrm{d}\overline{z}\wedge\delta v\wedge\delta\overline{v} - \int_{\widetilde{M}} G^{\overline{\alpha}\beta}\langle\varphi, \bigtriangledown_{\delta_\alpha} i(\delta_{\overline{\beta}})\psi\rangle \mathrm{d}V \\
&= \int_{\widetilde{M}} \big\{\langle\varphi, G^{\overline{\beta}\gamma}\Gamma^{\alpha}_{\gamma;\alpha} i(\delta_{\overline{\beta}})\psi\rangle - \langle\varphi, 2G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma;\alpha} i(\delta_{\overline{\beta}})\psi\rangle - \langle\varphi, G^{\overline{\beta}\alpha}\bigtriangledown_{\delta_\alpha} i(\delta_{\overline{\beta}})\psi\rangle\big\}\mathrm{d}V \\
&= \int_{\widetilde{M}} \langle\varphi, \{G^{\overline{\beta}\alpha}[\Gamma^{\gamma}_{\alpha;\gamma} - 2\Gamma^{\gamma}_{\gamma;\alpha}] i(\delta_{\overline{\beta}}) - G^{\overline{\beta}\alpha}\bigtriangledown_{\delta_\alpha} i(\delta_{\overline{\beta}})\}\psi\rangle \mathrm{d}V
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\widetilde{M}} \langle\overline{\Gamma^{\gamma}_{\beta;\alpha}} e(\mathrm{d}\overline{z}^{\alpha}) e(\mathrm{d}\overline{z}^{\beta}) i(\delta_{\overline{\gamma}})\varphi, \psi\rangle \mathrm{d}V \\
&= \int_{\widetilde{M}} \overline{\Gamma^{\gamma}_{\beta;\alpha}} G^{\overline{\alpha}\tau}\langle e(\mathrm{d}\overline{z}^{\beta}) i(\delta_{\overline{\gamma}})\varphi, i(\delta_{\overline{\tau}})\psi\rangle \mathrm{d}V \\
&= \int_{\widetilde{M}} \overline{\Gamma^{\gamma}_{\beta;\alpha}} G^{\overline{\alpha}\tau} G^{\overline{\beta}\sigma}\langle i(\delta_{\overline{\gamma}})\varphi, i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}})\psi\rangle \mathrm{d}V \\
&= \int_{\widetilde{M}} \overline{\Gamma^{\gamma}_{\beta;\alpha}} G^{\overline{\alpha}\tau} G^{\overline{\beta}\sigma} G_{\varepsilon\overline{\gamma}}\langle \varphi, e(\mathrm{d}\overline{z}^{\varepsilon}) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}})\psi\rangle \mathrm{d}V \\
&= \int_{\widetilde{M}} \langle \varphi, \Gamma^{\gamma}_{\beta;\alpha} G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^{\varepsilon}) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}})\psi\rangle \mathrm{d}V.
\end{aligned}
$$

Then

$$
\begin{aligned}
(\varphi, D^*_{\mathcal{H}}\psi) = \int_{\widetilde{M}} \langle\varphi, \{ & -G^{\overline{\beta}\alpha}\bigtriangledown_{\delta_\alpha} i(\delta_{\overline{\beta}}) + G^{\overline{\beta}\alpha}[\Gamma^{\gamma}_{\alpha;\gamma} - 2\Gamma^{\gamma}_{\gamma;\alpha}] i(\delta_{\overline{\beta}}) \\
& + \Gamma^{\gamma}_{\beta;\alpha} G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^{\varepsilon}) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}})\}\psi\rangle \mathrm{d}V,
\end{aligned}
$$

that is

$$
D^*_{\mathcal{H}} = -G^{\overline{\beta}\alpha}\bigtriangledown_{\delta_\alpha} i(\delta_{\overline{\beta}}) + G^{\overline{\beta}\alpha}[\Gamma^{\gamma}_{\alpha;\gamma} - 2\Gamma^{\gamma}_{\gamma;\alpha}] i(\delta_{\overline{\beta}}) + \Gamma^{\gamma}_{\beta;\alpha} G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^{\varepsilon}) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}}).
$$

Furthermore, we have

Theorem 4.3 *Let (M, F) be an m-dimensional strongly pseudoconvex compact complex Finsler manifold. Then for any $\varphi \in \mathcal{A}^{p,q;r,s}$, we have*

$$
\begin{aligned}
D^*_{\mathcal{H}}\varphi = & -\frac{1}{p!q!r!s!}\delta_{\alpha}(\varphi_{A_p\overline{B}_q C_r\overline{D}_s}) G^{\overline{\beta}\alpha} i(\delta_{\overline{\beta}})(\mathrm{d}z^{A_p}\wedge \mathrm{d}\overline{z}^{B_q}\wedge\delta v^{C_r}\wedge\delta\overline{v}^{D_s}) \\
& - G^{\overline{\beta}\alpha}\Gamma^{\mu}_{\sigma;\alpha} e(\mathrm{d}z^{\sigma}) i(\delta_{\overline{\beta}}) i(\delta_{\mu})\varphi - G^{\overline{\beta}\alpha}\Gamma^{\mu}_{\sigma;\alpha} e(\delta v^{\sigma}) i(\delta_{\overline{\beta}}) i(\dot{\partial}_{\mu})\varphi \\
& + G^{\overline{\beta}\alpha}[\Gamma^{\gamma}_{\alpha;\gamma} - 2\Gamma^{\gamma}_{\gamma;\alpha}] i(\delta_{\overline{\beta}})\varphi + \Gamma^{\gamma}_{\beta;\alpha} G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^{\varepsilon}) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}})\varphi, \\
\overline{D}^*_{\mathcal{H}}\varphi = & -\frac{1}{p!q!r!s!}\delta_{\overline{\beta}}(\varphi_{A_p\overline{B}_q C_r\overline{D}_s}) G^{\overline{\beta}\alpha} i(\delta_{\alpha})(\mathrm{d}z^{A_p}\wedge \mathrm{d}\overline{z}^{B_q}\wedge\delta v^{C_r}\wedge\delta\overline{v}^{D_s}) \\
& - G^{\overline{\beta}\alpha}\overline{\Gamma^{\mu}_{\sigma;\beta}} e(\mathrm{d}\overline{z}^{\sigma}) i(\delta_{\alpha}) i(\delta_{\overline{\mu}})\varphi - G^{\overline{\beta}\alpha}\overline{\Gamma^{\mu}_{\sigma;\beta}} e(\delta\overline{v}^{\sigma}) i(\delta_{\alpha}) i(\dot{\partial}_{\overline{\mu}})\varphi \\
& + G^{\overline{\beta}\alpha}[\overline{\Gamma^{\gamma}_{\beta;\gamma}} - 2\overline{\Gamma^{\gamma}_{\gamma;\beta}}] i(\delta_{\alpha})\varphi + \overline{\Gamma^{\gamma}_{\beta;\alpha}} G^{\overline{\beta}\sigma} G^{\overline{\alpha}\tau} G_{\varepsilon\overline{\gamma}} e(\mathrm{d}z^{\varepsilon}) i(\delta_{\sigma}) i(\delta_{\tau})\varphi, \\
D^*_{\mathcal{V}}\varphi = & -\frac{1}{p!q!r!s!}\dot{\partial}_{\alpha}(\varphi_{A_p\overline{B}_q C_r\overline{D}_s}) G^{\overline{\beta}\alpha} i(\dot{\partial}_{\overline{\beta}})(\mathrm{d}z^{A_p}\wedge \mathrm{d}\overline{z}^{B_q}\wedge\delta v^{C_r}\wedge\delta\overline{v}^{D_s})
\end{aligned}
$$

$$- G^{\overline{\beta}\alpha}\Gamma^{\mu}_{\gamma\alpha}e(\mathrm{d}z^{\sigma})i(\dot{\partial}_{\overline{\beta}})i(\delta_{\mu})\varphi - G^{\overline{\beta}\alpha}\Gamma^{\mu}_{\gamma\alpha}e(\delta\overline{v}^{\sigma})i(\dot{\partial}_{\overline{\beta}})i(\dot{\partial}_{\mu})\varphi$$
$$- G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}i(\dot{\partial}_{\overline{\beta}})\varphi + \Gamma^{\gamma}_{\alpha\beta}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\varepsilon})i(\delta_{\overline{\tau}})i(\dot{\partial}_{\overline{\sigma}})\varphi,$$
$$\overline{D}^{*}_{\mathcal{V}}\varphi = -\frac{1}{p!q!r!s!}\dot{\partial}_{\overline{\beta}}(\varphi_{A_p\overline{B}_qC_r\overline{D}_s})G^{\overline{\beta}\alpha}i(\dot{\partial}_{\alpha})(\mathrm{d}z^{A_p}\wedge \mathrm{d}\overline{z}^{B_q}\wedge\delta v^{C_r}\wedge\delta\overline{v}^{D_s})$$
$$- G^{\overline{\beta}\alpha}\overline{\Gamma^{\mu}_{\gamma\beta}}e(\mathrm{d}\overline{z}^{\sigma})i(\dot{\partial}_{\alpha})i(\delta_{\overline{\mu}})\varphi - G^{\overline{\beta}\alpha}\overline{\Gamma^{\mu}_{\gamma\beta}}e(\delta\overline{v}^{\sigma})i(\dot{\partial}_{\alpha})i(\dot{\partial}_{\overline{\mu}})\varphi$$
$$- G^{\overline{\beta}\alpha}\overline{\Gamma^{\gamma}_{\gamma\beta}}i(\dot{\partial}_{\alpha})\varphi + \overline{\Gamma^{\gamma}_{\alpha\beta}}G^{\overline{\alpha}\sigma}G^{\overline{\beta}\tau}G_{\varepsilon\overline{\gamma}}e(\mathrm{d}z^{\varepsilon})i(\delta_{\tau})i(\dot{\partial}_{\sigma})\varphi.$$

Definition 4.3 *We define the following differential operators:*

$$\Box_{\mathcal{H}} = \overline{D}_{\mathcal{H}}D^{*}_{\mathcal{H}} + D^{*}_{\mathcal{H}}\overline{D}_{\mathcal{H}}, \quad \Box_{\mathcal{V}} = \overline{D}_{\mathcal{V}}D^{*}_{\mathcal{V}} + D^{*}_{\mathcal{V}}\overline{D}_{\mathcal{V}},$$
$$\Box_{3} = \overline{D}_{3}D^{*}_{3} + D^{*}_{3}\overline{D}_{3}, \qquad \Box_{4} = \overline{D}_{4}D^{*}_{4} + D^{*}_{4}\overline{D}_{4}.$$

Associated to the Hermitian metric $\langle\, ,\, \rangle$ of holomorphic tangent bundle $\widetilde{M}$ induced by the strongly pseudoconvex Finsler metric on M, $\Box_{\mathcal{H}}$ and $\Box_{\mathcal{V}}$ are called the horizontal and vertical complex Laplacians respectively, while $\Box_3$ and $\Box_4$ are called the mixed complex Laplacians, and they are all type preserving operators from $\mathcal{A}^{p,q;r,s}$ to $\mathcal{A}^{p,q;r,s}$.

Theorem 4.4 *Let M be a strongly pseudoconvex compact complex Finsler manifold. Then for any $f \in C^{\infty}(\widetilde{M})$, by using the Chern Finsler connection, we have*

$$\triangle f = \mathrm{d}^{*}\mathrm{d}f = (\Box_{\mathcal{H}} + \overline{\Box}_{\mathcal{H}} + \Box_{\mathcal{V}} + \overline{\Box}_{\mathcal{V}})f.$$

Proof Since for any $f \in C^{\infty}(\widetilde{M})$, we have $\triangle f = \mathrm{d}^{*}\mathrm{d}f$. Thus by (4.7)–(4.14), we get

$$\begin{aligned}
D^{*}_{\mathcal{H}}\mathrm{d}f &= \{-G^{\overline{\beta}\alpha}i(\delta_{\overline{\beta}})\nabla_{\delta_{\alpha}} + G^{\overline{\beta}\alpha}[\Gamma^{\gamma}_{\alpha;\gamma} - 2\Gamma^{\gamma}_{\gamma;\alpha}]i(\delta_{\overline{\beta}}) + \Gamma^{\gamma}_{\beta;\alpha}G^{\overline{\tau}\alpha}G^{\overline{\sigma}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\varepsilon})i(\delta_{\overline{\sigma}})i(\delta_{\overline{\tau}})\}(\mathrm{d}f)\\
&= -G^{\overline{\beta}\alpha}\delta_{\alpha}\delta_{\overline{\beta}}f + G^{\overline{\beta}\alpha}[\Gamma^{\gamma}_{\alpha;\gamma} - 2\Gamma^{\gamma}_{\gamma;\alpha}]\delta_{\overline{\beta}}f = D^{*}_{\mathcal{H}}\overline{D}_{\mathcal{H}}f = \Box_{\mathcal{H}}f,\\
\overline{D}^{*}_{\mathcal{H}}\mathrm{d}f &= -G^{\overline{\beta}\alpha}\delta_{\overline{\beta}}\delta_{\alpha}f + G^{\overline{\beta}\alpha}[\overline{\Gamma^{\gamma}_{\beta;\gamma}} - 2\overline{\Gamma^{\gamma}_{\gamma;\beta}}]\delta_{\alpha}f = \overline{D}^{*}_{\mathcal{H}}D_{\mathcal{H}}f = \overline{\Box}_{\mathcal{H}}f,\\
D^{*}_{\mathcal{V}}\mathrm{d}f &= \{-G^{\overline{\beta}\alpha}i(\dot{\partial}_{\overline{\beta}})\nabla_{\dot{\partial}_{\alpha}} - G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}i(\dot{\partial}_{\overline{\beta}})\}(\mathrm{d}f) = -G^{\overline{\beta}\alpha}\dot{\partial}_{\alpha}\dot{\partial}_{\overline{\beta}}f - G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}\dot{\partial}_{\overline{\beta}}f\\
&= D^{*}_{\mathcal{V}}\overline{D}_{\mathcal{V}}f = \Box_{\mathcal{V}}f,\\
\overline{D}^{*}_{\mathcal{V}}\mathrm{d}f &= -G^{\overline{\beta}\alpha}\dot{\partial}_{\overline{\beta}}\dot{\partial}_{\alpha}f - G^{\overline{\beta}\alpha}\overline{\Gamma^{\gamma}_{\gamma\beta}}\dot{\partial}_{\alpha}f = \overline{D}^{*}_{\mathcal{V}}D_{\mathcal{V}}f = \overline{\Box}_{\mathcal{V}}f,\\
D^{*}_{3}\mathrm{d}f &= D^{*}_{3}\overline{D}_{3}f = 0, \quad \overline{D}^{*}_{3}\mathrm{d}f = \overline{D}^{*}_{3}D_{3}f = 0,\\
D^{*}_{4}\mathrm{d}f &= D^{*}_{4}\overline{D}_{4}f = 0, \quad \overline{D}^{*}_{4}\mathrm{d}f = \overline{D}^{*}_{4}D_{4}f = 0.
\end{aligned}$$

Then, by (4.6) we have

$$\triangle f = \mathrm{d}^{*}\mathrm{d}f = (\Box_{\mathcal{H}} + \overline{\Box}_{\mathcal{H}} + \Box_{\mathcal{V}} + \overline{\Box}_{\mathcal{V}})f.$$

Remark 4.1 [17, Theorem 5.7] also gives the same formula for a strongly pseudoconvex compact complex Finsler manifold with the associated complex Rund connection D.

Obviously, the type preserving component of Hodge-Laplace operator $\triangle = \mathrm{d}^{*}\mathrm{d} + \mathrm{d}\mathrm{d}^{*}$ is

$$\Box_{\mathcal{H}} + \overline{\Box}_{\mathcal{H}} + \Box_{\mathcal{V}} + \overline{\Box}_{\mathcal{V}} + \Box_{3} + \overline{\Box}_{3} + \Box_{4} + \overline{\Box}_{4}. \tag{4.15}$$

Then, for any $\varphi, \psi \in \mathcal{A}^{p,q;r,s}$, we have

$$(\triangle\varphi, \psi) = (\Box_{\mathcal{H}}\varphi, \psi) + (\overline{\Box}_{\mathcal{H}}\varphi, \psi) + (\Box_{\mathcal{V}}\varphi, \psi) + (\overline{\Box}_{\mathcal{V}}\varphi, \psi)$$

$$+ (\Box_3\varphi, \psi) + (\overline{\Box}_3\varphi, \psi) + (\Box_4\varphi, \psi) + (\overline{\Box}_4\varphi, \psi). \tag{4.16}$$

Thus, a form $\varphi \in \mathcal{A}^{p,q;r,s}$ is $\triangle$-harmonic if and only if

$$\begin{array}{llll} \Box_{\mathcal{H}}\varphi = 0, & \overline{\Box}_{\mathcal{H}}\varphi = 0, & \Box_{\mathcal{V}}\varphi = 0, & \overline{\Box}_{\mathcal{V}}\varphi = 0, \\ \Box_3\varphi = 0, & \overline{\Box}_3\varphi = 0, & \Box_4\varphi = 0, & \overline{\Box}_4\varphi = 0. \end{array} \tag{4.17}$$

Theorem 4.5 *Let M be a strongly pseudoconvex compact complex Finsler manifold. Then by using the Chern Finsler connection, we obtain*

$$\begin{aligned}
\Box_{\mathcal{H}} = & -G^{\overline{\beta}\alpha} \nabla_{\delta_\alpha} \nabla_{\delta_{\overline{\beta}}} + G^{\overline{\beta}\alpha} e(\mathrm{d}\overline{z}^\mu) i(\delta_{\overline{\beta}})(\nabla_{\delta_\alpha} \nabla_{\delta_{\overline{\mu}}} - \nabla_{\delta_{\overline{\mu}}} \nabla_{\delta_\alpha}) \\
& + G^{\overline{\beta}\alpha}(\Gamma^\gamma_{\alpha;\gamma} - 2\Gamma^\gamma_{\gamma;\alpha}) \nabla_{\delta_{\overline{\beta}}} + G^{\overline{\beta}\alpha}(\overline{\Gamma^\gamma_{\beta;\mu}} - \overline{\Gamma^\gamma_{\mu;\beta}}) e(\mathrm{d}\overline{z}^\mu) i(\delta_{\overline{\gamma}}) \nabla_{\delta_\alpha} \\
& + (\Gamma^\gamma_{\beta;\alpha} - \Gamma^\gamma_{\alpha;\beta}) G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) i(\delta_{\overline{\sigma}}) \nabla_{\delta_{\overline{\tau}}} \\
& + G^{\overline{\beta}\alpha}(\Gamma^\gamma_{\alpha;\gamma} - 2\Gamma^\gamma_{\gamma;\alpha})(\overline{\Gamma^\omega_{\mu;\beta}} - \overline{\Gamma^\omega_{\beta;\mu}}) e(\mathrm{d}\overline{z}^\mu) i(\delta_{\overline{\omega}}) \\
& + \Gamma^\gamma_{\beta;\alpha}(\overline{\Gamma^\omega_{\sigma;\tau}} - \overline{\Gamma^\omega_{\tau;\sigma}}) G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) i(\delta_{\overline{\omega}}) + G^{\overline{\beta}\alpha} \delta_{\overline{\mu}}(\Gamma^\gamma_{\alpha;\gamma} - 2\Gamma^\gamma_{\gamma;\alpha}) e(\mathrm{d}\overline{z}^\mu) i(\delta_{\overline{\beta}}) \\
& + (\Gamma^\gamma_{\beta;\alpha} - \Gamma^\gamma_{\alpha;\beta})(\overline{\Gamma^\omega_{\mu;\tau}} - \overline{\Gamma^\omega_{\tau;\mu}}) G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) e(\mathrm{d}\overline{z}^\mu) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\omega}}) \\
& + \delta_{\overline{\mu}}(\Gamma^\gamma_{\beta;\alpha}) G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\mu) e(\mathrm{d}\overline{z}^\varepsilon) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}}) \\
& + \Gamma^\gamma_{\beta;\alpha} \overline{\Gamma^\varepsilon_{\nu;\mu}} G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\mu) e(\mathrm{d}\overline{z}^\nu) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}}),
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\Box_{\mathcal{V}} = & -G^{\overline{\beta}\alpha} \nabla_{\dot{\partial}_\alpha} \nabla_{\dot{\partial}_{\overline{\beta}}} - G^{\overline{\beta}\alpha} \Gamma^\gamma_{\gamma\alpha} \nabla_{\dot{\partial}_{\overline{\beta}}} + G^{\overline{\beta}\alpha} e(\delta\overline{v}^\mu) i(\dot{\partial}_{\overline{\beta}})(\nabla_{\dot{\partial}_\alpha} \nabla_{\dot{\partial}_{\overline{\mu}}} - \nabla_{\dot{\partial}_{\overline{\mu}}} \nabla_{\dot{\partial}_\alpha}) \\
& - G^{\overline{\beta}\alpha} \dot{\partial}_\alpha(\overline{\Gamma^\omega_{\beta\nu}}) e(\mathrm{d}\overline{z}^\nu) i(\delta_{\overline{\omega}}) - G^{\overline{\beta}\alpha} \dot{\partial}_{\overline{\varepsilon}}(\Gamma^\gamma_{\gamma\alpha}) e(\delta\overline{v}^\varepsilon) i(\dot{\partial}_{\overline{\beta}}) - G^{\overline{\beta}\alpha} \overline{\Gamma^\omega_{\beta\nu}} e(\mathrm{d}\overline{z}^\nu) i(\delta_{\overline{\omega}}) \nabla_{\dot{\partial}_\alpha} \\
& - G^{\overline{\beta}\alpha} \dot{\partial}_\alpha(\overline{\Gamma^\omega_{\mu\nu}}) e(\delta\overline{v}^\mu) e(\mathrm{d}\overline{z}^\nu) i(\dot{\partial}_{\overline{\beta}}) i(\delta_{\overline{\omega}}) + \dot{\partial}_{\overline{\mu}}(\Gamma^\gamma_{\alpha\beta}) G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\varepsilon}} e(\delta\overline{v}^\mu) e(\mathrm{d}\overline{z}^\varepsilon) i(\delta_{\overline{\tau}}) i(\dot{\partial}_{\overline{\sigma}}) \\
& - G^{\overline{\beta}\alpha} \Gamma^\gamma_{\gamma\alpha} \overline{\Gamma^\omega_{\beta\nu}} e(\mathrm{d}\overline{z}^\nu) i(\delta_{\overline{\omega}}) + \Gamma^\gamma_{\alpha\beta} G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) i(\delta_{\overline{\tau}}) \nabla_{\dot{\partial}_{\overline{\sigma}}} \\
& + \Gamma^\gamma_{\alpha\beta} \overline{\Gamma^\omega_{\sigma\tau}} G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) i(\delta_{\overline{\omega}}) - \Gamma^\gamma_{\alpha\beta} \overline{\Gamma^\omega_{\sigma\nu}} G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) e(\mathrm{d}\overline{z}^\nu) i(\delta_{\overline{\tau}}) i(\delta_{\overline{\omega}}) \\
& - \Gamma^\gamma_{\alpha\beta} \overline{\Gamma^\omega_{\mu\tau}} G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\varepsilon}} e(\mathrm{d}\overline{z}^\varepsilon) e(\delta\overline{v}^\mu) i(\dot{\partial}_{\overline{\sigma}}) i(\delta_{\overline{\omega}}) \\
& + \Gamma^\gamma_{\alpha\beta} \overline{\Gamma^\varepsilon_{\mu\nu}} G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\varepsilon}} e(\delta\overline{v}^\mu) e(\mathrm{d}\overline{z}^\nu) i(\delta_{\overline{\tau}}) i(\dot{\partial}_{\overline{\sigma}}),
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\Box_3 = & \ \delta_\sigma(\Gamma^{\overline{\omega}}_{\overline{\varepsilon}}) \delta_{\overline{\beta}}(\Gamma^\alpha_\mu) G^{\overline{\beta}\sigma} G^{\overline{\varepsilon}\mu} G_{\tau\overline{\omega}} e(\delta v^\tau) i(\dot{\partial}_\alpha) \\
& - \delta_\sigma(\Gamma^{\overline{\omega}}_{\overline{\varepsilon}}) \delta_{\overline{\beta}}(\Gamma^\alpha_\mu) G^{\overline{\gamma}\sigma} G^{\overline{\varepsilon}\nu} G_{\alpha\overline{\omega}} e(\mathrm{d}z^\mu) e(\mathrm{d}\overline{z}^\beta) i(\delta_\nu) i(\delta_{\overline{\gamma}}) \\
& - \delta_\sigma(\Gamma^{\overline{\omega}}_{\overline{\varepsilon}}) \delta_{\overline{\beta}}(\Gamma^\alpha_\mu) G^{\overline{\beta}\sigma} G^{\overline{\varepsilon}\nu} G_{\tau\overline{\omega}} e(\mathrm{d}z^\mu) e(\delta v^\tau) i(\delta_\nu) i(\dot{\partial}_\alpha) \\
& + \delta_\sigma(\Gamma^{\overline{\omega}}_{\overline{\varepsilon}}) \delta_{\overline{\beta}}(\Gamma^\alpha_\mu) G^{\overline{\gamma}\sigma} G^{\overline{\varepsilon}\mu} G_{\tau\overline{\omega}} e(\mathrm{d}\overline{z}^\beta) e(\delta v^\tau) i(\delta_{\overline{\gamma}}) i(\dot{\partial}_\alpha),
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
\Box_4 = & \ \dot{\partial}_\beta(\Gamma^{\overline{\alpha}}_{\overline{\mu}}) \dot{\partial}_{\overline{\sigma}}(\Gamma^\varepsilon_\nu) G^{\overline{\sigma}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(\delta v^\tau) i(\dot{\partial}_\varepsilon) \\
& + \dot{\partial}_\beta(\Gamma^{\overline{\mu}}_{\overline{\alpha}}) \dot{\partial}_{\overline{\sigma}}(\Gamma^\varepsilon_\omega) G^{\overline{\sigma}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(\mathrm{d}z^\omega) e(\delta v^\tau) i(\delta_\nu) i(\dot{\partial}_\varepsilon) \\
& + \dot{\partial}_\beta(\Gamma^{\overline{\alpha}}_{\overline{\mu}}) \dot{\partial}_{\overline{\sigma}}(\Gamma^\varepsilon_\nu) G^{\overline{\gamma}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(\delta v^\tau) e(\delta\overline{v}^\sigma) i(\dot{\partial}_{\overline{\gamma}}) i(\dot{\partial}_\varepsilon) \\
& - \dot{\partial}_\beta(\Gamma^{\overline{\alpha}}_{\overline{\mu}}) \dot{\partial}_{\overline{\sigma}}(\Gamma^\tau_\omega) G^{\overline{\gamma}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(\mathrm{d}z^\omega) e(\delta\overline{v}^\sigma) i(\delta_\nu) i(\dot{\partial}_{\overline{\gamma}}).
\end{aligned} \tag{4.21}$$

Proof We only calculate $\Box_{\mathcal{V}}$, and the others are similar to obtain.

From (3.12) and (4.9), by a direct computation, we have

$$\begin{aligned}
D^*_{\mathcal{V}} \overline{D}_{\mathcal{V}} = & -G^{\overline{\beta}\alpha} \nabla_{\dot{\partial}_\alpha} \nabla_{\dot{\partial}_{\overline{\beta}}} + G^{\overline{\beta}\alpha} e(\delta\overline{v}^\mu) i(\dot{\partial}_{\overline{\beta}}) \nabla_{\dot{\partial}_\alpha} \nabla_{\dot{\partial}_{\overline{\mu}}} - G^{\overline{\beta}\alpha} \dot{\partial}_\alpha(\overline{\Gamma^\omega_{\beta\nu}}) e(\mathrm{d}\overline{z}^\nu) i(\delta_{\overline{\omega}}) \\
& - G^{\overline{\beta}\alpha} \overline{\Gamma^\omega_{\beta\nu}} e(\mathrm{d}\overline{z}^\nu) i(\delta_{\overline{\omega}}) \nabla_{\dot{\partial}_\alpha} - G^{\overline{\beta}\alpha} \dot{\partial}_\alpha(\overline{\Gamma^\omega_{\mu\nu}}) e(\delta\overline{v}^\mu) e(\mathrm{d}\overline{z}^\nu) i(\dot{\partial}_{\overline{\beta}}) i(\delta_{\overline{\omega}})
\end{aligned}$$

$$
\begin{aligned}
&- G^{\overline{\beta}\alpha}\overline{\Gamma^{\omega}_{\mu\nu}}e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\nu})i(\dot{\partial}_{\overline{\beta}})i(\delta_{\overline{\omega}})\nabla_{\dot{\partial}_{\alpha}} - G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}\nabla_{\dot{\partial}_{\overline{\beta}}} + G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}e(\delta\overline{v}^{\mu})i(\dot{\partial}_{\overline{\beta}})\nabla_{\dot{\partial}_{\overline{\mu}}}\\
&- G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}\overline{\Gamma^{\omega}_{\beta\nu}}e(\mathrm{d}\overline{z}^{\nu})i(\delta_{\overline{\omega}}) - G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}\overline{\Gamma^{\omega}_{\mu\nu}}e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\nu})i(\dot{\partial}_{\overline{\beta}})i(\delta_{\overline{\omega}})\\
&+ \Gamma^{\gamma}_{\alpha\beta}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\varepsilon})i(\delta_{\overline{\tau}})\nabla_{\dot{\partial}_{\overline{\sigma}}} + \Gamma^{\gamma}_{\alpha\beta}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\varepsilon})e(\delta\overline{v}^{\mu})i(\delta_{\overline{\tau}})i(\dot{\partial}_{\overline{\sigma}})\nabla_{\dot{\partial}_{\overline{\mu}}}\\
&+ \Gamma^{\gamma}_{\alpha\beta}\overline{\Gamma^{\omega}_{\sigma\tau}}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\varepsilon})i(\delta_{\overline{\omega}}) - \Gamma^{\gamma}_{\alpha\beta}\overline{\Gamma^{\omega}_{\mu\nu}}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\varepsilon})e(\mathrm{d}\overline{z}^{\nu})i(\delta_{\overline{\tau}})i(\delta_{\overline{\omega}})\\
&- \Gamma^{\gamma}_{\alpha\beta}\overline{\Gamma^{\omega}_{\mu\tau}}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\varepsilon})e(\delta\overline{v}^{\mu})i(\dot{\partial}_{\overline{\sigma}})i(\delta_{\overline{\omega}})\\
&- \Gamma^{\gamma}_{\alpha\beta}\overline{\Gamma^{\omega}_{\mu\nu}}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\varepsilon})e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\mu})i(\dot{\partial}_{\overline{\sigma}})i(\delta_{\overline{\tau}})i(\delta_{\overline{\omega}}),
\end{aligned}
$$

$$
\begin{aligned}
\overline{D}_{\mathcal{V}}D^{*}_{\mathcal{V}} = {}& -G^{\overline{\beta}\alpha}e(\delta\overline{v}^{\mu})i(\dot{\partial}_{\overline{\beta}})\nabla_{\dot{\partial}_{\overline{\mu}}}\nabla_{\dot{\partial}_{\alpha}} - G^{\overline{\beta}\alpha}\dot{\partial}_{\overline{\varepsilon}}(\Gamma^{\gamma}_{\gamma\alpha})e(\delta\overline{v}^{\varepsilon})i(\dot{\partial}_{\overline{\beta}})\\
&- G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}e(\delta\overline{v}^{\mu})i(\dot{\partial}_{\overline{\beta}})\nabla_{\dot{\partial}_{\overline{\mu}}} + \dot{\partial}_{\overline{\mu}}(\Gamma^{\gamma}_{\alpha\beta})G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\varepsilon})i(\delta_{\overline{\tau}})i(\dot{\partial}_{\overline{\sigma}})\\
&+ \Gamma^{\gamma}_{\alpha\beta}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\varepsilon})i(\delta_{\overline{\tau}})i(\dot{\partial}_{\overline{\sigma}})\nabla_{\dot{\partial}_{\overline{\mu}}} - G^{\overline{\beta}\alpha}\overline{\Gamma^{\omega}_{\mu\nu}}e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\nu})i(\delta_{\overline{\omega}})i(\dot{\partial}_{\overline{\beta}})\nabla_{\dot{\partial}_{\alpha}}\\
&- G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\gamma\alpha}\overline{\Gamma^{\omega}_{\mu\nu}}e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\nu})i(\delta_{\overline{\omega}})i(\dot{\partial}_{\overline{\beta}}) + \Gamma^{\gamma}_{\alpha\beta}\overline{\Gamma^{\varepsilon}_{\sigma\nu}}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\nu})i(\delta_{\overline{\tau}})i(\dot{\partial}_{\overline{\sigma}})\\
&- \Gamma^{\gamma}_{\alpha\beta}\overline{\Gamma^{\omega}_{\mu\nu}}G^{\overline{\sigma}\alpha}G^{\overline{\tau}\beta}G_{\gamma\overline{\varepsilon}}e(\delta\overline{v}^{\mu})e(\mathrm{d}\overline{z}^{\mu})e(\mathrm{d}\overline{z}^{\varepsilon})i(\delta_{\overline{\omega}})i(\delta_{\overline{\tau}})i(\dot{\partial}_{\overline{\sigma}}),
\end{aligned}
$$

and $\Box_{\mathcal{V}}$ can be obtained by plusing the above two formulas.

Remark 4.2 If M is a compact Kähler Finsler manifold, then $\Gamma^{\alpha}_{\mu;\nu} = \Gamma^{\alpha}_{\nu;\mu}$. Therefore (see [17])

$$
\begin{aligned}
\Box_{\mathcal{H}} = {}& -G^{\overline{\beta}\alpha}\nabla_{\delta_{\alpha}}\nabla_{\delta_{\overline{\beta}}} + G^{\overline{\beta}\alpha}(\nabla_{[\delta_{\alpha},\delta_{\overline{\mu}}]} + \Omega(\delta_{\alpha},\delta_{\overline{\mu}}))e(\mathrm{d}\overline{z}^{\mu})i(\delta_{\overline{\beta}}) - G^{\overline{\beta}\alpha}\Gamma^{\gamma}_{\alpha;\gamma}\nabla_{\delta_{\overline{\beta}}}\\
&- G^{\overline{\beta}\alpha}\delta_{\overline{\mu}}(\Gamma^{\gamma}_{\alpha;\gamma})e(\mathrm{d}\overline{z}^{\mu})i(\delta_{\overline{\beta}}) + \delta_{\overline{\mu}}(\Gamma^{\gamma}_{\beta;\alpha})G^{\overline{\tau}\alpha}G^{\overline{\sigma}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\mu})e(\mathrm{d}\overline{z}^{\varepsilon})i(\delta_{\overline{\sigma}})i(\delta_{\overline{\tau}})i(\delta_{\overline{\sigma}})i(\delta_{\overline{\tau}})\\
&+ \Gamma^{\gamma}_{\beta;\alpha}\overline{\Gamma^{\varepsilon}_{\nu;\mu}}G^{\overline{\tau}\alpha}G^{\overline{\sigma}\beta}G_{\gamma\overline{\varepsilon}}e(\mathrm{d}\overline{z}^{\mu})e(\mathrm{d}\overline{z}^{\nu})i(\delta_{\overline{\sigma}})i(\delta_{\overline{\tau}}).
\end{aligned}
$$

Since the Kähler Finsler condition shows no influence on $\Box_{\mathcal{V}}, \Box_3, \Box_4$ and their conjugations, the expressions of $\Box_{\mathcal{V}}, \Box_3, \Box_4$ are valid for a Kähler Finsler manifold.

Remark 4.3 By conjugations, we can also obtain the complex Laplacians $\overline{\Box}_{\mathcal{H}}, \overline{\Box}_{\mathcal{V}}, \overline{\Box}_3$ and $\overline{\Box}_4$.

5 A Vanishing Theorem

Let $\mathrm{d}V = (-1)^m\mathcal{G}^2\mathrm{d}z\wedge\mathrm{d}\overline{z}\wedge\delta v\wedge\delta\overline{v}$ be the volume form associated to the Hermitian metric $\widetilde{G}$ of $\widetilde{M}$, where $\mathcal{G} = \det(G_{\alpha\overline{\beta}})$. Denote $\mathcal{L}_X$ by the Lie derivative with respect to $X \in \mathcal{X}(T\widetilde{M})$. Then the divergence of X is defined by the equation

$$\mathcal{L}_X\mathrm{d}V = (\mathrm{div}\,X)\mathrm{d}V. \tag{5.1}$$

Lemma 5.1 *Let (M, F) be a strongly pseudoconvex compact complex Finsler manifold with $\dim_{\mathbb{C}} M = m$. Then for $X = X^{\mu}\delta_{\mu} + \overline{X}^{\nu}\delta_{\overline{\nu}} + \dot{X}^{\beta}\dot{\partial}_{\beta} + \overline{\dot{X}}^{\beta}\dot{\partial}_{\overline{\beta}} \in \mathcal{X}(T\widetilde{M})$, we have*

$$
\begin{aligned}
\mathrm{div}(X) = {}& \delta_{\alpha}(X^{\alpha}) + \delta_{\overline{\alpha}}(\overline{X}^{\alpha}) + \dot{\partial}_{\alpha}(\dot{X}^{\alpha}) + \dot{\partial}_{\overline{\alpha}}(\overline{\dot{X}}^{\alpha}) + X^{\mu}\Gamma^{\alpha}_{\alpha;\mu} + \overline{X}^{\nu}\overline{\Gamma}^{\alpha}_{\alpha;\nu}\\
&+ 2\dot{X}^{\alpha}\Gamma^{\gamma}_{\gamma\alpha} + 2\overline{\dot{X}}^{\alpha}\overline{\Gamma}^{\gamma}_{\gamma\alpha}.
\end{aligned} \tag{5.2}
$$

Proof By (2.3), we have

$$[X,\delta_{\alpha}] = -\delta_{\alpha}(X^{\mu})\delta_{\mu} - \overline{X}^{\nu}\delta_{\overline{\nu}}(\Gamma^{\sigma}_{\alpha})\dot{\partial}_{\sigma} + \overline{X}^{\nu}\delta_{\alpha}(\overline{\Gamma}^{\tau}_{\nu})\dot{\partial}_{\overline{\tau}} - \delta_{\alpha}(\overline{X}^{\nu})\delta_{\overline{\nu}} - \dot{X}^{\beta}\Gamma^{\sigma}_{\beta;\alpha}\dot{\partial}_{\sigma}$$

$$
\begin{aligned}
&\quad - \delta_\alpha(\dot{X}^\beta)\dot{\partial}_\beta - \overline{\dot{X}}^\beta \Gamma^{\sigma}_{\overline{\beta};\alpha}\dot{\partial}_\sigma - \delta_\alpha(\overline{\dot{X}}^\beta)\dot{\partial}_{\overline{\beta}},\\
[X, \delta_{\overline{\alpha}}] &= X^\nu \delta_\alpha(\Gamma^\sigma_\nu)\dot{\partial}_\sigma - \delta_{\overline{\alpha}}(X^\mu)\delta_\mu - X^\nu \delta_\nu(\overline{\Gamma}^\tau_\alpha)\dot{\partial}_{\overline{\tau}} - \delta_{\overline{\alpha}}(\overline{X}^\nu)\delta_{\overline{\nu}} - \dot{X}^\beta \Gamma^{\overline{\sigma}}_{\beta;\overline{\alpha}}\dot{\partial}_{\overline{\sigma}}\\
&\quad - \delta_{\overline{\alpha}}(\dot{X}^\beta)\dot{\partial}_\beta - \overline{\dot{X}}^\beta \overline{\Gamma}^{\sigma}_{\beta;\alpha}\dot{\partial}_{\overline{\sigma}} - \delta_{\overline{\alpha}}(\overline{\dot{X}}^\beta)\dot{\partial}_{\overline{\beta}},\\
[X, \dot{\partial}_\alpha] &= X^\mu \Gamma^\sigma_{\alpha;\mu}\dot{\partial}_\sigma - \dot{\partial}_\alpha(X^\mu)\delta_\mu + \overline{X}^\nu \Gamma^{\overline{\sigma}}_{\alpha;\overline{\nu}}\dot{\partial}_{\overline{\sigma}} - \dot{\partial}_\alpha(\overline{X}^\nu)\delta_{\overline{\nu}} - \dot{\partial}_\alpha(\dot{X}^\beta)\dot{\partial}_\beta\\
&\quad - \dot{\partial}_\alpha(\overline{\dot{X}}^\beta)\dot{\partial}_{\overline{\beta}},\\
[X, \dot{\partial}_{\overline{\alpha}}] &= X^\mu \Gamma^{\sigma}_{\overline{\alpha};\mu}\dot{\partial}_\sigma - \dot{\partial}_{\overline{\alpha}}(X^\mu)\delta_\mu + \overline{X}^\nu \overline{\Gamma}^{\sigma}_{\alpha;\nu}\dot{\partial}_{\overline{\sigma}} - \dot{\partial}_{\overline{\alpha}}(\overline{X}^\nu)\delta_{\overline{\nu}} - \dot{\partial}_{\overline{\alpha}}(\dot{X}^\beta)\dot{\partial}_\beta\\
&\quad - \dot{\partial}_{\overline{\alpha}}(\overline{\dot{X}}^\beta)\dot{\partial}_{\overline{\beta}},
\end{aligned}
$$

and then

$$
\begin{aligned}
\mathrm{div}(X)(-1)^m\mathcal{G}^2 &= \mathrm{div}(X)\mathrm{d}V(\delta_1, \cdots, \delta_m, \delta_{\overline{1}}, \cdots, \delta_{\overline{m}}, \dot{\partial}_1, \cdots, \dot{\partial}_m, \dot{\partial}_{\overline{1}}, \cdots, \dot{\partial}_{\overline{m}})\\
&= \mathcal{L}_X \mathrm{d}V(\delta_1, \cdots, \delta_m, \delta_{\overline{1}}, \cdots, \delta_{\overline{m}}, \dot{\partial}_1, \cdots, \dot{\partial}_m, \dot{\partial}_{\overline{1}}, \cdots, \dot{\partial}_{\overline{m}})\\
&= X(\mathrm{d}V(\delta_1, \cdots, \delta_m, \delta_{\overline{1}}, \cdots, \delta_{\overline{m}}, \dot{\partial}_1, \cdots, \dot{\partial}_m, \dot{\partial}_{\overline{1}}, \cdots, \dot{\partial}_{\overline{m}}))\\
&\quad - \sum_{\alpha=1}^m \mathrm{d}V(\delta_1, \cdots, \delta_{\alpha-1}, [X, \delta_\alpha], \delta_{\alpha+1}, \cdots, \delta_m, \cdots, \cdots, \cdots)\\
&\quad - \sum_{\alpha=1}^m \mathrm{d}V(\cdots, \delta_{\overline{1}}, \cdots, \delta_{\overline{\alpha-1}}, [X, \delta_{\overline{\alpha}}], \delta_{\overline{\alpha+1}}, \cdots, \delta_{\overline{m}}, \cdots, \cdots)\\
&\quad - \sum_{\alpha=1}^m \mathrm{d}V(\cdots, \cdots, \dot{\partial}_1, \cdots, \dot{\partial}_{\alpha-1}, [X, \dot{\partial}_\alpha], \dot{\partial}_{\alpha+1}, \cdots, \dot{\partial}_m, \cdots)\\
&\quad - \sum_{\alpha=1}^m \mathrm{d}V(\cdots, \cdots, \cdots, \dot{\partial}_{\overline{1}}, \cdots, \dot{\partial}_{\overline{\alpha-1}}, [X, \dot{\partial}_{\overline{\alpha}}], \dot{\partial}_{\overline{\alpha+1}}, \cdots, \dot{\partial}_{\overline{m}})\\
&= (-1)^m\mathcal{G}^2\{\delta_\alpha(X^\alpha) + \delta_{\overline{\alpha}}(\overline{X}^\alpha) + \dot{\partial}_\alpha(\dot{X}^\alpha) + \dot{\partial}_{\overline{\alpha}}(\overline{\dot{X}}^\alpha)\\
&\quad + X^\mu \Gamma^\alpha_{\alpha;\mu} + \overline{X}^\nu \overline{\Gamma}^\alpha_{\alpha;\nu} + 2\dot{X}^\alpha \Gamma^\gamma_{\gamma\alpha} + 2\overline{\dot{X}}^\alpha \overline{\Gamma}^\gamma_{\gamma\alpha}\},
\end{aligned}
$$

that is,

$$
\begin{aligned}
\mathrm{div}(X) &= \delta_\alpha(X^\alpha) + \delta_{\overline{\alpha}}(\overline{X}^\alpha) + \dot{\partial}_\alpha(\dot{X}^\alpha) + \dot{\partial}_{\overline{\alpha}}(\overline{\dot{X}}^\alpha) + X^\mu \Gamma^\alpha_{\alpha;\mu} + \overline{X}^\nu \overline{\Gamma}^\alpha_{\alpha;\nu}\\
&\quad + 2\dot{X}^\alpha \Gamma^\gamma_{\gamma\alpha} + 2\overline{\dot{X}}^\alpha \overline{\Gamma}^\gamma_{\gamma\alpha}.
\end{aligned}
$$

Theorem 5.1 *Let (M, F) be a strongly pseudoconvex compact complex Finsler manifold with $\dim_{\mathbb{C}} M = m$. Then for any function $f \in C^\infty(\widetilde{M})$, we have*

$$
\int_{\widetilde{M}} \{\triangle f + \overline{\mathcal{B}^\beta}\delta_{\overline{\beta}} f + \mathcal{B}^\alpha \delta_\alpha f\}\mathrm{d}V = 0, \tag{5.3}
$$

where $\mathcal{B}^\alpha = G^{\overline{\beta}\alpha}\overline{\Gamma}^\gamma_{\gamma;\beta}$.

Proof Let $X = G^{\overline{\beta}\alpha}\delta_{\overline{\beta}} f\delta_\alpha + G^{\overline{\beta}\alpha}\delta_\alpha f\delta_{\overline{\beta}} + G^{\overline{\beta}\alpha}\dot{\partial}_{\overline{\beta}} f\dot{\partial}_\alpha + G^{\overline{\beta}\alpha}\dot{\partial}_\alpha f\dot{\partial}_{\overline{\beta}}$. By (5.2), we have

$$
\begin{aligned}
\mathrm{div}(X) &= \delta_\alpha(G^{\overline{\beta}\alpha}\delta_{\overline{\beta}} f) + \delta_{\overline{\beta}}(G^{\overline{\beta}\alpha}\delta_\alpha f) + \dot{\partial}_\alpha(G^{\overline{\beta}\alpha}\dot{\partial}_{\overline{\beta}} f) + \dot{\partial}_{\overline{\beta}}(G^{\overline{\beta}\alpha}\dot{\partial}_\alpha f)\\
&\quad + G^{\overline{\beta}\alpha}\delta_{\overline{\beta}} f\Gamma^\gamma_{\gamma;\alpha} + G^{\overline{\beta}\alpha}\delta_\alpha f\overline{\Gamma}^\gamma_{\gamma;\beta} + 2G^{\overline{\beta}\alpha}\dot{\partial}_{\overline{\beta}} f\Gamma^\gamma_{\gamma\alpha} + 2G^{\overline{\beta}\alpha}\dot{\partial}_\alpha f\overline{\Gamma}^\gamma_{\gamma\beta}
\end{aligned}
$$

$$= G^{\overline{\beta}\alpha}(\delta_\alpha\delta_{\overline{\beta}} + \delta_{\overline{\beta}}\delta_\alpha)f + G^{\overline{\beta}\alpha}(\dot{\partial}_\alpha\dot{\partial}_{\overline{\beta}} + \dot{\partial}_{\overline{\beta}}\dot{\partial}_\alpha)f + G^{\overline{\beta}\alpha}(\Gamma^\gamma_{\gamma;\alpha} - \Gamma^\gamma_{\alpha;\gamma})\delta_{\overline{\beta}}f$$
$$+ G^{\overline{\beta}\alpha}(\overline{\Gamma}^\gamma_{\gamma;\beta} - \overline{\Gamma}^\gamma_{\beta;\gamma})\delta_\alpha f + G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma\alpha}\dot{\partial}_{\overline{\beta}}f + G^{\overline{\beta}\alpha}\overline{\Gamma}^\gamma_{\gamma\beta}\dot{\partial}_\alpha f,$$

and by Theorem 4.4, we have

$$(\Box_{\mathcal{H}} + \overline{\Box}_{\mathcal{H}})f = -G^{\overline{\beta}\alpha}(\delta_\alpha\delta_{\overline{\beta}} + \delta_{\overline{\beta}}\delta_\alpha)f + G^{\overline{\beta}\alpha}(\Gamma^\gamma_{\alpha;\gamma} - 2\Gamma^\gamma_{\gamma;\alpha})\delta_{\overline{\beta}}f + G^{\overline{\beta}\alpha}(\overline{\Gamma}^\gamma_{\beta;\gamma} - 2\overline{\Gamma}^\gamma_{\gamma;\beta})\delta_\alpha f,$$
$$(\Box_{\mathcal{V}} + \overline{\Box}_{\mathcal{V}})f = G^{\overline{\beta}\alpha}(\dot{\partial}_\alpha\dot{\partial}_{\overline{\beta}} + \dot{\partial}_{\overline{\beta}}\dot{\partial}_\alpha)f + G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma\alpha}\dot{\partial}_{\overline{\beta}}f + G^{\overline{\beta}\alpha}\overline{\Gamma}^\gamma_{\gamma\beta}\dot{\partial}_\alpha f,$$

therefore,

$$\triangle f + G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma;\alpha}\delta_{\overline{\beta}}f + G^{\overline{\beta}\alpha}\overline{\Gamma}^\gamma_{\gamma;\beta}\delta_\alpha f = -\mathrm{div}(X),$$

that is,

$$\int_{\widetilde{M}}\{\triangle f + G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma;\alpha}\delta_{\overline{\beta}}f + G^{\overline{\beta}\alpha}\overline{\Gamma}^\gamma_{\gamma;\beta}\delta_\alpha f\}\mathrm{d}V = 0.$$

Theorem 5.2 *Let (M,F) be a strongly pseudoconvex compact complex Finsler manifold with* $\dim_{\mathbb{C}} M = m$*. For any $\varphi \in \mathcal{A}^{p,0,r,0}$, if*

$$\mathrm{Re}(\langle\varphi, \mathcal{M}\varphi\rangle) > 0, \tag{5.4}$$

where $\mathcal{M} = \mathcal{B}^\beta\triangledown_{\dot{\partial}_\beta}$, then there exists no non-zero $\varphi \in \mathcal{A}^{p,0,r,0}$ such that $\overline{\partial}\varphi = 0$.

Proof For all $\varphi \in \mathcal{A}^{p,0,r,0}$ such that $\overline{\partial}\varphi = 0$, we have $\triangledown_{\delta_{\overline{\alpha}}}\varphi = 0$, $\triangledown_{\dot{\partial}_{\overline{\alpha}}}\varphi = 0$.

If there exists a non-zero $\varphi \in \mathcal{A}^{p,0,r,0}$ such that $\overline{\partial}\varphi = 0$, then

$$\Box_{\mathcal{H}}\varphi = -G^{\overline{\beta}\alpha}\triangledown_{\delta_\alpha}\triangledown_{\delta_{\overline{\beta}}}\varphi - G^{\overline{\beta}\alpha}\Gamma^\gamma_{\alpha;\gamma}\triangledown_{\delta_{\overline{\beta}}}\varphi = 0,$$
$$\Box_{\mathcal{V}}\varphi = -G^{\overline{\beta}\alpha}\triangledown_{\dot{\partial}_\alpha}\triangledown_{\dot{\partial}_{\overline{\beta}}}\varphi - G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma\alpha}\triangledown_{\dot{\partial}_{\overline{\beta}}}\varphi = 0.$$

Let $f = |\varphi|^2 = \langle\varphi,\varphi\rangle$. We have

$$G^{\overline{\beta}\alpha}(\delta_\alpha\delta_{\overline{\beta}} + \delta_{\overline{\beta}}\delta_\alpha)|\varphi|^2 = 2|D_{\mathcal{H}}\varphi|^2 + 2|\overline{D}_{\mathcal{H}}\varphi|^2,$$
$$G^{\overline{\beta}\alpha}\dot{\partial}_\alpha\dot{\partial}_{\overline{\beta}} = |D_{\mathcal{V}}\varphi|^2 + |\overline{D}_{\mathcal{V}}\varphi|^2,$$

where $|D_{\mathcal{H}}\varphi|^2 = G^{\overline{\beta}\alpha}\langle\triangledown_{\delta_\alpha}\varphi, \triangledown_{\delta_\beta}\varphi\rangle$, $|\overline{D}_{\mathcal{H}}\varphi|^2 = G^{\overline{\beta}\alpha}\langle\triangledown_{\delta_{\overline{\alpha}}}\varphi, \triangledown_{\delta_{\overline{\beta}}}\varphi\rangle = 0$, $|D_{\mathcal{V}}\varphi|^2 = G^{\overline{\beta}\alpha}\langle\triangledown_{\dot{\partial}_\alpha}\varphi, \triangledown_{\dot{\partial}_\beta}\varphi\rangle$, $|\overline{D}_{\mathcal{V}}\varphi|^2 = G^{\overline{\beta}\alpha}\langle\triangledown_{\dot{\partial}_{\overline{\alpha}}}\varphi, \triangledown_{\dot{\partial}_{\overline{\beta}}}\varphi\rangle = 0$, and therefore, by (5.3)

$$\int_{\widetilde{M}}\{\triangle f + G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma;\alpha}\delta_{\overline{\beta}}f + G^{\overline{\beta}\alpha}\overline{\Gamma}^\gamma_{\gamma;\beta}\delta_\alpha f\}\mathrm{d}V$$
$$= \int_{\widetilde{M}}\{2|D_{\mathcal{H}}\varphi|^2 + 2|\overline{D}_{\mathcal{H}}\varphi|^2 + 2|D_{\mathcal{V}}\varphi|^2 + 2|\overline{D}_{\mathcal{V}}\varphi|^2 + G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma\alpha}\,\dot{\partial}_{\overline{\beta}}|\varphi|^2 + G^{\overline{\beta}\alpha}\overline{\Gamma}^\gamma_{\gamma\beta}\dot{\partial}_\alpha|\varphi|^2\}\mathrm{d}V$$
$$= \int_{\widetilde{M}} 2\{|D_{\mathcal{H}}\varphi|^2 + |D_{\mathcal{V}}\varphi|^2 + \mathrm{Re}(G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma\alpha}\,\dot{\partial}_{\overline{\beta}}|\varphi|^2)\}\mathrm{d}V$$
$$= \int_{\widetilde{M}} 2\{|D_{\mathcal{H}}\varphi|^2 + |D_{\mathcal{V}}\varphi|^2 + \mathrm{Re}(G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma\alpha}\,\langle\varphi, \triangledown_{\dot{\partial}_\beta}\varphi\rangle)\}\mathrm{d}V$$
$$= 0.$$

Thus, if

$$\mathrm{Re}(G^{\overline{\beta}\alpha}\Gamma^\gamma_{\gamma\alpha}\langle\varphi, \triangledown_{\dot{\partial}_\beta}\varphi\rangle) > 0,$$

then there is no non-zero $\varphi \in \mathcal{A}^{p,0,r,0}$ such that $\overline{\partial}\varphi = 0$.

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