

# Gröbner-Shirshov Basis of Quantum Group of Type $\mathbb{D}_4^*$

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**Abstract** The authors take all isomorphism classes of indecomposable representations as new generators, and obtain all skew-commutators between these generators by using the Ringel-Hall algebra method. Then they prove that the set of these skew-commutators is a Gröbner-Shirshov basis for quantum group of type  $\mathbb{D}_4$ .

**Keywords** Ringel-Hall algebra, Indecomposable modules, Gröbner-Shirshov basis, Compositions

**2000 MR Subject Classification** 16S15, 13P10, 17B37

## 1 Introduction

The Gröbner basis theory for commutative algebras was introduced by Buchberger [4], and provided a solution to the reduction problem for commutative algebras. It gives an algorithm of computing a set of generators for a given ideal of a commutative ring which can be used to determine the reduced elements with respect to the relations given by the ideal. In [1], Bergman generalized the Gröbner basis theory to associative algebras by providing the Diamond Lemma.

The Gröbner basis theory for Lie algebras was developed by Shirshov [12]. The key ingredient of the theory is the so-called Composition Lemma which characterizes the leading terms of elements in the given ideal. In [2], Bokut noticed that Shirshov's method works for associative algebras as well. For this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the Gröbner-Shirshov basis theory.

In [3], Bokut and Malcolmson developed the theory of Gröbner-Shirshov basis for the quantum enveloping algebras, or the so-called quantum groups, and by using the Jimbo relations given by Yamane [13], they explicitly constructed the basis for the quantum group of type  $\mathbb{A}_n$  for  $q^8 \neq 1$ . The Gröbner-Shirshov basis for quantum groups of other types is not known. The main reason for this, from our point of view, may be that the construction of the so-called Jimbo relations for other types by the method of Yamane is very difficult.

In [10], for constructing a PBW type basis for quantum groups, Ringel constructed a generating sequence for Ringel-Hall algebras and some skew commutator relations for these generators by using the Auslander-Reiten theory. In this paper, by using the Ringel's method, we compute all skew-commutator relations for the quantum group of type  $\mathbb{D}_4$ . Then using the canonical

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isomorphism between the positive parts of quantum groups and the Ringel-Hall algebras, we give a Gröbner-Shirshov basis for quantum group of type  $\mathbb{D}_4$ . This may give a new idea to get a Gröbner-Shirshov basis for the quantum group of type  $\mathbb{D}_n$ .

## 2 Some Preliminaries

In this section, we recall some notions about Gröbner-Shirshov basis theory of quantum groups and the Ringel-Hall algebras, respectively.

First, we recall some basic notions about Gröbner-Shirshov basis theory from [3]. Let  $k$  be a field and  $X$  a non-empty set of alphabets. Let  $\langle X \rangle$  and  $k\langle X \rangle$  be the free semigroup with 1 and the free algebra generated by  $X$ , respectively. We choose a monomial ordering  $<$  on  $\langle X \rangle$  in order to determine the leading term  $\bar{f}$  for each element  $f \in k\langle X \rangle$ . An element  $f \in k\langle X \rangle$  is called monic if the coefficient of the leading term  $\bar{f}$  is  $1 \in k$ . If  $f$  and  $g$  are monic elements in  $k\langle X \rangle$  with leading terms  $\bar{f}$  and  $\bar{g}$ , there is a so-called composition of intersection if there are  $a$  and  $b$  in  $\langle X \rangle$  such that  $\bar{f}a = b\bar{g} = \omega$  with the total length of  $\bar{f}$  being larger than that of  $b$ . We write  $(f, g)_\omega = fa - bg$  in that case and note that the leading term  $\overline{(f, g)_\omega} < \omega$ . There is a composition of inclusion if there are  $a$  and  $b$  in  $\langle X \rangle$  such that  $\bar{f} = a\bar{g}b = \omega$ . We write  $(f, g)_\omega = f - agb$  in that case and again note that the leading term is less than  $\omega$ .

Let us take some sets of relations  $S \subseteq k\langle X \rangle$  (which, we assume, consists of monic elements). Let us denote by  $(S)$  the ideal generated by  $S$  in  $k\langle X \rangle$ . Let  $p, q \in k\langle X \rangle$  and  $\omega \in \langle X \rangle$ . We define an equivalence relation on  $k\langle X \rangle$  as follows:  $p \equiv q \pmod{(S; \omega)}$  if and only if  $p - q = \sum \alpha_i a_i s_i b_i$ , where  $\alpha_i \in k$ ,  $a_i, b_i \in \langle X \rangle$ ,  $s_i \in S$ ,  $\overline{a_i s_i b_i} < \omega$ . We say that  $S$  is closed under composition if for any  $f, g \in S$  we have  $(f, g)_\omega \equiv 0 \pmod{(S; \omega)}$ , whenever the composition  $(f, g)_\omega$  is defined. In this case, we say that the composition  $(f, g)_\omega$  is trivial with respect to  $S$ . If  $S$  is not closed under composition, then we need to expand  $S$  by including all nontrivial compositions (inductively) to obtain a completion  $S^c$ . If  $S$  is complete (i.e., closed under composition) in this sense ( $S^c = S$ ), then Shirshov's Lemma (see [12]) tells us that any monic element  $f \in (S)$  has a reducible leading term  $\bar{f} = a\bar{s}b$ , where  $s \in S$  and  $a, b \in \langle X \rangle$ . That lemma also tells us that a linear basis for the factor algebra  $k\langle X \rangle / (S)$  (i.e., as a vector space over  $k$ ) may be obtained by taking the set of irreducible monomials in  $\langle X \rangle$ .

The set  $S$  is then referred to as a Gröbner-Shirshov basis for the ideal  $(S)$ . By abusing the definition, we may also refer to  $S$  as a Gröbner-Shirshov basis for the factor algebra  $k\langle X \rangle / (S)$ . The set  $S$  is called a minimal Gröbner-Shirshov basis if there is no inclusion composition in  $S$ .

Next, we recall the definition of quantum groups from [6] and [8].

Let  $A = (a_{ij})$  be an integral symmetrizable  $N \times N$  Cartan matrix, so that  $a_{ii} = 2$ ,  $a_{ij} \leq 0$  ( $i \neq j$ ), and there exists a diagonal matrix  $D$  with nonzero integer diagonal entries  $d_i$  such that the product  $DA$  is symmetric. Let  $q$  be a nonzero element of  $k$  so that  $q^{4d_i} \neq 1$  for each  $i$ . Then the quantum group  $U_q(A)$  is the  $k$ -algebra generated by  $4N$  elements  $E_i, K_i^{\pm 1}, F_i$ , subject to the following set of relations (for  $1 \leq i, j \leq N$ ):

$$\begin{aligned} K &= \{K_i K_j - K_j K_i, K_i K_i^{-1} - 1, K_i^{-1} K_i - 1, E_j K_i^{\pm 1} - q^{\pm d_i a_{ij}} K_i^{\pm 1} E_j, \\ &\quad K_i^{\pm 1} F_j - q^{\pm d_i a_{ij}} F_j K_i^{\pm 1}\}, \\ T &= \left\{ E_i F_j - F_j E_i - \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \right\}, \end{aligned}$$

$$S^+ = \left\{ \sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t E_i^{1-a_{ij}-\mu} E_j E_i^\mu \mid i \neq j, t = q^{2d_i} \right\},$$

$$S^- = \left\{ \sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t F_i^{1-a_{ij}-\mu} F_j F_i^\mu \mid i \neq j, t = q^{2d_i} \right\},$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{cases} \prod_{i=1}^n \frac{t^{m-i+1} - t^{i-m-1}}{t^i - t^{-i}} & \text{for } m > n > 0, \\ 1 & \text{for } n = 0 \text{ or } n = m. \end{cases}$$

Let  $U_q^0(A)$  be the subalgebra of  $U_q(A)$  generated by  $K_i^{\pm 1}$ . Let  $U_q^+(A)$  (resp.  $U_q^-(A)$ ) be the subalgebra of  $U_q(A)$  generated by  $E_i$  (resp.  $F_i$ ). Then we have the following triangular decomposition of  $U_q(A)$  (see [11]):

$$U_q(A) \cong U_q^+(A) \otimes U_q^0(A) \otimes U_q^-(A).$$

The main result in [3] is as follows.

**Theorem 2.1** *If the set  $S^{+c}$  (resp.  $S^{-c}$ ) is a Gröbner-Shirshov basis of  $U_q^+(A)$  (resp.  $U_q^-(A)$ ), then the set  $S^{+c} \cup K \cup T \cup S^{-c}$  is a Gröbner-Shirshov basis of  $U_q(A)$ .*

Finally, we recall some basic notions about the twisted generic Ringel-Hall algebras. Because we only consider the quantum group of type  $\mathbb{D}_4$  in this paper, we recall the relevant notions directly for the finite dimensional hereditary algebra of Dynkin type from [5].

Let  $\mathbb{F}$  be a finite field,  $\vec{Q}$  a (connected) quiver with the underlying graph  $Q$  of Dynkin type, that is,  $Q \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ . Then it is well-known that the path algebra  $\Lambda(\mathbb{F}, \vec{Q}) = \mathbb{F}\vec{Q}$  is a finite dimensional hereditary  $\mathbb{F}$ -algebra of finite representation type. By  $\Lambda(\mathbb{F}, \vec{Q})\text{-mod}$ , we denote the category of finite dimensional right  $\Lambda(\mathbb{F}, \vec{Q})$ -modules. For  $M, N_1, \dots, N_t \in \Lambda(\mathbb{F}, \vec{Q})\text{-mod}$ , let  $F_{N_1, \dots, N_t}^M$  be the number of filtrations

$$M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_{t-1} \supseteq M_t = 0,$$

such that  $M_{i-1}/M_i \cong N_i$  for all  $1 \leq i \leq t$ .

For each  $M \in \Lambda(\mathbb{F}, \vec{Q})\text{-mod}$ , we denote by  $[M]$  the isomorphism class of  $M$  and by  $\mathbf{dim} M$  the dimension vector of the  $\Lambda(\mathbb{F}, \vec{Q})$ -module  $M$ . We have the well-known Euler form  $\langle -, - \rangle$  defined by

$$\langle \mathbf{dim} M, \mathbf{dim} N \rangle = \mathbf{dim} \operatorname{Hom}_\Lambda(M, N) - \mathbf{dim} \operatorname{Ext}_\Lambda^1(M, N).$$

Note that  $(-, -)$  is the symmetrization of  $\langle -, - \rangle$ .

Let  $v$  be an indeterminate and  $\mathbb{Q}(v)$  be the rational function field of  $v$  over the field  $\mathbb{Q}$  of rational numbers and set  $v^2 = q$ . In order to define the twisted generic Ringel-Hall algebra, we recall the notion of Hall polynomials.

For a Dynkin diagram  $Q$ , there is the corresponding semisimple Lie algebra  $\mathfrak{g}$ . Let  $\Phi^+$  be the set of positive roots of  $\mathfrak{g}$ . According to [7],  $\mathbf{dim}$  is a bijection between the set of the isomorphism classes of the indecomposable modules and the set of positive roots  $\Phi^+$  of  $\mathfrak{g}$ . For each  $\alpha \in \Phi^+$ , let  $M_{\mathbb{F}}(\alpha)$  denote the corresponding indecomposable  $\Lambda(\mathbb{F}, \vec{Q})$ -module; thus  $\mathbf{dim} M_{\mathbb{F}}(\alpha) = \alpha$ . By the

Krull-Schmidt theorem, every  $\Lambda(\mathbb{F}, \vec{Q})$ -module  $M_{\mathbb{F}}$  is isomorphic to  $M_{\mathbb{F}}(\lambda) = \bigoplus_{\alpha \in \Phi^+} \lambda(\alpha) M_{\mathbb{F}}(\alpha)$

for some function  $\lambda : \Phi^+ \longrightarrow \mathbb{N}$ . Thus, isoclasses of  $\Lambda(\mathbb{F}, \vec{Q})$ -modules are indexed by the set

$$\mathfrak{B} = \mathfrak{B}(\vec{Q}) =: \{\lambda \mid \lambda : \Phi^+ \longrightarrow \mathbb{N}\},$$

which is independent of the finite field  $\mathbb{F}$ . To be consistent, we view each  $\alpha \in \Phi^+$  as the function  $\Phi^+ \longrightarrow \mathbb{N}$ ,  $\beta \longmapsto \delta_{\alpha, \beta}$ . For later use, we denote by  $\alpha_i$  the  $i$ th simple root in  $\Phi^+$  and  $\lambda_i$  the function  $\Phi^+ \longrightarrow \mathbb{N}$ ,  $\beta \longmapsto \delta_{\alpha_i, \beta}$ . For any finite field  $\mathbb{F}$  and  $\lambda, \mu \in \mathfrak{B}(\vec{Q})$ , we define

$$\langle \lambda, \mu \rangle = \langle \mathbf{dim} M_{\mathbb{F}}(\lambda), \mathbf{dim} M_{\mathbb{F}}(\mu) \rangle.$$

Then we have the result below.

**Theorem 2.2** (see [9]) *Assume that  $\vec{Q}$  is a Dynkin quiver. For any  $\lambda, \mu, \rho \in \mathfrak{B} = \mathfrak{B}(\vec{Q})$ , there exists a polynomial  $\varphi_{\mu, \rho}^{\lambda}(T) \in \mathbb{Z}[T]$ , such that*

$$\varphi_{\mu, \rho}^{\lambda}(|\mathbb{F}|) = F_{M_{\mathbb{F}}(\mu), M_{\mathbb{F}}(\rho)}^{M_{\mathbb{F}}(\lambda)}$$

holds for each finite field  $\mathbb{F}$ .

Now, we are ready to define the twisted generic Ringel-Hall algebra.

**Definition 2.1** *The twisted generic Ringel-Hall algebra  $\mathcal{H}(\vec{Q})$  of Dynkin quiver  $\vec{Q}$  is the free  $\mathbb{Q}(v)$ -module having basis  $\{u_{\lambda} \mid \lambda \in \mathfrak{B}(\vec{Q})\}$  with multiplication defined by*

$$u_{\mu} u_{\rho} = v^{\langle \mu, \rho \rangle} \sum_{\lambda \in \mathfrak{B}(\vec{Q})} \varphi_{\mu, \rho}^{\lambda}(v^2) u_{\lambda}.$$

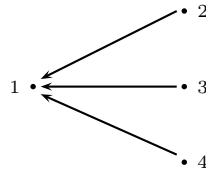
Then  $\mathcal{H}(\Lambda)$  is an associative algebra with identity  $1 = u_0$ , where 0 denotes the zero function in  $\mathfrak{B}(\vec{Q})$ .

From now on, we fix  $k = \mathbb{Q}(v)$ . Let  $\vec{Q}$  be a Dynkin quiver with an underlying graph  $Q$  and  $\mathfrak{g}$  the corresponding semisimple Lie algebra. Then the main result in [9] is as follows.

**Theorem 2.3** *The map  $\eta : U_q^+(\mathfrak{g}) \longrightarrow \mathcal{H}(\vec{Q})$  given by  $\eta(E_i) = u_{[\lambda_i]}$  is a  $\mathbb{Q}(v)$ -algebra isomorphism.*

### 3 Gröbner-Shirshov Basis of Quantum Group of Type $\mathbb{D}_4$

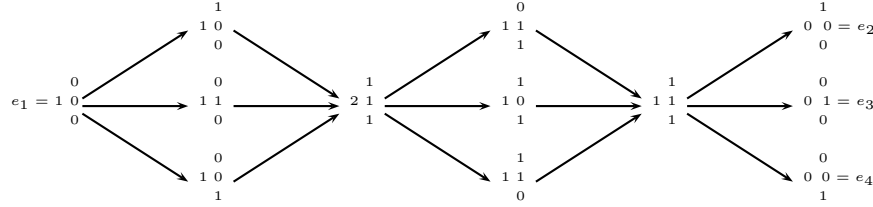
Throughout this section, quantum group  $U_q(\mathfrak{g})$  means the quantum group  $U_q$  of type  $\mathbb{D}_4$ :



The corresponding Cartan matrix  $A$  is

$$A = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

and the Auslander-Reiten quiver of the algebra given by  $\mathbb{D}_4$  is the following:



where  $e_1, e_2, e_3, e_4$  are the dimension vectors of simple representations. By abuse of notations, we also denote by  $e_1, e_2, e_3, e_4$  the simple representations.

In the Ringel-Hall algebra  $\mathcal{H}(\vec{\mathbb{D}}_4)$ , we consider the following elements:

$$\begin{aligned}
 M_{11} &= v^{\frac{1}{2}(\mathbf{dim} e_1, \mathbf{dim} e_1) - \dim_k e_1} u_{[e_1]} = u_{[e_1]}, \\
 M_{12} &= v^{\frac{1}{2}(\mathbf{dim}(e_1+e_2), \mathbf{dim}(e_1+e_2)) - \dim_k(e_1+e_2)} u_{[e_1+e_2]} = v^{-1} u_{[e_1+e_2]}, \\
 M_{13} &= v^{\frac{1}{2}(\mathbf{dim}(e_1+e_3), \mathbf{dim}(e_1+e_3)) - \dim_k(e_1+e_3)} u_{[e_1+e_3]} = v^{-1} u_{[e_1+e_3]}, \\
 M_{14} &= v^{\frac{1}{2}(\mathbf{dim}(e_1+e_4), \mathbf{dim}(e_1+e_4)) - \dim_k(e_1+e_4)} u_{[e_1+e_4]} = v^{-1} u_{[e_1+e_4]}, \\
 M_{21} &= v^{\frac{1}{2}(\mathbf{dim}(2e_1+e_2+e_3+e_4), \mathbf{dim}(2e_1+e_2+e_3+e_4)) - \dim_k(2e_1+e_2+e_3+e_4)} u_{[2e_1+e_2+e_3+e_4]} \\
 &= v^{-4} u_{[2e_1+e_2+e_3+e_4]}, \\
 M_{22} &= v^{\frac{1}{2}(\mathbf{dim}(e_1+e_3+e_4), \mathbf{dim}(e_1+e_3+e_4)) - \dim_k(e_1+e_3+e_4)} u_{[e_1+e_3+e_4]} = v^{-2} u_{[e_1+e_3+e_4]}, \\
 M_{23} &= v^{\frac{1}{2}(\mathbf{dim}(e_1+e_2+e_4), \mathbf{dim}(e_1+e_2+e_4)) - \dim_k(e_1+e_2+e_4)} u_{[e_1+e_2+e_4]} = v^{-2} u_{[e_1+e_2+e_4]}, \\
 M_{24} &= v^{\frac{1}{2}(\mathbf{dim}(e_1+e_2+e_3), \mathbf{dim}(e_1+e_2+e_3)) - \dim_k(e_1+e_2+e_3)} u_{[e_1+e_2+e_3]} = v^{-2} u_{[e_1+e_2+e_3]}, \\
 M_{31} &= v^{\frac{1}{2}(\mathbf{dim}(e_1+e_2+e_3+e_4), \mathbf{dim}(e_1+e_2+e_3+e_4)) - \dim_k(e_1+e_2+e_3+e_4)} u_{[e_1+e_2+e_3+e_4]} \\
 &= v^{-3} u_{[e_1+e_2+e_3+e_4]}, \\
 M_{32} &= v^{\frac{1}{2}(\mathbf{dim} e_2, \mathbf{dim} e_2) - \dim_k e_2} u_{[e_2]} = u_{[e_2]}, \\
 M_{33} &= v^{\frac{1}{2}(\mathbf{dim} e_3, \mathbf{dim} e_3) - \dim_k e_3} u_{[e_3]} = u_{[e_3]}, \\
 M_{34} &= v^{\frac{1}{2}(\mathbf{dim} e_4, \mathbf{dim} e_4) - \dim_k e_4} u_{[e_4]} = u_{[e_4]}.
 \end{aligned}$$

For convenience, we use the following notations:

$$\begin{aligned}
 C_1 &= \{((m, n)(i, j)) \mid m = i \in \{1, 2, 3\}, n \in \{3, 4\}, j \in \{2, 3\} \text{ and } n > j\}, \\
 C_2 &= \{((m, n)(i, j)) \mid m = i \in \{1, 2, 3\}, n \in \{2, 3, 4\}, j = 1\}, \\
 C_3 &= \{((m, n)(i, j)) \mid m = 3, i = 1, n = j \in \{2, 3, 4\}\}, \\
 C_4 &= \{((m, n)(i, j)) \mid m \in \{2, 3\}, i = m - 1, n \in \{1, 2, 3, 4\}, j \in \{2, 3, 4\} \text{ and } n \neq j\}, \\
 C_5 &= \{((m, n)(i, j)) \mid m = 3, n \in \{2, 3, 4\}, i = j = 1\},
 \end{aligned}$$

$$C_6 = \{((m, n)(i, j)) \mid m = 3, i = 1, n, j \in \{2, 3, 4\} \text{ and } n \neq j\},$$

$$C_7 = \{((m, n)(i, j)) \mid m \in \{2, 3\}, i = m - 1, n = j \in \{2, 3, 4\}\},$$

$$C_8 = \{((m, n)(i, j)) \mid m = 3, n = i = 1, j \in \{2, 3, 4\}\},$$

$$C_9 = \{((m, n)(i, j)) \mid m \in \{2, 3\}, i = m - 1, n \in \{2, 3, 4\}, j = 1\},$$

$$C_{10} = \{((m, n)(i, j)) \mid m \in \{2, 3\}, i = m - 1, n = j = 1\},$$

$$C_{11} = \{((m, n)(i, j)) \mid m = 3, i = n = j = 1\}.$$

Then by using the Auslander-Reiten quiver, we get the following relations:

$$\begin{aligned} M_{mn}M_{ij} &= M_{ij}M_{mn}, & ((m, n)(i, j)) &\in C_1, \\ M_{mn}M_{ij} &= vM_{ij}M_{mn}, & ((m, n)(i, j)) &\in C_2 \cup C_3 \cup C_4, \\ M_{mn}M_{ij} &= v^{-1}M_{ij}M_{mn} + M_{1n}, & ((m, n)(i, j)) &\in C_5, \\ M_{mn}M_{ij} &= v^{-1}M_{ij}M_{mn} + M_{2r}, & ((m, n)(i, j)) &\in C_6, \end{aligned}$$

where  $r \in \{2, 3, 4\}$  and  $r \neq n, r \neq j$ ,

$$\begin{aligned} M_{mn}M_{ij} &= v^{-1}M_{ij}M_{mn} + M_{m1}, & ((m, n)(i, j)) &\in C_7, \\ M_{mn}M_{ij} &= M_{ij}M_{mn} + (v - v^{-1})M_{2r}M_{2s}, & ((m, n)(i, j)) &\in C_8, \end{aligned}$$

where  $r, s \in \{2, 3, 4\}$ , and  $j \neq r, j \neq s, r < s$ ,

$$M_{mn}M_{ij} = M_{ij}M_{mn} + (v - v^{-1})M_{ir}M_{is}, \quad ((m, n)(i, j)) \in C_9,$$

where  $r, s \in \{2, 3, 4\}$  and  $n \neq r, n \neq s, r < s$ ,

$$\begin{aligned} M_{mn}M_{ij} &= vM_{ij}M_{mn} + (v^2 - 2 + v^{-2})M_{i2}M_{i3}M_{i4}, & ((m, n)(i, j)) &\in C_{10}, \\ M_{mn}M_{ij} &= v^{-1}M_{ij}M_{mn} + (v - 2v^{-1})M_{21} + (1 - v^{-2})M_{12}M_{22} \\ &\quad + (1 - v^{-2})M_{13}M_{23} + (1 - v^{-2})M_{14}M_{24}, & ((m, n)(i, j)) &\in C_{11}. \end{aligned}$$

Since  $e_1, e_2, e_3$  and  $e_4$  are the simple modules corresponding to vertices 1, 2, 3 and 4, respectively, it follows that  $M_{11} = u_{[e_1]}, M_{32} = u_{[e_2]}, M_{33} = u_{[e_3]}, M_{34} = u_{[e_4]}$ . Let

$$\begin{aligned} E_1 &= E_{11} = \eta^{-1}(M_{11}), & E_4 &= E_{34} = \eta^{-1}(M_{34}), & E_{14} &= \eta^{-1}(M_{14}), & E_{23} &= \eta^{-1}(M_{23}), \\ E_2 &= E_{32} = \eta^{-1}(M_{32}), & E_{12} &= \eta^{-1}(M_{12}), & E_{21} &= \eta^{-1}(M_{21}), & E_{24} &= \eta^{-1}(M_{24}), \\ E_3 &= E_{33} = \eta^{-1}(M_{33}), & E_{13} &= \eta^{-1}(M_{13}), & E_{22} &= \eta^{-1}(M_{22}), & E_{31} &= \eta^{-1}(M_{31}), \end{aligned}$$

where  $\eta$  is the isomorphism in Theorem 2.3, and let

$$X = \{E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}, E_{31}, E_{32}, E_{33}, E_{34}\}.$$

Then we have the following relations (under the isomorphism  $\eta^{-1}$  in Theorem 2.3):

$$\begin{aligned}
E_{mn}E_{ij} &= E_{ij}E_{mn}, & ((m,n)(i,j)) &\in C_1, \\
E_{mn}E_{ij} &= vE_{ij}E_{mn}, & ((m,n)(i,j)) &\in C_2, \\
E_{mn}E_{ij} &= vE_{ij}E_{mn}, & ((m,n)(i,j)) &\in C_3, \\
E_{mn}E_{ij} &= vE_{ij}E_{mn}, & ((m,n)(i,j)) &\in C_4, \\
E_{mn}E_{ij} &= v^{-1}E_{ij}E_{mn} + E_{1n}, & ((m,n)(i,j)) &\in C_5, \\
E_{mn}E_{ij} &= v^{-1}E_{ij}E_{mn} + E_{2r}, & ((m,n)(i,j)) &\in C_6, \\
E_{mn}E_{ij} &= v^{-1}E_{ij}E_{mn} + E_{m1}, & ((m,n)(i,j)) &\in C_7, \\
E_{mn}E_{ij} &= E_{ij}E_{mn} + (v - v^{-1})E_{2r}E_{2s}, & ((m,n)(i,j)) &\in C_8, \\
E_{mn}E_{ij} &= E_{ij}E_{mn} + (v - v^{-1})E_{ir}E_{is}, & ((m,n)(i,j)) &\in C_9, \\
E_{mn}E_{ij} &= vE_{ij}E_{mn} + (v^2 - 2 + v^{-2})E_{i2}E_{i3}E_{i4}, & ((m,n)(i,j)) &\in C_{10}, \\
E_{mn}E_{ij} &= v^{-1}E_{ij}E_{mn} + (v - 2v^{-1})E_{21} + (1 - v^{-2})E_{12}E_{22} \\
&\quad + (1 - v^{-2})E_{13}E_{23} + (1 - v^{-2})E_{14}E_{24}, & ((m,n)(i,j)) &\in C_{11}.
\end{aligned} \tag{3.1}$$

Note that the relations (3.1) include the Serre relations  $S^+$ . So  $U_q^+(A)$  can be viewed as a factor algebra  $\mathbb{Q}(v)\langle X \rangle / I$ , where  $I$  is the ideal generated by the relations (3.1).

We define an ordering

$$E_{11} < E_{12} < E_{13} < E_{14} < E_{21} < E_{22} < E_{23} < E_{24} < E_{31} < E_{32} < E_{33} < E_{34}$$

for the elements  $E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}, E_{31}, E_{32}, E_{33}, E_{34}$ , and then this ordering induces a degree-lexicographical ordering on the monomials of these elements. For convenience, we denote by  $r_1, \dots, r_{11}$ , respectively, the polynomials obtained from the relations in (3.1) by subtracting the right-hand side from the left-hand side, and let  $S^{+c} = \{r_1, r_2, \dots, r_{11}\}$ . Then, of course,  $S^+ \subset S^{+c}$ , and we have the following theorem.

**Theorem 3.1** *The set  $S^{+c}$  is a Gröbner-Shirshov basis of the algebra  $U_q^+(A)$ .*

**Proof** The possible compositions between the elements of  $S^{+c}$  can be divided into 32 cases. We only prove the triviality of three cases, and the proofs of other cases are similar.

**Case 1** Let  $f = r_4 = E_{mn}E_{ij} - vE_{ij}E_{mn}$ ,  $g = r_4 = E_{ij}E_{kl} - vE_{kl}E_{ij}$ ,  $\omega = E_{mn}E_{ij}E_{kl}$ , where  $((m,n)(i,j)), ((i,j)(k,l)) \in C_4$  and  $((m,n)(k,l)) \in C_3, C_6$  or  $C_8$ . We consider the following different cases:

(1.1) If  $((m,n)(i,j)), ((i,j)(k,l)) \in C_4$  and  $((m,n)(k,l)) \in C_3$ , then

$$\begin{aligned}
(f, g)_\omega &\equiv -vE_{ij}E_{mn}E_{kl} + vE_{mn}E_{kl}E_{ij} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
&\equiv -v^2E_{ij}E_{kl}E_{mn} + v^2E_{kl}E_{mn}E_{ij} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
&\equiv -v^3E_{kl}E_{ij}E_{mn} + v^3E_{kl}E_{ij}E_{mn} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
&\equiv 0 \mod(S^{+c}, E_{mn}E_{ij}E_{kl}).
\end{aligned}$$

**(1.2)** If  $((m, n)(i, j)), ((i, j)(k, l)) \in C_4$  and  $((m, n)(k, l)) \in C_6$ , then

$$\begin{aligned}
 (f, g)_\omega &\equiv -vE_{ij}E_{mn}E_{kl} + vE_{mn}E_{kl}E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -vE_{ij}[v^{-1}E_{kl}E_{mn} + E_{2r}] + v[v^{-1}E_{kl}E_{mn} + E_{2r}]E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -vE_{kl}E_{ij}E_{mn} - vE_{ij}E_{2r} + vE_{kl}E_{ij}E_{mn} + vE_{2r}E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv 0 \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}},
 \end{aligned}$$

where  $i = 2, j = r$ .

**(1.3)** If  $((m, n)(i, j)), ((i, j)(k, l)) \in C_4, ((i, j)(2, r)), ((i, j)(2, s)) \in C_1$  and  $((m, n)(k, l)) \in C_8$ , then

$$\begin{aligned}
 (f, g)_\omega &\equiv -vE_{ij}E_{mn}E_{kl} + vE_{mn}E_{kl}E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -vE_{ij}[E_{kl}E_{mn} + (v - v^{-1})E_{2r}E_{2s}] \\
 &\quad + v[E_{kl}E_{mn} + (v - v^{-1})E_{2r}E_{2s}]E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -v^2E_{kl}E_{ij}E_{mn} - (v^2 - 1)E_{ij}E_{2r}E_{2s} \\
 &\quad + v^2E_{kl}E_{ij}E_{mn} + (v^2 - 1)E_{2r}E_{2s}E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -(v^2 - 1)E_{2r}E_{2s}E_{ij} + (v^2 - 1)E_{2r}E_{2s}E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv 0 \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}}.
 \end{aligned}$$

**Case 2** Let  $f = r_9 = E_{mn}E_{ij} - E_{ij}E_{mn} - (v - v^{-1})E_{ir}E_{is}$ ,  $g = r_4 = E_{ij}E_{kl} - vE_{kl}E_{ij}$ ,  $\omega = E_{mn}E_{ij}E_{kl}$ , where  $((m, n)(i, j)) \in C_9, ((i, j)(k, l)) \in C_4$  and  $((m, n)(k, l)) \in C_3$  or  $C_6$ . We consider the following different cases:

**(2.1)** If  $((m, n)(k, l)) \in C_3, ((2, r)(k, l)), ((2, s)(k, l)) \in C_4$ , then

$$\begin{aligned}
 (f, g)_\omega &\equiv -E_{ij}E_{mn}E_{kl} - (v - v^{-1})E_{ir}E_{is}E_{kl} + vE_{mn}E_{kl}E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -vE_{ij}E_{kl}E_{mn} - (v - v^{-1})E_{ir}E_{is}E_{kl} + v^2E_{kl}E_{mn}E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -v^2E_{kl}E_{ij}E_{mn} - (v - v^{-1})E_{ir}E_{is}E_{kl} + v^2E_{kl}E_{ij}E_{mn} \\
 &\quad + (v^3 - v)E_{kl}E_{ir}E_{is} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -(v - v^{-1})vE_{ir}E_{kl}E_{is} + (v^3 - v)E_{kl}E_{ir}E_{is} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -(v - v^{-1})v^2E_{kl}E_{ir}E_{is} + (v^3 - v)E_{kl}E_{ir}E_{is} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv 0 \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}}.
 \end{aligned}$$

**(2.2)** If  $((m, n)(k, l)) \in C_6$ , then

$$\begin{aligned}
 (f, g)_\omega &\equiv -E_{ij}E_{mn}E_{kl} - (v - v^{-1})E_{ir}E_{is}E_{kl} + vE_{mn}E_{kl}E_{ij} \pmod{S^{+c}, E_{mn}E_{ij}E_{kl}} \\
 &\equiv -E_{ij}[v^{-1}E_{kl}E_{mn} + E_{2t}] - (v - v^{-1})E_{ir}E_{is}E_{kl}
 \end{aligned}$$



$$\begin{aligned}
& + v[v^{-1}E_{kl}E_{mn} + E_{2t}]E_{ij} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
& \equiv -E_{kl}E_{ij}E_{mn} - E_{ij}E_{2t} - (v - v^{-1})E_{ir}E_{is}E_{kl} + E_{kl}E_{ij}E_{mn} \\
& + (v - v^{-1})E_{kl}E_{ir}E_{is} + vE_{2t}E_{ij} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
& \equiv -E_{ij}E_{2t} - (v - v^{-1})E_{ir}E_{is}E_{kl} + (v - v^{-1})E_{kl}E_{ir}E_{is} \\
& + vE_{2t}E_{ij} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}).
\end{aligned}$$

If  $((i, r)(k, l)) \in C_7$ ,  $((i, s)(k, l)) \in C_4$ ,  $((2, t)(i, j)) \in C_2$ , then

$$\begin{aligned}
(f, g)_\omega & \equiv -E_{ij}E_{2t} - (v^2 - 1)E_{ir}E_{kl}E_{is} + (v - v^{-1})E_{kl}E_{ir}E_{is} \\
& + v^2E_{ij}E_{2t} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
& \equiv -(v - v^{-1})E_{kl}E_{ir}E_{is} - (v^2 - 1)E_{i1}E_{is} + (v - v^{-1})E_{kl}E_{ir}E_{is} \\
& + (v^2 - 1)E_{ij}E_{2t} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
& \equiv 0 \mod(S^{+c}, E_{mn}E_{ij}E_{kl}),
\end{aligned}$$

where  $i = 2$ ,  $j = 1$ ,  $t = s$ .

If  $((i, r)(k, l)) \in C_4$ ,  $((i, s)(k, l)) \in C_7$ ,  $((2, t)(i, j)) \in C_2$ ,  $((i, r)(i, 1)) \in C_2$ , then

$$\begin{aligned}
(f, g)_\omega & \equiv -E_{ij}E_{2t} - (v - v^{-1})E_{ir}[v^{-1}E_{kl}E_{is} + E_{i1}] \\
& + (v - v^{-1})E_{kl}E_{ir}E_{is} + v^2E_{ij}E_{2t} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
& \equiv -E_{ij}E_{2t} - (v - v^{-1})E_{kl}E_{ir}E_{is} - (v - v^{-1})E_{ir}E_{i1} \\
& + (v - v^{-1})E_{kl}E_{ir}E_{is} + v^2E_{ij}E_{2t} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
& \equiv -E_{ij}E_{2t} - (v^2 - 1)E_{i1}E_{ir} + v^2E_{ij}E_{2t} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
& \equiv 0 \mod(S^{+c}, E_{mn}E_{ij}E_{kl}),
\end{aligned}$$

where  $i = 2$ ,  $j = 1$ ,  $t = r$ .

**Case 3** Let

$$f = r_{10} = E_{31}E_{21} - vE_{21}E_{31} - (v^2 - 2 + v^{-2})E_{22}E_{23}E_{24}$$

and

$$g = r_{10} = E_{21}E_{11} - vE_{11}E_{21} - (v^2 - 2 + v^{-2})E_{12}E_{13}E_{14}, \quad w = E_{31}E_{21}E_{11}.$$

Then

$$\begin{aligned}
(f, g)_\omega & \equiv -vE_{21}E_{31}E_{11} - (v^2 - 2 + v^{-2})E_{22}E_{23}E_{24}E_{11} + vE_{31}E_{11}E_{21} \\
& + (v^2 - 2 + v^{-2})E_{31}E_{12}E_{13}E_{14} \mod(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
& \equiv -vE_{21}[v^{-1}E_{11}E_{31} + (v - 2v^{-1})E_{21} + (1 - v^{-2})E_{12}E_{22} + (1 - v^{-2})E_{13}E_{23} \\
& + (1 - v^{-2})E_{14}E_{24}] - (v^2 - 2 + v^{-2})E_{22}E_{23}[E_{11}E_{24} + (v - v^{-1})E_{12}E_{13}]
\end{aligned}$$

$$\begin{aligned}
& + v[v^{-1}E_{11}E_{31} + (v - 2v^{-1})E_{21} + (1 - v^{-2})E_{12}E_{22} \\
& + (1 - v^{-2})E_{13}E_{23} + (1 - v^{-2})E_{14}E_{24}]E_{21} \\
& + (v^2 - 2 + v^{-2})[E_{12}E_{31} + (v - v^{-1})E_{23}E_{24}]E_{13}E_{14} \pmod{(S^{+c}, E_{mn}E_{ij}E_{kl})} \\
\equiv & -E_{21}E_{11}E_{31} - (v^2 - 2)E_{21}E_{21} - (v - v^{-1})E_{21}E_{12}E_{22} - (v - v^{-1})E_{21}E_{13}E_{23} \\
& - (v - v^{-1})E_{21}E_{14}E_{24} - (v^2 - 2 + v^{-2})E_{22}[E_{11}E_{23} + (v - v^{-1})E_{12}E_{14}]E_{24} \\
& - (v^3 - 3v + 3v^{-1} - v^{-3})E_{22}E_{23}E_{12}E_{13} + E_{11}E_{31}E_{21} + (v^2 - 2)E_{21}E_{21} \\
& + (v - v^{-1})E_{12}E_{22}E_{21} + (v - v^{-1})E_{13}E_{23}E_{21} + (v - v^{-1})E_{14}E_{24}E_{21} \\
& + (v^2 - 2 + v^{-2})E_{12}[E_{13}E_{31} + (v - v^{-1})E_{22}E_{24}]E_{14} \\
& + (v^3 - 3v + 3v^{-1} - v^{-3})E_{23}E_{24}E_{13}E_{14} \pmod{(S^{+c}, E_{mn}E_{ij}E_{kl})} \\
\equiv & -vE_{11}E_{21}E_{31} - (v^2 - 2 + v^{-2})E_{12}E_{13}E_{14}E_{31} - (v^2 - 1)E_{12}E_{21}E_{22} \\
& - (v^2 - 1)E_{13}E_{21}E_{23} - (v^2 - 1)E_{14}E_{21}E_{24} - (v^2 - 2 + v^{-2})[E_{11}E_{22} \\
& + (v - v^{-1})E_{13}E_{14}]E_{23}E_{24} - (v^3 - 3v + 3v^{-1} - v^{-3})E_{22}E_{12}E_{14}E_{24} \\
& - (v^4 - 3v^2 + 3 - v^{-2})E_{22}E_{12}E_{23}E_{13} + vE_{11}E_{21}E_{31} \\
& + (v^2 - 2 + v^{-2})E_{11}E_{22}E_{23}E_{24} + (v^2 - 1)E_{12}E_{21}E_{22} + (v^2 - 1)E_{13}E_{21}E_{23} \\
& + (v^2 - 1)E_{14}E_{21}E_{24} + (v^2 - 2 + v^{-2})E_{12}E_{13}[E_{14}E_{31} + (v - v^{-1})E_{22}E_{23}] \\
& - (v^3 - 3v + 3v^{-1} - v^{-3})E_{12}E_{22}E_{24}E_{14} \\
& + (v^4 - 3v^2 + 3 - v^{-2})E_{23}E_{13}E_{24}E_{14} \pmod{(S^{+c}, E_{mn}E_{ij}E_{kl})} \\
\equiv & -(v^2 - 2 + v^{-2})E_{12}E_{13}E_{14}E_{31} - (v^2 - 2 + v^{-2})E_{11}E_{22}E_{23}E_{24} \\
& - (v^3 - 3v + 3v^{-1} - v^{-3})E_{13}E_{14}E_{23}E_{24} - (v^3 - 3v + 3v^{-1} - v^{-3})[v^{-1}E_{12}E_{22} \\
& + E_{21}]E_{14}E_{24} - (v^4 - 3v^2 + 3 - v^{-2})[v^{-1}E_{12}E_{22} + E_{21}][v^{-1}E_{13}E_{23} + E_{21}] \\
& + (v^2 - 2 + v^{-2})E_{11}E_{22}E_{23}E_{24} + (v^2 - 2 + v^{-2})E_{12}E_{13}E_{14}E_{31} \\
& + (v^3 - 3v + 3v^{-1} - v^{-3})E_{12}E_{13}E_{22}E_{23} \\
& + (v^3 - 3v + 3v^{-1} - v^{-3})E_{12}E_{22}[v^{-1}E_{14}E_{24} + E_{21}] \\
& - (v^4 - 3v^2 + 3 - v^{-2})[v^{-1}E_{13}E_{23} + E_{21}][v^{-1}E_{14}E_{24} + E_{21}] \pmod{(S^{+c}, E_{mn}E_{ij}E_{kl})} \\
\equiv & -(v^3 - 3v + 3v^{-1} - v^{-3})E_{13}E_{14}E_{23}E_{24} - (v^3 - 3v + 3v^{-1} - v^{-3})E_{12}E_{14}E_{22}E_{24} \\
& - (v^4 - 3v^2 + 3 - v^{-2})E_{14}E_{21}E_{24} - (v^3 - 3v + 3v^{-1} - v^{-3})E_{12}E_{13}E_{22}E_{23} \\
& - (v^4 - 3v^2 + 3 - v^{-2})E_{12}E_{21}E_{22} - (v^4 - 3v^2 + 3 - v^{-2})E_{13}E_{21}E_{23}
\end{aligned}$$

$$\begin{aligned}
& - (v^4 - 3v^2 + 3 - v^{-2})E_{21}E_{21} + (v^3 - 3v + 3v^{-1} - v^{-3})E_{12}E_{13}E_{22}E_{23} \\
& + (v^3 - 3v + 3v^{-1} - v^{-3})E_{12}E_{14}E_{22}E_{24} + (v^4 - 3v^2 + 3 - v^{-2})E_{12}E_{21}E_{22} \\
& + (v^3 - 3v + 3v^{-1} - v^{-3})E_{13}E_{14}E_{23}E_{24} + (v^4 - 3v^2 + 3 - v^{-2})E_{13}E_{21}E_{23} \\
& + (v^4 - 3v^2 + 3 - v^{-2})E_{14}E_{21}E_{24} + (v^4 - 3v^2 + 3 - v^{-2})E_{21}E_{21} \\
& \equiv 0 \pmod{(S^{+c}, E_{mn}E_{ij}E_{kl})}.
\end{aligned}$$

Dually, by replacing all  $E$ 's in (3.1) by  $F$ 's, we get similar relations, say (3.1)', for the generators

$$Y = \{F_{11}, F_{12}, F_{13}, F_{14}, F_{21}, F_{22}, F_{23}, F_{24}, F_{31}, F_{32}, F_{33}, F_{34}\}$$

of subalgebra  $U_q^-(A)$ . It is easy to see that relations (3.1)' include the Serre relations  $S^-$ . So, similarly, if  $J$  is the ideal generated by the relations (3.1)', then the negative part  $U_q^-(A)$  of quantum group  $U_q(A)$  can be viewed as a factor algebra  $\mathbb{Q}(v)\langle Y \rangle / J$  of the free algebra  $\mathbb{Q}(v)\langle Y \rangle$  generated by the set  $Y$ .

We define an ordering

$$F_{11} < F_{12} < F_{13} < F_{14} < F_{21} < F_{22} < F_{23} < F_{24} < F_{31} < F_{32} < F_{33} < F_{34}$$

for the elements  $F_{11}, F_{12}, F_{13}, F_{14}, F_{21}, F_{22}, F_{23}, F_{24}, F_{31}, F_{32}, F_{33}, F_{34}$ . Then this ordering induces a degree-lexicographical ordering on the monomials of these elements. In a way similar to the discussions in the positive part, we denote the polynomials obtained from the relations in (3.1)' by  $f_1, \dots, f_{11}$ , and let  $S^{-c} = \{f_1, f_2, \dots, f_{11}\}$ . Then, of course,  $S^- \subset S^{-c}$ , and we have the following theorem.

**Theorem 3.2** *The set  $S^{-c}$  is a Gröbner-Shirshov basis of the algebra  $U_q^-(A)$ .*

If we define an ordering

$$\begin{aligned}
& E_{11} < E_{12} < E_{13} < E_{14} < E_{21} < E_{22} < E_{23} < E_{24} < E_{31} < E_{32} < E_{33} < E_{34} < K_1 < K_2 \\
& < K_3 < K_4 < F_{11} < F_{12} < F_{13} < F_{14} < F_{21} < F_{22} < F_{23} < F_{24} < F_{31} < F_{32} < F_{33} < F_{34}
\end{aligned}$$

for the elements  $E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}, E_{31}, E_{32}, E_{33}, E_{34}, K_1, K_2, K_3, K_4, F_{11}, F_{12}, F_{13}, F_{14}, F_{21}, F_{22}, F_{23}, F_{24}, F_{31}, F_{32}, F_{33}, F_{34}$ , then this ordering induces a degree-lexicographical ordering on the monomials of these elements. Now, by [3, Theorem 2.7], we are able to state our main result.

**Theorem 3.3** *The set  $S^{+c} \cup K \cup T \cup S^{-c}$  is a Gröbner-Shirshov basis of the quantum group  $U_q(A)$ .*

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## References

- [1] Bergman, G. M., The diamond lemma for ring theory, *Adv. Math.*, **29**, 1978, 178–218.
- [2] Bokut, L. A., Imbeddings into simple associative algebras, *Algebra Logic*, **15**, 1976, 117–142.

- [3] Bokut, L. A. and Malcolmson, P., Gröbner-Shirshov bases for quantum enveloping algebras, *Israel J. Math.*, **96**, 1996, 97–113.
- [4] Buchberger, B., An algorithm for finding a basis for the residue class ring of a zero-dimensional ideal (in German), Ph. D. Thesis, University of Innsbruck, Austria, 1965.
- [5] Deng, B. M., Du, J., Parshal, B. and Wang, J. P., Finite Dimensional Algebras and Quantum Groups, Mathematical Surveys and Monographs, **150**, A. M. S., Providence, RI, 2008.
- [6] Drinfel'd, V. G., Hopf algebras and the quantum Yang-Baxter equation, *Dokl. Akad. Nauk SSSR*, **283**(5), 1985, 1060–1064.
- [7] Gabriel, P., Unzerlegbare Darstellungen I (in German), *Manuscripta Math.*, **6**, 1972, 71–103.
- [8] Jimbo, M., A  $q$ -difference analogue of  $U(g)$  and the Yang-Baxter equation, *Lett. Math. Phys.*, **10**(1), 1985, 63–69.
- [9] Ringel, C. M., Hall algebras and quantum groups, *Invent. Math.*, **101**, 1990, 583–592.
- [10] Ringel, C. M., PBW-bases of quantum groups, *J. Reine Angew. Math.*, **470**, 1996, 51–88.
- [11] Rosso, M., Finite dimensional representations of the quantum analogue of the enveloping algebra of a complex simple Lie algebra, *Comm. Math. Phys.*, **117**, 1988, 581–593.
- [12] Shirshov, A. I., Some algorithmic problems for Lie algebras, *Siberian Math. J.*, **3**, 1962, 292–296.
- [13] Yamane, H., A Poincaré-Birkhoff-Witt theorem for quantized universal enveloping algebras of type  $A_n$ , *Publ. RIMS. Kyoto Univ.*, **25**, 1989, 503–520.