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Remarks on the Operator Norm Localization Property*

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Abstract The author studies the metric spaces with operator norm localization property. It is proved that the operator norm localization property is coarsely invariant and is preserved under certain infinite union. In the case of finitely generated groups, the operator norm localization property is also preserved under the direct limits.

Keywords Metric space, Asymptotic dimension, Operator norm localization, Coarse invariance, Roe algebra
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1 Introduction

Let X be a discrete metric space with bounded geometry. Associated to X, there is a C^*-algebra $C^*(X)$ called Roe algebra. The coarse Baum-Connes conjecture states that the coarse assembly map $\mu : KX_*(X) \to K_*(C^*(X))$ is an isomorphism. It plays a very important role in topology and geometry. In [11], Yu showed that the coarse Baum-Connes conjecture holds in the case that X is uniformly embedded into a Hilbert space. Inspired by Gromov's expander graph structure, Higson [6] gave a counterexample to the coarse Baum-Connes conjecture. The relevant construction is the box space $X(\Gamma)$ of an infinite group Γ with property T, the residually finite and linear type, that is, the coarse disjoint union of the quotient groups Γ/Γ_n. In Higson's original construction (see [7]), there is an algebraic lifting principle, that is, an operator $T \in C^*_{\rm alg}(X(\Gamma))$ will be restricted to an operator on $C^*_{\rm alg}(\Gamma/\Gamma_n)$ for all but finitely many n, and such an operator can then be lifted to a Γ_n-invariant element of the Roe algebra of Γ. In general, such lifting can be extended to the maximal norm closure. Using a certain type of localization estimation of the operator norm in the case of asymptotic finite dimension, Higson proved that the lifting can also be extended to the reduced norm closure. This was important in his original construction of counterexamples to the coarse Baum-Connes conjecture. In fact, such lifting can be extended to the maximal norm closure for every discrete metric space (see [5]). The natural question is what kinds of conditions can guarantee the algebraic level lifting to be extended to the reduced norm level. Generalizing the local estimation method in the case of finite asymptotic dimension, Yu [5] introduced the concept of the operator norm localization property. It is easy to prove that if the metric space has the operator norm localization property, then the algebraic level lifting can be extended to the reduced norm level. In this paper, we study

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its basic properties and get some results on preserving properties of operator norm localization property.

2 Preliminaries

In this section, we introduce the definition and basic properties of the operator norm localization property for a discrete metric space.

Definition 2.1 (see [8]) *Let X be a discrete metric space, and H be a separable and infinite dimensional Hilbert space. A bounded operator $T : \ell^2(X)\otimes H \to \ell^2(X)\otimes H$, is said to have propagation at most r (abbr.* $\mathrm{Prop}(T) \le r$*) if for all $\varphi, \psi \in \ell^2(X)\otimes H$ with $d(\mathrm{Supp}(\varphi), \mathrm{Supp}(\psi)) > r$ such that $\langle T\varphi, \psi\rangle = 0$.*

Note that X is a discrete metric space. Then we can write

$$\ell^2(X) \otimes H = \bigoplus_{x\in X}(\delta_x \otimes H),$$

where δ_x is the Dirac function at x. Every bounded operator acting on $\ell^2(X) \otimes H$ has a corresponding matrix representation $T = (T_{x,y})_{x,y\in X}$, where $T_{x,y} : \delta_y \otimes H \to \delta_x \otimes H$ is a bounded operator. The support of T (abbr. $\mathrm{Supp}(T)$) is the complement (in $X \times Y$) of the set of all points $(x, y) \in X \times Y$ such that $T_{x,y} = 0$. We call that T is locally compact if $T_{x,y}$ is a compact operator for all x, y in X. For T to have propagation r, it is equivalent to saying that the matrix coefficient $T_{x,y}$ of T vanishes when $d(x, y) > r$. The space of operators acting on $\ell^2(X) \otimes H$ with propagation at most r will be denoted by $\mathcal{A}_r(X)$.

Let $\|T\|$ denote the operator norm of a bounded linear operator T.

Definition 2.2 *The collection of all locally compact, finite propagation operators on $\ell^2 \otimes H$ is a $*$-subalgebra of $\mathcal{B}(\ell^2(X) \otimes H)$. Its norm-completion, denoted by $C^*(X)$, is the Roe algebra of X.*

Definition 2.3 *Let X be a discrete metric space. Let $f : \mathbb{N} \to \mathbb{N}$ be a (non-decreasing) function. We say that X has the operator norm localization property relative to f with constant $c \ge 1$, if for any $k > 0$ and every $T \in \mathcal{A}_k(X)$, the following inequality holds:*

$$\|T\| \le c \sup\{\|Tv\| : v \in \ell^2(X, H), \|v\| \le 1, \mathrm{diam}(\mathrm{Supp}(v)) \le f(k)\}.$$

The infimum over all possible c is called the operator localization number of X.

Recall that a discrete metric space X has the bounded geometry, if for every $R > 0$, there is a uniform bound on the number of elements in the ball of radius R in X. It is not necessary to assume the metric space to be with the bounded geometry in the above definition. But our interest is in the bounded geometric case.

Definition 2.4 *A Borel map f from a metric space X to another metric space Y is called* coarse *if*

(1) *f is metrically proper, i.e., the inverse image of any bounded set is bounded,*

(2) *for every $R > 0$, there exists an $R' > 0$ such that $d(f(x), f(y)) \le R'$ for all $x, y \in X$ satisfying $d(x, y) \le R$.*

Definition 2.5 *We say that the metric spaces X and Y are coarsely equivalent if there exists an $r > 0$ and coarse maps $\varphi : X \to Y$, $\psi : Y \to X$ such that $d_Y(\varphi \circ \psi(y), y) \le r$ and $d_X(\psi \circ \varphi(x), x) \le r$ for all $x \in X$ and $y \in Y$.*

Definition 2.6 *Let Γ be a countable discrete group. A length function on Γ is a non-negative real-valued function l satisfying that for all x and y in Γ,*

(1) $l(xy) \le l(x) + l(y)$,

(2) $l(x^{-1}) = l(x)$,

(3) $l(x) = 0$ *if and only if* $x = 1$.

This defines a metric on Γ by $d_\Gamma(f, g) = l(f^{-1}g)$. A length function l is proper, if for all $C > 0$, the subset $l^{-1}([0, C])$ is finite which induces a proper metric on Γ. If Γ is a finitely generated group and its generating set S is symmetric, i.e., $S = S^{-1}$, then the length $l(g)$ of an element $g \in \Gamma$ is defined to be the length of the shortest word in S representing g. In this case, d_Γ is the left invariant in the sense that $d_\Gamma(hf, hg) = d_\Gamma(f, g)$.

From the definition, we immediately have the following two propositions.

Proposition 2.1 *Let d and d' be equivalent metrics on Γ. Then Γ has the operator norm localization property with respect to d if and only if it has the operator norm localization property with respect to d'.*

Proof There are $m_1, m_2 > 0$ such that $m_1 d'(x, y) \le d(x, y) \le m_2 d'(x, y)$, since d and d' are equivalent metrics on Γ. Then the proposition holds from the definition.

Proposition 2.2 *Let $Y \subset X$. If X has the operator norm localization property, then Y has the operator norm localization property.*

Proof It is obvious.

Proposition 2.3 *Let Γ be a countable finitely generated group and Γ_1 be a finitely generated subgroup of Γ. If Γ has the operator norm localization property with a constant c, then Γ_1 also has the operator norm localization property with a constant c.*

Proof Choose finite generating sets S and S_1 for Γ and Γ_1, such that $S_1 \subseteq S$, S and S_1 are closed under the inverse operation. Then the induced metrics d_S and d_{S_1} satisfy $d_{S_1}(x, y) \ge d_S(x, y)$ for all $x, y \in \Gamma_1$. Note that Γ_1 is a subset of Γ. Then Γ_1 has the operator norm localization property with respect to metric d_S. Therefore, for each operator $T \in \mathcal{B}(\ell^2(\Gamma_1) \otimes H)$ with $\mathrm{Prop}(T) \le r$, there are a constant c and a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, such that

$$\|T\| \le c \sup\{\|Tv\| : v \in \ell^2(\Gamma_1, H), \|v\| \le 1, \mathrm{Diam}_\Gamma(\mathrm{Supp}(v)) \le f(r)\}.$$

Hence, for each v and for any $x \in \mathrm{Supp}(v)$, we can write $x = x_0 g_1 \cdots g_k$ for $k \le f(r)$, where $g_l \in S$ for all $0 \le l \le k$, and x_0 is a fixed element in $\mathrm{Supp}(v)$. Let $B_{f(r)}(e) = \{g \in \Gamma_1 : g = g_1 \cdots g_j,\ g_i \in S,\ j \le f(r)\}$. It is not difficult to see that $B_{f(r)}(e)$ is a finite set. Therefore, there exists an $R(r) > 0$ such that $\sup\{d_{S_1}(g, e) : g \in B_{f(r)}(e)\} \le R(r)$. Note that $\mathrm{Supp}(v) \subset x_0 B_{f(r)}(e)$. Then the diameter of $\mathrm{Supp}(v)$ is at most $2R(r)$ with respect to d_{S_1} for all v. Let $f_{\Gamma_1}(r) = 2R(r)$ for each r. Hence,

$$\|T\| \le c \sup\{\|Tv\| : v \in \ell^2(\Gamma_1, H), \|v\| \le 1, \mathrm{Diam}_{\Gamma_1}(\mathrm{Supp}(v)) \le f_{\Gamma_1}(r)\}.$$

This completes the proof.

Definition 2.7 *Let X be a metric space. For every $d > 0$, $x, y \in X$, we say that $x \sim_d y$ if and only if there is a chain of points $x_0, x_1, \cdots, x_k$ with $x_0 = x$, $x_k = y$ such that $d(x_i, x_{i+1}) \leq d$ for all $i < k$. We call that the subset M of X is d-connected if and only if any two elements of M are equivalent. A d-component of X is a d-connected subset, that is not contained in any larger d-connected subset.*

Proposition 2.4 *Let Y be a discrete metric space with the bounded geometry which has the operator norm localization property with a constant c. If there exists an injective map f from X to Y satisfying that, for every $R > 0$, there exists $R' > 0$ such that $d(f(x), f(y)) \leq R'$ whenever $d(x, y) \leq R$, then X has the operator norm localization property with a constant c.*

Proof For every operator $T \in \mathcal{A}_r(X)$, we let

$$r' = \sup\{d(f(x), f(y)) : (x, y) \in \mathrm{Supp}(T)\},$$

and set

$$S_{y_1,y_2} = \begin{cases} T_{f^{-1}(y_1), f^{-1}(y_2)}, & \text{if } y_1, y_2 \in \mathrm{Im}(f), \\ 0, & \text{otherwise}, \end{cases}$$

which defines an operator $S = (S_{y_1,y_2}) \in \mathcal{B}(\ell^2(Y) \otimes H)$. Then $\mathrm{Prop}(S) \leq r'$ and $\|S\| = \|T\|$. Since Y has the operator norm localization property, we have

$$\|S\| \leq c \sup\{\|Sv\| : v \in \ell^2(Y, H), \|v\| \leq 1, \mathrm{Diam}_Y(\mathrm{Supp}(v)) \leq f_Y(r')\}.$$

For each v, we set $A_v = \mathrm{Supp}(v)$. We now define the operators $S^v = (S^v_{y_1,y_2}) \in \mathcal{B}(\ell^2(Y) \otimes H)$ by

$$S^v_{y_1,y_2} = \begin{cases} S_{y_1,y_2}, & \text{if } y_1 \in A_v, \\ 0, & \text{otherwise}, \end{cases}$$

and $T^v \in \mathcal{B}(\ell^2(X) \otimes H)$ by $T^v_{x_1,x_2} = S^v_{f(x_1),f(x_2)}$, $\forall (x_1, x_2) \in X \times X$. Note that $\|T^v\| = \|S^v\|$, $\|Sv\| = \|S^v v\|$ and $\mathrm{Prop}(T^v) \leq r$, whence

$$\begin{aligned} \|T\| = \|S\| &\leq c \sup\{\|S^v v\| : v \in \ell^2(Y, H), \|v\| \leq 1, \mathrm{Diam}_Y(\mathrm{Supp}(v)) \leq f_Y(r')\} \\ &\leq c \sup\{\|S^v\| : v \in \ell^2(Y, H), \|v\| \leq 1, \mathrm{Diam}_Y(\mathrm{Supp}(v)) \leq f_Y(r')\} \\ &\leq c \sup\{\|T^v\| : v \in \ell^2(Y, H), \|v\| \leq 1, \mathrm{Diam}_Y(\mathrm{Supp}(v)) \leq f_Y(r')\}. \end{aligned}$$

Let B_v be the r'-neighborhood of A_v. Then $\mathrm{Supp}(S^v) \subset B_v \times B_v$ and $\mathrm{Diam}_Y(B_v) \leq f_Y(r') + 2r'$. Therefore, there exists a constant $M(r) > 0$, such that $\#B_v \leq M(r)$, where $\#B_v$ is the number of elements in B_v. Let $D_v = f^{-1}(B_v)$. Since f is injective, we have $\#D_v \leq \#B_v \leq M(r)$. We now present the set D_v as the union of $3r$-components, and note that each component has a diameter at most $3rM(r)$ and every two $3r$-components are $3r$-separated. So

$$\|T^v\| = \sup\{\|T^v \xi\| : \xi \in \ell^2(X, H), \|\xi\| \leq 1, \mathrm{Diam}_X(\mathrm{Supp}(\xi)) \leq 3rM(r)\}.$$

Note that $\|T^v \xi\| \leq \|T\xi\|$. Then the proposition follows if we set $f_X(r) = 3rM(r)$.

Let Γ be a group acting on a metric space X. For every $k \geq 0$, the k-stabilizer $W_k(x_0)$ of a point $x_0 \in X$ is defined to be the set of all $g \in \Gamma$ with $gx_0 \in B(x_0, k)$, where $B(x_0, k)$ is the closed ball with the center x_0 and the radius k. The concept of k-stabilizer is introduced by Bell and Dranishnikov in their work on permanence properties of asymptotic dimension (see [1]).

Proposition 2.5 *Let Γ be a finitely generated group acting freely and isometrically on a metric space X (without assuming X to have the bounded geometry). If X has the operator norm localization property with a constant c_X and there exists an $x_0 \in X$, such that for every $k > 0$, $W_k(x_0)$ has the operator norm localization property with a constant c_Γ, where c_Γ is independent of k. Then Γ has the operator norm localization property with the constant $c = c_X c_\Gamma$.*

Proof We define a map $\pi : \Gamma \to X$ by the formula $\pi(g) = gx_0$. Then we have $W_k(x_0) = \pi^{-1}(B(x_0, k))$. Let $\lambda = \max\{d(sx_0, x_0) : s \in S\}$, where S is a generating set of Γ. Now we show that π is λ-Lipschitz. In fact, for all $g_1, g_2 \in \Gamma$, we have $d(\pi(g_1), \pi(g_2)) = d(g_1x_0, g_2x_0) = d(g_2^{-1}g_1x_0, x_0)$. Let $g = g_2^{-1}g_1 = s_1 \cdots s_n$, $s_i \in S$ $(1 \le i \le n)$ be the shortest word to represent g. Then

$$d(\pi(g_1), \pi(g_2)) = d(s_1 \cdots s_n x_0, x_0) \le \sum_{j=1}^{n} d(s_1 \cdots s_j x_0, s_1 \cdots s_{j-1} x_0) \le n\lambda = \lambda d(g_1, g_2).$$

Let $Y = \pi(\Gamma) \subset X$. For each operator T in $\mathcal{A}_k(\Gamma)$, we define an operator $S \in \mathcal{B}(\ell^2(Y) \otimes H)$ by

$$S_{y_1,y_2} = \begin{cases} T_{g_1,g_2}, & \text{if } \pi^{-1}(y_1) = g_1, \pi^{-1}(y_2) = g_2. \\ 0, & \text{otherwise.} \end{cases}$$

Then $S \in \mathcal{A}_{\lambda k}(Y)$ and $\|S\| = \|T\|$. Since X has the operator norm localization property with the constant c_X, we have

$$\|S\| \le c_X \sup\{\|Sv\| : v \in \ell^2(Y, H), \|v\| \le 1, \mathrm{Diam}_X(\mathrm{Supp}(v)) \le f_X(\lambda k)\}.$$

For each v, let A_v be the λk-neighborhood of $\mathrm{Supp}(v)$ in Y, and define $S^v \in \mathcal{B}(\ell^2(Y) \otimes H)$ by

$$S^v_{y_1,y_2} = \begin{cases} S_{y_1,y_2}, & \text{if } y_1 \in A_v, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|Sv\| = \|S^v v\|$. Similarly, we define an operator $T^v \in \mathcal{B}(\ell^2(\Gamma) \otimes H)$ by

$$T^v_{g_1,g_2} = \begin{cases} S^v_{x,y}, & \text{if } \pi(g_1) = x,\ \pi(g_2) = y, \\ 0, & \text{otherwise.} \end{cases}$$

Let B_v be the λk-neighborhood of A_v, and let $R := f_X(\lambda k) + 2\lambda k + 2\lambda k$. Then $\mathrm{Prop}(T^v) \le k$, $\mathrm{Supp}(T^v) \subset \pi^{-1}(B_v) \times \pi^{-1}(B_v)$ and $\mathrm{Diam}_X(B_v) \le R$. Note that Γ acts on Y transitively. Then there are $x \in Y$ and $g \in \Gamma$, such that $B_v \subset B^Y(x, R) = gB^Y(x_0, R)$, where $B^Y(x_0, R)$ is a ball in Y with the center x_0 and the radius R. Hence, $T^v \in \mathcal{B}(\ell^2(g.W_R(x_0)) \otimes H)$. By the assumption that $W_R(x_0)$ has the operator norm localization property with the constant c_Γ, there exists a non-decreasing function $g_R : \mathbb{N} \to \mathbb{N}$, such that

$$\|T^v\| \le c_\Gamma \sup\{\|T^v\xi\| : \xi \in \ell^2(\Gamma, H), \|\xi\| \le 1, \mathrm{Diam}_\Gamma(\mathrm{Supp}(\xi)) \le g_R(r)\}.$$

Note that $\|S^v\| = \|T^v\|$. Then

$$\begin{aligned}
\|T\| = \|S\| &\le c_X \sup\{\|Sv\| : v \in \ell^2(X, H), \|v\| \le 1, \mathrm{Diam}_X(\mathrm{Supp}(v)) \le f_X(\lambda r)\} \\
&\le c_X \sup\{\|S^v v\| : v \in \ell^2(X, H), \|v\| \le 1, \mathrm{Diam}_X(\mathrm{Supp}(v)) \le f_X(\lambda r)\} \\
&\le c_X \sup\{\|S^v\| : v \in \ell^2(X, H), \|v\| \le 1, \mathrm{Diam}_X(\mathrm{Supp}(v)) \le f_X(\lambda r)\} \\
&\le c_X \sup\{\|T^v\| : v \in \ell^2(X, H), \|v\| \le 1, \mathrm{Diam}_X(\mathrm{Supp}(v)) \le f_X(\lambda r)\} \\
&\le c_\Gamma c_X \sup\{\|T^v\xi\| : v \in \ell^2(X, H), \|v\| \le 1, \mathrm{Diam}_X(\mathrm{Supp}(v)) \le f_X(\lambda r), \\
&\quad \xi \in \ell^2(\Gamma, H), \|\xi\| \le 1, \mathrm{Diam}_\Gamma(\mathrm{Supp}(\xi)) \le g_R(r)\}.
\end{aligned}$$

Let $f_\Gamma(r) = g_R(r)$. Since $\|T^v\xi\| \le \|T\xi\|$ for each v and ξ, we obtain

$$\|T\| \le c_\Gamma c_X \sup\{\|T\xi\| : \xi \in \ell^2(\Gamma, H), \|\xi\| \le 1, \mathrm{Diam}_\Gamma(\mathrm{Supp}(\xi)) \le f_\Gamma(r)\}.$$

Then Γ has the operator norm localization property with the constant $c = c_\Gamma c_X$.

Remark 2.1 From the proof of Proposition 2.5, it is easy to see that when Γ has the bounded geometry, the assumption that Γ is finitely generated in Proposition 2.5 can be replaced by the condition that if the map $\pi : \Gamma \to X$ satisfies for every $R > 0$ and all $g_1, g_2 \in \Gamma$ with $d(g_1, g_2) \le R$, then there exists an $R' > 0$ such that $d(\pi(g_1), \pi(g_2)) \le R'$.

3 Coarse Invariance of the Operator Norm Localization Property

Coarse invariance is a basic property in the coarse geometry. Therefore, it is important to know whether the operator norm localization property is coarsely invariant. Indeed we have the following result.

Theorem 3.1 *Let X and Y be metric spaces with the bounded geometry. If X is coarse equivalent to Y, then X has the operator norm localization property with a constant c if and only if Y has the operator norm localization property with a constant c.*

We need some preparations to prove the above theorem. Recall that the metric space X is proper if every close ball in the metric space is compact.

Lemma 3.1 (see [10]) *Assume that f is a coarse map from a proper metric space X to another metric space Y. Let $H_X = \ell^2(X) \otimes H$, $H_Y = \ell^2(Y) \otimes H$. Then for any $\epsilon > 0$, there exists an isometry $V_f : H_X \to H_Y$, such that* $\mathrm{Supp}(V_f) \subseteq \{(y, x) \in Y \times X : d(f(x), y) \le \epsilon\}$.

Lemma 3.2 *Let X be a metric space with bounded geometry and $\Gamma \subset X$ be an ϵ-net of X. If Γ has the operator norm localization property with a constant c, then X has the operator norm localization property with the constant c.*

Remark 3.1 If we choose Γ and X as in Lemma 3.2, then the isometry can be chosen to be unitary. The observation is due to [3].

Proof of Lemma 3.2 Let $H_\Gamma = \ell^2(\Gamma) \otimes H$ and $H_X = \ell^2(X) \otimes H$. Let $f : \Gamma \to X$ be the inclusion. Then f is a coarse map and V_f can be chosen to be unitary. Then V_f gives a rise to a homomorphism $\mathrm{Ad}(V_f^*)$ from $\mathcal{B}(\ell^2(X) \otimes H)$ to $\mathcal{B}(\ell^2(\Gamma) \otimes H)$ defined by $\mathrm{Ad}(V_f^*)(T) = V_f^* T V_f$. For each $T \in \mathcal{B}(\ell^2(X) \otimes H)$ with $\mathrm{Prop}(T) \le r$, we let $S = \mathrm{Ad}(V_f^*)(T)$. Note that

$$\mathrm{Supp}(V_f^* T V_f) \subseteq \left\{ (y, x) \in \Gamma \times \Gamma : \begin{matrix} \exists x' \in X,\ y' \in X : (x', x) \in \mathrm{Supp}(V_f), \\ (y', x') \in \mathrm{Supp}(T),\ (y, y') \in \mathrm{Supp}(V_f^*) \end{matrix} \right\},$$

which implies that

$$\begin{aligned} \mathrm{Prop}(S) &\le \sup\{d(y, x) : (y, x) \in \mathrm{Supp}(S)\} \\ &\le \sup\{d(y, y') + d(y', x') + d(x', x) : (x', x) \in \mathrm{Supp}(V_f), \\ &\quad (y', x') \in \mathrm{Supp}(T),\ (y, y') \in \mathrm{Supp}(V_f^*)\} \\ &\le 2\epsilon + r. \end{aligned}$$

Let $r' = 2\epsilon + r$. Since Γ has the operator norm localization property with a constant c, there is a non-decreasing function $f_\Gamma : \mathbb{N} \to \mathbb{N}$, such that

$$\|S\| \le c \sup\{\|S\xi\| : \xi \in \ell^2(\Gamma, H), \mathrm{Diam}_\Gamma(\mathrm{Supp}(\xi)) \le f_\Gamma(r')\}.$$

Note that

$$\|S\|^2 = \|V_f^* T V_f\|^2 = \sup_{\|\xi\|\le 1} \|V_f^* T V_f \xi\|^2 = \sup_{\|\xi\|\le 1} \langle TV_f\xi, TV_f\xi\rangle = \|T\|^2.$$

Therefore,

$$\|T\| = \|S\| \le c \sup\{\|TV_f\xi\| : \xi \in \ell^2(\Gamma.H), \mathrm{Diam}(\mathrm{Supp}(V_f\xi)) \le f_\Gamma(r') + 2\epsilon\}.$$

Let $f_X(r) = f_\Gamma(r + 2\epsilon) + 2\epsilon$. Then

$$\|T\| \le c \sup\{\|T\xi\| : \xi \in \ell^2(X, H), \mathrm{Diam}(\mathrm{Supp}(\xi)) \le f_X(r)\}.$$

Hence, X has the operator norm localization property with a constant c.

Remark 3.2 From the proof of Proposition 3.2, we have that if Γ is an ϵ-net of X with the operator norm localization property, then for all $r > 0$, $f_X(r) = f_\Gamma(r + 2\epsilon) + 2\epsilon$ depends only on ϵ and f_Γ.

Proof of Theorem 3.1 Assume that X has the operator norm localization property with a constant c. Since X and Y are coarsely equivalent, there are a constant $r_0 > 0$ and coarse maps $\phi : X \to Y$, $\psi : Y \to X$ such that $d(\phi \circ \psi(y), y) \le r_0$ and $d(\psi \circ \phi(x), x) \le r_0$. Let $Y' = \phi(X)$. Note that $d(\phi \circ \psi(y), y) \le r_0,\ y \in Y$. Then Y' is an r_0-net of Y. By Lemma 3.2, it is sufficient to show that Y' has the operator norm localization property with a constant c.

We claim that $\mathrm{Diam}_X(\phi^{-1}(B(y, r))),\ y \in Y'$, is uniformly bounded, where $B(y, r)$ is a ball in Y' with the center y and the radius r. In fact, since ψ is a coarse map, there is an $R(r) > 0$, such that $\mathrm{Diam}_X(\psi(B(y, r))) \le R(r)$. Furthermore, for all x_1, x_2 in $\phi^{-1}(B(y, r))$, we have

$$\begin{aligned} d_X(x_1, x_2) &\le d_X(x_1, \psi \circ \phi(x_1)) + d_X(\psi \circ \phi(x_1), \psi \circ \phi(x_2)) + d_X(\psi \circ \phi(x_2), x_2) \\ &\le R(r) + 2r_0. \end{aligned}$$

So $\mathrm{Diam}_X(\phi^{-1}(B(y, r))) \le R(r) + 2r_0$ for all $y \in Y'$.

Let X' be the maximal subset of X, such that $\phi|_{X'}$ is injective. Then $\phi : X' \to Y'$ is a one to one map. Hence, for any operator T in $\mathcal{B}(\ell^2(Y') \otimes H)$ with $\mathrm{Prop}(T) \le r$, one can define an operator $S \in \mathcal{B}(\ell^2(X') \otimes H)$ by $S_{x_1,x_2} = T_{\phi(x_1),\phi(x_2)},\ \forall x_1, x_2 \in X'$. Let $r_1 = R(r) + 2r_0$. Note that $\|S\| = \|T\|$, $\mathrm{Prop}(S) \le r_1$, and X' is a subset of X with the operator norm localization property. Then we obtain

$$\|T\| = \|S\| \le c \sup\{\|Sv\| : \|v\| \le 1, v \in \ell^2(X', H), \mathrm{Diam}_X(\mathrm{Supp}(v)) \le f_X(r_1)\}.$$

For each v, we define $\xi^v = (\xi^v_y) \in \ell^2(Y', H)$ with $\xi^v_y = v_{\phi^{-1}(y)},\ y \in Y'$. Note that $\|Sv\| = \|T\xi^v\|$ and ϕ is a coarse map. Then there exists a $\kappa(f_X(r_1)) > 0$, such that $\mathrm{Diam}_{Y'}(\mathrm{Supp}(\xi^v)) \le \kappa(f_X(r_1))$. Let $f_{Y'}(r) = \kappa(f_X(R(r) + 2r_0))$. Hence, we have

$$\|T\| \le c \sup\{\|T\xi\| : \|\xi\| \le 1, \xi \in \ell^2(Y', H), \mathrm{Diam}_{Y'}(\mathrm{Supp}(\xi)) \le f_{Y'}(r)\}.$$

Therefore, Y' has the operator norm localization property.

4 Some Examples

In this section, we will give some examples to show that there are many metric spaces with the operator norm localization property.

Definition 4.1 (see [4, Definition 2.1]) *A discrete, bounded geometry metric space X is called a simple core, if for any $R > 0$, there is a compact subset $K \subset X$ such that $d(x, y) > R$ whenever $(x, y) \in X \times X \backslash K \times K$.*

Proposition 4.1 *Simple cores have the operator norm localization property with a constant at most* 2.

Proof It is easy to check by the definitions of the simple core and the operator norm localization property.

Recall that two sets U_1, U_2 in a metric space are called d-disjoint if they are at least d-apart, i.e., $\inf\{d(x_1, x_2) : x_1 \in U_1, x_2 \in U_2\} \geq d$.

Definition 4.2 (see [2]) *We call that the metric space X has an asymptotic dimension less than n, if for any number $d > 0$, one can find $n+1$ uniformly bounded families $\mathcal{U}^0, \mathcal{U}^1, \cdots, \mathcal{U}^n$ of d-disjoint sets in X, such that $\cup\mathcal{U}^i$ is a cover of X. We denote it* $\operatorname{asym}\dim X \leq n$.

Note that simple cores have finite asymptotic dimension. In general, the metric spaces with finite asymptotic dimensions also have the operator norm localization property.

Proposition 4.2 (see [9]) *If Γ has finite asymptotic dimension,* $\operatorname{asym}\dim \Gamma \leq n$, *then Γ has the operator norm localization property with constant $n+1$.*

For the readers convenience, we provide the following proof.

Proof For every $r > 0$, let $d = 10r$. Since $\operatorname{asym}\dim \Gamma \leq n$, there are R-bounded families $\mathcal{U}^0, \mathcal{U}^1, \cdots, \mathcal{U}^n$ of d-disjoint sets in Γ, such that $\bigcup\limits_i \mathcal{U}^i$ is a cover of Γ. For each operator T in $\mathcal{B}(\ell^2(\Gamma) \otimes H)$ with $\operatorname{Prop}(T) \leq r$, we define operators $T^i \in \mathcal{B}(\ell^2(X) \otimes H)$ by

$$T^i_{x,y} = \begin{cases} T_{x,y}, & x \in A \in \mathcal{V}^i, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathcal{V}^i = \{A - B_i : A \in \mathcal{U}^i\}$, $B_0 = \emptyset$ and $B_i = \cup\{B : B \in \mathcal{U}^j, 0 \leq j < i\}$ $(i \leq n)$. Then $T = \sum\limits_{i=0}^{n} T^i$. Since $d(U, V) > d$ for all $V, U \in \mathcal{U}^i$, we have

$$\|T^i\| = \sup\{\|T^i\xi\| : \|\xi\| \leq 1, \operatorname{Diam}(\operatorname{Supp}(\xi)) \leq R + 2r\}.$$

Let $f(r) = R + 2r$. Then we get

$$\begin{aligned} \|T\| &\leq \sum_{i=0}^{n} \|T^i\| \leq (n+1) \max_i\{\|T^i\|\} \\ &\leq (n+1) \sup\{\|T^i\xi\| : \|\xi\| \leq 1, \operatorname{Diam}(\operatorname{Supp}(\xi)) \leq f(r)\} \\ &\leq (n+1) \sup\{\|T\xi\| : \|\xi\| \leq 1, \operatorname{Diam}(\operatorname{Supp}(\xi)) \leq f(r)\}. \end{aligned}$$

So Γ has the operator norm localization property with the constant $n+1$.

Proposition 4.3 *Let X and Y be discrete metric spaces with the bounded geometry. If X has a finite asymptotic dimension and Y has the operator norm localization property, then $X \times Y$ has the operator norm localization property.*

Proof Let d_X and d_Y be metrics on X and Y, respectively. Since any two metrics d and d' on $X \times Y$ with $d((x_1, y), (x_2, y)) = d'((x_1, y), (x_2, y)) = d_X(x_1, x_2)$ and $d((x, y_1), (x, y_2)) = d'((x, y_1), (x, y_2)) = d_Y(y_1, y_2)$ for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ are equivalent, we equip $X \times Y$ with the ℓ^1 metric, i.e., $d(x, y) = d_X(x_1, x_2) + d_Y(y_1, y_2)$, where $x = (x_1, y_1)$ and $y = (x_2, y_2)$ both in $X \times Y$.

Assume $\operatorname{asym\,dim}(X) \le n$. For every $r > 0$, we let $d = 10r$. Then there are R-bounded families $\mathcal{U}^0, \cdots, \mathcal{U}^n$ of d-disjoint sets in X, such that $\bigcup_{j=1}^{n} \mathcal{U}^j$ is a cover of X. Let π be the projection from $X \times Y$ to the first coordinate. For each operator T in $\mathcal{B}(\ell^2(X \times Y) \otimes H)$ with $\operatorname{Prop}(T) \le r$, we define operators $T^i \in \mathcal{B}(\ell^2(X \times Y) \otimes H)$ by

$$T^i_{x,y} = \begin{cases} T_{x,y}, & \pi(x) \in A \in \mathcal{V}^i, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathcal{V}^i = \{A - B_i : A \in \mathcal{U}^i\}$, $B_0 = \emptyset$ and $B_i = \cup\{B : B \in \mathcal{U}^j, 0 \le j < i\}$ $(i \le n)$. Then $\|T\| = \Big\| \sum_{i=0}^{n} T^i \Big\| \le (n+1) \max_{0 \le i \le n} \{\|T^i\|\}$. For all $0 \le i \le n$ and every $A \in \mathcal{U}^i$, let

$$T^{i,A}_{x,y} = \begin{cases} T^i_{x,y}, & \text{if } \pi(x) \in A, \\ 0, & \text{otherwise}, \end{cases}$$

which defines an operator $T^{i,A} = (T^{i,A}_{x,y}) \in \mathcal{B}(\ell^2(X \times Y) \otimes H)$. Then we have $\|T^i\| = \sup\{\|T^{i,A}\| : A \in \mathcal{U}^i\}$. Note that Y has the operator norm localization property and Y is the R-net of $A \times Y$ for all $A \in \mathcal{U}^i$. If we regard Y as a subset of $A \times Y$, by Remark 3.2, there are a $c > 0$ and a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, such that

$$\begin{aligned} \|T^{i,A}\| &\le c \sup\{\|T^{i,A}\xi\| : \|\xi\| \le 1, \xi \in \ell^2(A \times Y, H), \operatorname{Diam}(\operatorname{supp}(\xi)) \le f(r)\} \\ &\le c \sup\{\|T\xi\| : \|\xi\| \le 1, \xi \in \ell^2(X \times Y, H), \operatorname{Diam}(\operatorname{supp}(\xi)) \le f(r)\} \end{aligned}$$

for all $A \in \mathcal{U}^i$ $(1 \le i \le n)$. Therefore,

$$\begin{aligned} \|T\| &\le (n+1) \sup\{\|T^{i,A}\| : A \in \mathcal{U}^i\} \\ &\le (n+1)c \sup\{\|T\xi\| : \xi \in \ell^2(X \times Y, H), \|\xi\| \le 1, \operatorname{Diam}(\operatorname{supp}(\xi)) \le f(r)\}. \end{aligned}$$

Let $c_{X \times Y} = (n+1)c$. The proof is completed.

The class of finitely generated groups with finite asymptotic dimension contains hyperbolic groups, $\mathbb{Z}^n$, free groups with finite generators F_n and so on. So the category of the metric spaces with the operator norm localization property is very large.

5 Union Property

Let Γ_1 and Γ_2 be metric spaces. If both Γ_1 and Γ_2 have the operator norm localization property, a natural question is whether $\Gamma_1 \cup \Gamma_2$ has the operator norm localization property. In this section, we show that the operator norm localization property is preserved by any finite union, certain infinite union and direct limit of the group.

Definition 5.1 *A family of metric spaces $\{\Gamma_\alpha\}_{\alpha\in J}$ is said to have the operator norm localization property uniformly, if there exist a common constant $c \geq 1$ and a common (non-decreasing) function $f : \mathbb{N} \to \mathbb{N}$, such that for each $\alpha \in J$, Γ_α has the operator norm localization property relative to f with a constant c.*

Theorem 5.1 *Let $\{\Gamma_\alpha\}_{\alpha\in J}$ be a family of metric spaces which has the operator norm localization property uniformly with a constant c. Let $X = \bigcup_{\alpha\in J} \Gamma_\alpha$. If X has the bounded geometry, and for all $t > 0$, there exists a $Y_t \subset X$, such that $\{\Gamma_\alpha \backslash Y_t\}$ is t-disjoint and Y_t has the operator norm localization property with a constant c, then X has the operator norm localization property with a constant $2c$.*

Proof For every $r > 0$, we let $t = 10r$, such that $\{\Gamma_\alpha \backslash Y_t\}$ is $10r$-disjoint. For all operators $T \in \mathcal{A}_r(X)$ and each $\alpha \in J$, we define an operator $T^\alpha = (T^\alpha_{x,y}) \in \mathcal{B}(\ell^2(X) \otimes H)$ by

$$T^\alpha_{x,y} = \begin{cases} T_{x,y}, & \text{if } x \in \Gamma_\alpha \backslash V_r(Y_t), \\ 0, & \text{otherwise,} \end{cases}$$

where $V_r(Y_t)$ is an r-neighborhood of Y_t. Then $\mathrm{Supp}(T^\alpha) \subset (\Gamma_\alpha \backslash Y_t) \times (\Gamma_\alpha \backslash Y_t)$.

Now let

$$T'_{x,y} = \begin{cases} T_{x,y}, & \text{if } x \in V_r(Y_t), \\ 0, & \text{otherwise,} \end{cases}$$

which defines an operator $T' = (T'_{x,y}) \in \mathcal{B}(\ell^2(X) \otimes H)$. Moreover, $\mathrm{Supp}(T') \subset V_{2r}(Y_t) \times V_{2r}(Y_t)$. Let $S = \sum_{\alpha\in J} T^\alpha$. It is clear that $\mathrm{Prop}(S) \leq r$. Since $\{\Gamma_\alpha\}_{\alpha\in J}$ have the operator norm localization property uniformly, we have

$$\|S\| \leq \sup_{\alpha\in J} \|T^\alpha\| \leq c \sup_{\alpha\in J} \{\|T^\alpha \xi\| : \|\xi\| \leq 1, \xi \in \ell^2(\Gamma_\alpha, H), \mathrm{Diam}(\mathrm{Supp}(\xi)) \leq f(r)\}.$$

By Theorem 3.1, $V_{2r}(Y_t)$ has the operator norm localization property with a constant c. Then there exists a nondecreasing function $f_t : \mathbb{N} \to \mathbb{N}$, such that

$$\|T'\| \leq c \sup\{\|T'\xi\| : \|\xi\| \leq 1, \xi \in \ell^2(V_{2r}(Y_t), H), \mathrm{Diam}(\mathrm{Supp}(\xi)) \leq f_t(r)\}.$$

Let $f_X(r) = \max\{f(r), f_t(r)\}$. Note that $\|T^\alpha \xi\| \leq \|T\xi\|$ and $\|T'\xi\| \leq \|T\xi\|$, $\xi \in \ell^2(X, H)$. Then

$$\begin{aligned} \|T\| &= \|S + T'\| \leq \|S\| + \|T'\| \\ &\leq 2 \max\{\|S\|, \|T'\|\} \\ &\leq 2c \sup\{\|T\xi\| : \xi \in \ell^2(X, H), \|\xi\| \leq 1, \mathrm{Diam}_X(\mathrm{Supp}(\xi)) \leq f_X(r)\}. \end{aligned}$$

Hence, X has the operator norm localization property with constant $2c$.

Corollary 5.1 *Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a discrete metric space with the bounded geometry. If Γ_1 and Γ_2 have the operator norm localization property, then Γ has the operator norm localization property.*

Proof Take the family $\{\Gamma_\alpha\}$ consisting of the sets Γ_1 and Γ_2. For each $t > 0$, put $Y_t = \Gamma_1$. Then it satisfies all the conditions of Theorem 5.1.

Lemma 5.1 *Let Γ be a discrete metric space with the bounded geometry. For all $\epsilon > 0$ and every operator T in $\mathcal{B}(\ell^2(\Gamma) \otimes H)$ with $\mathrm{Prop}(T) \leq r$, there exists an operator $T' \in \mathcal{B}(\ell^2(\Gamma) \otimes H)$, such that*

(1) $\mathrm{Prop}(T') \leq r$;
(2) $\mathrm{Supp}(T')$ *is a bounded subset of* $\Gamma \times \Gamma$;
(3) $\|T'\| \leq \|T\| \leq (1+\epsilon)\|T'\|$ *and* $\|T'\xi\| \leq \|T\xi\|$ *for all* $\xi \in \ell^2(\Gamma, H)$.

Proof For all $0 < \delta_0 < \frac{\|T\|^2}{2}$, we choose a vector $v_0 \in \ell^2(\Gamma, H)$ with $\|v_0\| \leq 1$, such that $\|T\|^2 \leq \|Tv_0\|^2 + \delta_0$. Let $Tv_0 = \xi^{v_0} = (\xi_x^{v_0})_{x\in\Gamma}$. Then, $\|Tv_0\|^2 = \sum_{x\in\Gamma} \|\xi_x^{v_0}\|^2 \leq \|T\|^2$. Therefore, for all $0 < \delta_1 < \frac{\|T\|^2}{2}$, there exists a finite subset F_{v_0} of Γ, such that

$$\sum_{x\in\Gamma} \|\xi_x^{v_0}\|^2 \leq \sum_{x\in F_{v_0}} \|\xi_x^{v_0}\|^2 + \delta_1.$$

Let A_{v_0} be the r-neighborhood of F_{v_0}, and define $T^{v_0} = (T_{x,y}^{v_0}) \in \mathcal{B}(\ell^2(X) \otimes H)$ by

$$T_{x,y}^{v_0} = \begin{cases} T_{x,y}, & \text{if } x \in A_{v_0}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathrm{Prop}(T^{v_0}) \leq r$, $\sum_{x\in F_{v_0}} \|\xi_x^{v_0}\|^2 \leq \|T^{v_0}v_0\|^2$ and $\mathrm{Supp}(T^{v_0})$ is a bounded subset of $\Gamma \times \Gamma$. Let $\delta_{v_0} = \delta_0 + \delta_1$ and choose δ_0, δ_1 small enough, such that

$$\Big(1 + \frac{\delta_{v_0}}{\|T\|^2 - \delta_{v_0}}\Big) \leq (1+\epsilon)^2.$$

Let $T' = T^{v_0}$. Then we obtain

$$\begin{aligned} \|T'\|^2 \leq \|T\|^2 \leq \|Tv_0\|^2 + \delta_0 &\leq \sum_{x\in F_{v_0}} \|\xi_x^{v_0}\|^2 + \delta_0 + \delta_1 \\ &\leq \|T^{v_0}v_0\|^2 + \delta_{v_0} \leq \|T^{v_0}\|^2 + \delta_{v_0} = \Big(1 + \frac{\delta_{v_0}}{\|T^{v_0}\|^2}\Big)\|T^{v_0}\|^2 \\ &\leq \Big(1 + \frac{\delta_{v_0}}{\|T\|^2 - \delta_{v_0}}\Big)\|T^{v_0}\|^2 \leq (1+\epsilon)^2\|T'\|^2. \end{aligned}$$

Hence $\|T'\| \leq \|T\| \leq (1+\epsilon)\|T'\|$, as desired. The proof is now completed.

Theorem 5.2 *Let* Γ *be the limit of a direct system of countable discrete groups*

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3 \xrightarrow{\phi_3} \cdots,$$

in which the maps $G_n \xrightarrow{\phi_n} G_{n+1}$ *are injective. If* $\{G_n\}_{n\in N}$ *has the operator norm localization property uniformly with a constant* c*, then* Γ *has the operator norm localization property with a constant* c.

Proof Let Γ be a direct limit as in the statement of the theorem. Without loss of generality, we assume that G_i is a subgroup of Γ for $i \in \mathbb{N}$. Then $\Gamma = \bigcup_{i\in\mathbb{N}} G_i$. Equip Γ with a proper length function l_Γ and the associated metric d_Γ. Metrize each of the subgroups G_n as subspaces of Γ. The metric and length function on G_n are simply the restrictions of d_Γ and l_Γ. By Lemma 5.1, for any $\epsilon > 0$ and for any operator $T \in \mathcal{B}(\ell^2(\Gamma) \otimes H)$ with $\mathrm{Prop}(T) \leq r$, there exists an operator T' in $\mathcal{B}(\ell^2(\Gamma) \otimes H)$ with $\mathrm{Prop}(T') \leq r$ such that $\mathrm{Supp}(T')$ is a bounded subset of $\Gamma \times \Gamma$. So there exists a $j \in \mathbb{N}$ such that $\mathrm{Supp}(T') \subset G_j \times G_j$. Since $\{G_i\}$ have the operator norm localization property uniformly with a constant c, we have

$$\|T\| \leq (1+\epsilon)\|T'\| \leq c(1+\epsilon)\sup\{\|T'\xi\| : \xi \in \ell^2(G_j, H), \|\xi\| \leq 1, \mathrm{Diam}(\mathrm{Supp}(\xi)) \leq fr\}.$$

Note that $\|T'\xi\| \le \|T\xi\|$ for all $\xi \in \ell^2(\Gamma, H)$. Therefore,

$$\|T\| \le c(1+\epsilon)\sup\{\|T\xi\| : \|\xi\| \le 1,\ \xi \in \ell^2(\Gamma, H),\ \mathrm{Diam}(\mathrm{Supp}(\xi)) \le f(r)\}.$$

So Γ has the operator norm localization property.

Theorem 5.3 *Let Γ be a discrete metric space with bounded geometry. Then the following are equivalent:*

(1) *Γ has the operator norm localization property;*

(2) *$\mathcal{A} = \{F : F \subset \Gamma, \#F < \infty\}$ has the operator norm localization property uniformly.*

Proof It is obvious that (1) implies (2).

Now let us prove that (2) implies (1). For each operator $T \in \mathcal{B}(\ell^2(\Gamma)\otimes H)$ with $\mathrm{Prop}(T) \le r$, by Lemma 5.1, for all $\epsilon > 0$, there is an operator $T' \in \mathcal{B}(\ell^2(\Gamma)\otimes H)$ with $\mathrm{Prop}(T') \le r$, such that $\mathrm{Supp}(T')$ is a bounded subset of $\Gamma \times \Gamma$. By the bounded geometry of Γ, there exists an $F \in \mathcal{A}$ such that $\mathrm{Supp}(T') \subset F \times F$. By the assumption that $\mathcal{A}$ has the operator norm localization property uniformly, there are a constant c and a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, such that

$$\|T\| \le (1+\epsilon)\|T'\| \le c(1+\epsilon)\sup\{\|T'\xi\| : \xi \in \ell^2(\Gamma, H), \|\xi\| \le 1, \mathrm{Diam}(\mathrm{Supp}(\xi)) \le f(r)\}.$$

Note that $\|T\xi\| \ge \|T'\xi\|$ for all $\xi \in \ell^2(\Gamma, H)$. Therefore,

$$\|T\| \le c(1+\epsilon)\sup\{\|T\xi\| : \xi \in \ell^2(\Gamma, H), \|\xi\| \le 1, \mathrm{Diam}(\mathrm{Supp}(\xi)) \le f(r)\}.$$

So Γ has the operator norm localization property. This completes the proof.

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