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Existence of Nontrivial Solutions for p-Laplacian Variational Inclusion Systems in $\mathbb{R}^N$*

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Abstract The authors study the existence of nontrivial solutions to p-Laplacian variational inclusion systems

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \partial_1 F(u,v), & \text{in } \mathbb{R}^N, \\ -\Delta_p v + |v|^{p-2}v \in \partial_2 F(u,v), & \text{in } \mathbb{R}^N, \end{cases}$$

where $N \geq 2,\ 2 \leq p \leq N$ and $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function. Under some growth conditions on F, and by Mountain Pass Theorem and the principle of symmetric criticality, the existence of such solutions is guaranteed.

Keywords Mountain pass theorem, p-Laplacian, Principle of symmetric criticality, Variational inclusion systems, (PS)-condition, Locally Lipschitz functions
2000 MR Subject Classification 35J20, 35J25

1 Introduction

This paper is devoted to the study of existence of nontrivial solutions to a class of variational inclusion systems of the form

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \partial_1 F(u,v), & \text{in } \mathbb{R}^N, \\ -\Delta_p v + |v|^{p-2}v \in \partial_2 F(u,v), & \text{in } \mathbb{R}^N, \\ u, v \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{S}$$

where $N \geq 2,\ 2 \leq p \leq N,\ F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz not necessarily smooth potential function and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We denote by $\partial_1 F(u,v)$ the partial generalized gradient of $F(\,\cdot\,,v)$ at the point u and by $\partial_2 F(u,v)$ that of $F(u,\,\cdot\,)$ at v.

In recent years, nonlinear elliptic systems have been the object of intensive investigations by many authors, for their theoretical and practical importance (see [1, 2]).

Yang and Shen [16] considered a quasilinear elliptic system with Neumann boundary condition, and obtained a multiplicity result by Ricceri's three critical points theorems.

Costa [5] studied a class of semilinear elliptic systems of the form

$$\begin{cases} -\Delta u + a(x)u = F_u(x,u,v), & \text{in } \mathbb{R}^N, \\ -\Delta v + b(x)u = F_v(x,u,v), & \text{in } \mathbb{R}^N. \end{cases} \tag{C}$$

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Due to the coercivity of functions a and b, the subspace $E_{a,b}$ is compactly embedded in $L^2(\mathbb{R}^N, \mathbb{R}^2)$ (see [5]). This makes possible the application of classical minimax theorems with suitable compactness condition. Then weak solutions to (C) in the usual sense are obtained in this manner.

When $p = 2$, A. Kristály [8] considered the following variational inclusion system:

$$\begin{cases} -\Delta u + u \in \partial_1 F(u,v), & \text{in } \mathbb{R}^N, \\ -\Delta u + u \in \partial_2 F(u,v), & \text{in } \mathbb{R}^N \end{cases} \tag{K}$$

with $u, v \in H^1(\mathbb{R}^N)$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function and $\partial_i F(u,v)$ $(i \in \{1,2\})$ are the partial generalized gradients in the sense of Clarke (see [4]). Using the principle of symmetry critically and locally on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ to recover the compactness, we obtain the weak solutions to (K).

The motivation is to investigate the variational inclusion systems arising from the recent works of Kristály [8, 9, 11, 12]. It is interesting to study the existence of weak solutions to (K) when $p > 2$. Here, the potential function $F(u,v)$ is not necessarily smooth, only locally Lipschitz. Consequently, our approach, which is variational, is based on the nonsmooth critical point theory as was developed by Chang [3]. In addition, the compactness is lost due to the unbounded domain, and we will use the principle of Symmetric Critically of Krawcewicz and Marzantowicz [7], and restrict the energy function on a subspace of $W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$ to recover the compact embedding.

The paper is organized as follows. In Section 2, we give the hypotheses on F, the basic notions and preliminary results. In Section 3, we investigate the Palais-Smale and the geometric conditions in Mountain Pass Theorem, and prove the main theorem.

2 The Main Results

As we have already mentioned in Section 1, our approach is variational and is based on the nonsmooth critical point theory of Chang [3]. The tools of the nonsmooth critical point theory come from the subdifferential theory for locally Lipschitz functions due to Clarke [4]. For easy reference, we recall the basic definitions and facts from this theory.

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every bounded set $B \subseteq X$ there exists $L > 0$, such that $|\varphi(x) - \varphi(y)| \le L\|x - y\|$ for all $x, y \in B$. This is a slightly more restrictive version of local Lipschitzness than the one used in the literature where φ is locally Lipschitz, if for every $u \in X$, there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|\varphi(x) - \varphi(y)| \le L_u \|x - y\| \quad \text{for all } x, y \in U.$$

We define the generalized directional derivative of a locally Lipschitz function ϕ at $x \in X$ in the direction $h \in X$ by

$$\phi^\circ(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\phi(x' + \lambda h) - \phi(x')}{\lambda}.$$

It is easy to see that the function $h \mapsto \phi^\circ(x; h)$ is sublinear and continuous on X, so by Hahn-Banach Theorem, it is the support function of a nonempty, convex, w^*-compact set

$$\partial\phi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \phi^\circ(x; h) \text{ for all } h \in X\}.$$

The multifunction $x \to \partial\phi(x)$ is known as the generalized (or Clarke) subdifferential of ϕ. Now, we list some fundamental properties of the generalized grandient and directional derivative which will be used through the paper.

Lemma 2.1 (i) $(-\phi)^{\circ}(u;z) = \phi^{\circ}(u;-z)$ *for all* $u, z \in X$;

(ii) $\phi^{\circ}(u;z) = \max\{\langle x^*, z\rangle : x^* \in \partial\phi(u)\}$ *for all* $u, z \in X$;

(iii) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$, $j^{\circ}(u;z) = \langle j'(u), z\rangle$, *and* $(\phi + j)^{\circ}(u;z) = \phi^{\circ}(u;z) + \langle j'(u), z\rangle$ *for all* $u, z \in X$;

(iv) (Leborug's Mean Value Theorem) *Let* u *and* v *be two points in* X. *Then there exists a point* w *in the open segment between* u *and* v, *and* $x_w^* \in \partial\phi(w)$ *such that*

$$\phi(u) - \phi(v) = \langle x_w^*, u - v\rangle;$$

(v) (Second Chain Rule) *Let* Y *be a Banach space and* $j : Y \to X$ *a continuously differentiable function. Then* $\phi \circ j$ *is locally Lipschitz and*

$$\partial(\phi \circ j)(y) \subseteq \partial\phi(j(y)) \circ j'(y) \quad \text{for all } y \in Y;$$

(vi) *If* $\phi, \psi : X \to \mathbb{R}$ *are locally Lipschitz, then* $\partial(\phi + \psi)(x) \subseteq \partial\phi(x) + \partial\psi(x)$ *for all* $x \in X$.

A point $u \in X$ is a critical point of ϕ if $0 \in \partial\phi(u)$, i.e., $\phi^{\circ}(u;w) \geq 0$ for all $w \in X$. In this case, $\phi(u)$ is a critical value of ϕ. We define $\lambda_\phi(u) = \inf\{\|x^*\|_{X^*} : x^* \in \partial\phi(u)\}$. Of course, this infimum is attained, since $\partial\phi(u)$ is w^*-compact.

The function ϕ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (denoted $(\mathrm{PS})_c$), if every sequence $\{x_n\} \subseteq X$ such that $\phi(x_n) \to c$ and $\lambda_\phi(x_n) \to 0$ contains a convergent subsequence in X.

The function ϕ satisfies the Cerami condition at level $c \in \mathbb{R}$ (denoted $(\mathrm{C})_c$), if every sequence $\{x_n\} \subseteq X$ such that $\phi(x_n) \to c$ and $(1 + \|x_n\|)\lambda_\phi(x_n) \to 0$ contains a convergent subsequence in X. It is clear that $(\mathrm{PS})_c$ implies $(\mathrm{C})_c$.

We say that ϕ is regular at $u \in X$ (in the sense of [4]) if for all $z \in X$ the usual one-sided directional derivative

$$\phi'(u;z) = \lim_{t \to 0^+} \frac{\phi(u + tz) - \phi(u)}{t}$$

exists and $\phi'(u;z) = \phi^{\circ}(u;z)$. ϕ is regular on X, if it is regular in every point $u \in X$.

Lemma 2.2 *Let* $\phi : X \times X \to \mathbb{R}$ *be a locally Lipschitz function which is regular at* $(u, v) \in X \times X$. *Then*

(i) $\partial\phi(u,v) \subseteq \partial_1\phi(u,v) \times \partial_2(u,v)$, *where* $\partial_1\phi(u,v)$ *denotes the partial generalized gradient of* $\phi(\,\cdot\,, v)$ *at* u, *and* $\partial_2\phi(u,v)$ *that of* $\phi(u, \,\cdot\,)$ *at* v;

(ii) $\phi^{\circ}(u, v; w, z) \leq \phi_1^{\circ}(u, v; w) + \phi_2^{\circ}(u, v; z)$ *for all* $w, z \in X$.

Definition 2.1 *We say that* (u, v) *is a weak solution to* (S), *if* $(u, v) \in W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$ *satisfies the following hemivariational inequality systems* (HIS):

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla w + |u|^{p-2}uw)\mathrm{d}x + \int_{\mathbb{R}^N} F_1^{\circ}(u(x), v(x); -w(x))\mathrm{d}x \geq 0 \quad \text{for all } w \in W^{1,p}(\mathbb{R}^N);$$

$$\int_{\mathbb{R}^N} (|\nabla v|^{p-2}\nabla v \cdot \nabla y + |v|^{p-2}vy)\mathrm{d}x + \int_{\mathbb{R}^N} F_2^{\circ}(u(x), v(x); -y(x))\mathrm{d}x \geq 0 \quad \text{for all } y \in W^{1,p}(\mathbb{R}^N).$$

Here $F_1^{\circ}(u, v; -w)$ *denotes the partial generalized directional derivative of* $F(\,\cdot\,, v)$ *at the point* $u \in \mathbb{R}$ *in the direction* $w \in \mathbb{R}$, *and* $F_2^{\circ}(u, v; -w)$ *is defined in a similar way.*

Throughout the paper, we will make the following assumptions on F:

(F^1) There exist $c_1 > 0$ and $q \in (p, p^*)$ such that

$$|w_1| + |w_2| \leq c_1(|u|^{p-1} + |v|^{p-1} + |u|^{q-1} + |v|^{q-1}) \tag{2.1}$$

for all $(u,v)\in\mathbb{R}^2$ and $w_i\in\partial_i F(u,v)$, $i\in\{1,2\}$;

(F^2) F is regular on $\mathbb{R}^2$;

(F^3) $\displaystyle\lim_{|u|+|v|\to 0}\frac{\max\{\partial_1 F(u,v),\partial_2 F(u,v)\}}{|u|^{p-1}+|v|^{p-1}}=0;$

(F^4) There exists an $\alpha>p$ such that

$$\alpha F(u,v)+F_1^\circ(u,v;-u)+F_2^\circ(u,v;-v)\le 0\quad\text{for all } (u,v)\in\mathbb{R};$$

(F^5) $F\ge 0$ and $F(u,v)>0$ for all $(u,v)\ne 0$.

Theorem 2.1 *If $F:\mathbb{R}^2\to\mathbb{R}$ is a locally Lipschitz function satisfying (F^1)–(F^5), then system* (S) *possesses at least one nontrivial weak solution.*

Next we collect some preliminary results and investigate the Palais-Smale and the geometric conditions from Mountain Pass Theorem for an appropriate function.

Lemma 2.3 *Suppose that $F:\mathbb{R}^2\to\mathbb{R}$ is a locally Lipschitz function which satisfies (F^1) and (F^2). Then $\mathcal{F}:W^{1,p}(\mathbb{R}^N)\times W^{1,p}(\mathbb{R}^N)\to\mathbb{R}$ defined by*

$$\mathcal{F}(u,v)=\int_{\mathbb{R}^N}F(u(x),v(x))\mathrm{d}x$$

is locally Lipschitz. Moreover,

$$\mathcal{F}^\circ(u,v;w,y)\le\int_{\mathbb{R}^N}F^\circ(u(x),v(x);w(x),y(x))\mathrm{d}x\quad\textit{for all } u,v,w,y\in X=W^{1,p}(\mathbb{R}^N).$$

Proof First, let $(u_1,u_2),(u_3,u_4)\in\mathbb{R}^2$ be fixed elements. Applying Lebourg's mean value theorem, we obtain a $w\in\partial F(\xi,\eta)$ such that

$$F(u_1,u_2)-F(u_3,u_4)=\langle w,(u_1-u_3,u_2-u_4)\rangle,\tag{2.2}$$

where (ξ,η) is in the open line segment between (u_1,u_2) and (u_3,u_4). By using the regularity of F at (ξ,η) and Lemma 2.2(i), we see that there exist $w_i\in\partial_i F(\xi,\eta)$ such that

$$F(u_1,u_2)-F(u_3,u_4)=w_1(u_1-u_3)+w_2(u_2-u_4).\tag{2.3}$$

From relations (2.1) and (2.3), we obtain

$$|F(u_1,u_2)-F(u_3,u_4)|\le c_3\sum_1^4(|u_i|^{p-1}+|u_i|^{q-1})(|u_1-u_3|+|u_2-u_4|).\tag{2.4}$$

Using Hölder inequality and the embedding $W^{1,p}(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N)$, we have

$$\begin{aligned}|\mathcal{F}(u_1,u_2)-\mathcal{F}(u_3,u_4)|&\le c_3\sum_1^4(\|u_i\|_{1,p}^{p-1}+c_{1,p}^q\|u_i\|_{1,p}^{q-1})(\|u_1-u_3\|_{1,p}+\|u_2-u_4\|_{1,p})\\&=c_3\sum_1^4(\|u_i\|_{1,p}^{p-1}+c_{1,p}^q\|u_i\|_{1,p}^{q-1})\|(u_1-u_3,u_2-u_4)\|_{1,p}.\end{aligned}$$

From this relation, it follows that $\mathcal{F}$ is locally Lipschitz on $W^{1,p}(\mathbb{R}^N)$.

Since F is continuous, $F^\circ(u(x),v(x);w(x),y(x))$ can be expressed as the upper limit of

$$\frac{F(z^1+tw,z^2+ty)-F(z^1,z^2)}{t},$$

where $t \to 0$ and $(z^1, z^2) \to (u, v)$.

$W^{1,p}(\mathbb{R}^N)$ being a Banach space, there exist functions $z_n^1, z_n^2 \in W^{1,p}(\mathbb{R}^N)$ and numbers $t_n \to 0^+$ such that

$$(z_n^1, z_n^2) \to (u, v), \quad \text{in } W^{1,p}(\mathbb{R}^N)$$

and

$$\mathcal{F}^\circ(u, v; w, y) = \lim_{n\to\infty} \frac{\mathcal{F}(z_n^1 + t_n w, z_n^2 + t_n y) - \mathcal{F}(z_n^1, z_n^2)}{t_n}.$$

We define $g_n : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned} g_n(x) = &-\frac{F(z_n^1(x) + t_n w(x), z_n^2(x) + t_n y(x)) - F(z_n^1(x), z_n^2(x))}{t_n} + c_3(|w(x)| + |y(x)|) \\ &\times [|z_n^1(x)|^{p-1} + |z_n^2(x)|^{p-1} + |z_n^1 + t_n w(x)|^{p-1} + |z_n^2 + t_n y(x)|^{p-1} + |z_n^1(x)|^{q-1} \\ &+ |z_n^2(x)|^{q-1} + |z_n^1 + t_n w(x)|^{q-1} + |z_n^2 + t_n y(x)|^{q-1}]. \end{aligned}$$

The function g_n is measurable, and due to (3.1) in the next section, it is non-negative. From Fatou's lemma, we have

$$A = \int_{\mathbb{R}^N} \limsup_{n\to\infty} [-g_n(x)] \mathrm{d}x \geq \limsup_{n\to\infty} \int_{\mathbb{R}^N} [-g_n(x)] \mathrm{d}x = B.$$

Let $g_n = -C_n + D_n$, where

$$C_n = -\frac{F(z_n^1(x) + t_n w(x), z_n^2(x) + t_n y(x)) - F(z_n^1(x), z_n^2(x))}{t_n}$$

and

$$\begin{aligned} D_n(x) = &\, c_3(|w(x)| + |y(x)|)[|z_n^1(x)|^{p-1} + |z_n^2(x)|^{p-1} + |z_n^1 + t_n w(x)|^{p-1} + |z_n^2 + t_n y(x)|^{p-1} \\ &+ |z_n^1(x)|^{q-1} + |z_n^2(x)|^{q-1} + |z_n^1 + t_n w(x)|^{q-1} + |z_n^2 + t_n y(x)|^{q-1}]. \end{aligned}$$

Let $d_n = \int_{\mathbb{R}^N} D_n \mathrm{d}x$. Then we obtain the following estimation:

$$\begin{aligned} &\left| d_n - 2c_3 \int_{\mathbb{R}^N} (|w| + |y|)(|u|^{p-1} + |v|^{p-1} + |u|^{q-1} + |v|^{q-1}) \mathrm{d}x \right| \\ &\leq c_3\{(p-1)2^{p-1}(\|w\|_{1,p} + \|y\|_{1,p})(2\|z_n^1 - u\|_{1,p} + t_n\|w\|_{1,p} + 2\|z_n^2 - v\|_{1,p} + t_n\|y\|_{1,p}) \\ &\quad + (q-1)2^{q-2}(\|w\|_q + \|y\|_q)[\|z_n^1 - u\|_q(\|z_n^1\|_q^{q-2} + \|u\|_q^{q-2}) + (\|z_n^1 - u\|_q + t_n\|w\|_q) \\ &\quad \times ((\|z_n^1\|_p + t_n\|w\|_p)^{p-2} + \|u\|_p^{p-2}) + \|z_n^2 - v\|_q(\|z_n^2\|_q^{q-2} + \|v\|_q^{q-2}) \\ &\quad + (\|z_n^2 - v\|_q + t_n\|y\|_q) \times ((\|z_n^2\|_q + t_n\|y\|_q)^{q-2} + \|v\|_q^{q-2})]\}. \end{aligned}$$

Since the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ $(s \in (p, p^*))$ is continuous, $\|z_n^1 - u\|_{1,p} \to 0$, $\|z_n^2 - v\|_{1,p} \to 0$ and $t_n \to 0^+$, we obtain that the sequence d_n is convergent, with its limit being

$$\lim_{n\to\infty} d_n = 2c_3 \int_{\mathbb{R}^N} (|w| + |y|)(|u|^{p-1} + |v|^{p-1} + |u|^{q-1} + |v|^{q-1}) \mathrm{d}x.$$

We have

$$\begin{aligned} B &= \lim_{n\to\infty} \sup \int_{\mathbb{R}^N} [-g_n(x)] \mathrm{d}x \\ &= \lim_{n\to\infty} \sup \frac{\mathcal{F}(z_n^1 + t_n w, z_n^2 + t_n y) - \mathcal{F}(z_n^1, z_n^2)}{t_n} - \lim_{n\to\infty} d_n \\ &= \mathcal{F}^\circ(u, v; w, y) - \lim_{n\to\infty} d_n. \end{aligned}$$

On the other hand, $A \le A_1 - A_2$, where

$$A_1 = \int_{\mathbb{R}^N} \limsup_{n\to\infty} C_n(x)\mathrm{d}x, \quad A_2 = \int_{\mathbb{R}^N} \liminf_{n\to\infty} D_n(x)\mathrm{d}x = \lim_{n\to\infty} d_n.$$

Then

$$\begin{aligned} A_1 &= \int_{\mathbb{R}^N} \limsup_{n\to\infty} \frac{F(z_n^1 + t_n w, z_n^2 + t_n y) - F(z_n^1, z_n^2)}{t_n}\mathrm{d}x \\ &\le \int_{\mathbb{R}^N} \limsup_{\substack{(z^1,z^2)\to(u,v)\\ t\to 0^+}} \frac{F(z^1 + tw, z^2 + ty) - F(z^1, z^2)}{t_n}\mathrm{d}x \\ &= \int_{\mathbb{R}^N} F^\circ(u(x), v(x); w(x), y(x))\mathrm{d}x. \end{aligned}$$

This completes the proof.

Since $W^{1,p}(\mathbb{R}^N) \ni u \mapsto \frac{1}{p}\|u\|_{1,p}^p$ is of class C^1, the energy function $\mathcal{I}$ defined by

$$\mathcal{I}(u,v) = \frac{1}{p}\|u\|_E^p + \frac{1}{p}\|v\|_E^p - \int_{\mathbb{R}^N} F(u,v)\mathrm{d}x$$

is locally Lipschitz on E. Now we are in the position to establish the following.

Lemma 2.4 *If function F satisfies (F^1) and (F^2), then every critical point $(u,v) \in E$ of $\mathcal{I}$ is a weak solution to* (S).

Proof Since (u,v) is a critical point of $\mathcal{I}$ for every $w, y \in W^{1,p}(\mathbb{R}^N)$, applying Lemma 2.3, we have

$$\begin{aligned} 0 &\le \mathcal{I}^\circ(u,v;w,y) \\ &= \langle |\nabla u|^{p-2}\nabla u \cdot \nabla + |u|^{p-2}u, w\rangle + \langle |\nabla v|^{p-2}\nabla v \cdot \nabla + |v|^{p-2}v, y\rangle + (-\mathcal{I})^\circ(u,v;w,y) \\ &= \langle |\nabla u|^{p-2}\nabla u \cdot \nabla + |u|^{p-2}u, w\rangle + \langle |\nabla v|^{p-2}\nabla v \cdot \nabla + |v|^{p-2}v, y\rangle + (\mathcal{I})^\circ(u,v;-w,-y) \\ &\le \langle |\nabla u|^{p-2}\nabla u \cdot \nabla + |u|^{p-2}u, w\rangle + \langle |\nabla v|^{p-2}\nabla v \cdot \nabla + |v|^{p-2}v, y\rangle \\ &\quad + \int_{\mathbb{R}^N} F^\circ(u(x), v(x); -w(x), -y(x))\mathrm{d}x. \end{aligned}$$

By the regularity of F, we obtain

$$\begin{aligned} 0 \le \mathcal{I}^\circ(u,v;w,y) &= \langle |\nabla u|^{p-2}\nabla u \cdot \nabla + |u|^{p-2}u, w\rangle + \langle |\nabla v|^{p-2}\nabla v \cdot \nabla + |v|^{p-2}v, y\rangle \\ &\quad + \int_{\mathbb{R}^N} F_1^\circ(u(x), v(x); -w(x))\mathrm{d}x + \int_{\mathbb{R}^N} F_2^\circ(u(x), v(x); -y(x))\mathrm{d}x. \end{aligned}$$

Taking $y = 0$, respectively $w = 0$, in the above inequality, we are led to the required inequalities from (HIS), i.e., (u,v) is a weak solution to (S).

The compactness is lost due to the unbounded domain, and it suffices to consider invariant functions in order to recover compactness. Now we recall some notions which will be used in Section 3. Let G be a compact Lie group which acts linear isometrically on the real Banach space $(X, \|\cdot\|)$, i.e., the action $G \times X \to X : [g,u] \mapsto gu$ is continuous, $1 \cdot u = u$, $(g_1g_2)u = g_1(g_2u)$ for every $g_1, g_2 \in G$, and the map $u \mapsto gu$ is linear such that $\|gu\| = \|u\|$ for every $g \in G$ and $u \in X$.

A function $h : X \to \mathbb{R}$ is G-invariant if $h(gu) = h(u)$ for all $g \in G$, $u \in X$. The action on X induces an action of the same type on the dual space X^*, defined by $(gx^*)(u) = x^*(gu)$ for all $g \in G,\ u \in X$ and $x^* \in X^*$. We have $\|gx^*\| = \|x^*\|$ for all $g \in G,\ x^* \in X^*$. Suppose that $h(gu) = h(u)$ is a G-invariant, locally Lipschitz function. Then $g\partial h(u) = \partial h(gu) = \partial h(u)$ for all $g \in G,\ u \in X$. Therefore, the function $u \mapsto \lambda_h(u)$ is G-invariant. Let

$$X^G = \{u \in X : gu = u \text{ for all } g \in G\}.$$

We recall the Principle of Symmetric Criticality of Krawcewicz and Marzantowicz [7, p. 1045], which will be crucial in the proof of our theorems.

Lemma 2.5 *Assume that a compact Lie group G acts linearly isometrically on a Banach space X. If $h : X \to \mathbb{R}$ is a G-invariant, locally Lipschitz function and if $u \in X^G$ is a critical point of h restricted to X^G, then u is a critical point of h.*

Let E be the Cartesian product $W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$ with the norm $\|(u,v)\|_E^p = \|u\|_{1,p}^p + \|v\|_{1,p}^p,\ u,v \in W^{1,p}(\mathbb{R}^N)$. Then G acts linearly isometrically on E, where the action $G \times E \mapsto E$ is defined by

$$g(u,v) = (gu, gv)$$

for all $g \in G$ and $u,v \in W^{1,p}(\mathbb{R}^N)$. Moreover,

$$\begin{aligned} E^G &= (W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N))^G = \{(u,v) \in E : g(u,v) = (u,v) \text{ for all } g \in G\} \\ &= W^{1,p}(\mathbb{R}^N)^G \times W^{1,p}(\mathbb{R}^N)^G. \end{aligned}$$

Theorem 2.2 *Let X be a Banach space, and $h : X \to \mathbb{R}$ be a locally Lipschitz function with $h(0) = 0$. Suppose that there exist a point $e \in X$ and constants $\rho, \eta > 0$ such that*

(i) *$h(u) \geq \eta$ for all $u \in X$ with $\|u\| = \rho$;*
(ii) *$\|e\| > \rho$ and $h(e) \leq 0$;*
(iii) *h satisfies $(\mathrm{PS})_c$ with*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} h(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

Then $c \geq \eta$ and $c \in \mathbb{R}$ is a critical value of h.

3 Proof of the Main Theorem

In this section, we study the Palais-Smale condition for $\mathcal{I}$ which will be restricted to a certain subspace of $W^{1,p}(\mathbb{R}^N)$, investigate the geometric conditions of Mountain Pass Theorem, and prove the main theorem.

In order to study the Palais-Smale condition for $\mathcal{I}$, we need the following estimation first:

$$|F(u_1,u_2) - F(u_3,u_4)| \leq \Big(\varepsilon \sum_1^4 |u_i|^{p-1} + c_\varepsilon \sum_1^4 |u_i|^{q-1}\Big)(|u_1 - u_3| + |u_2 - u_4|) \tag{3.1}$$

for all $u_i \in \mathbb{R}$ $(i \in \{1,2,3,4\})$.

Indeed, (F^3) implies that for all $\varepsilon \geq 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|w_1| + |w_2| \leq \varepsilon(|u|^{p-1} + |v|^{p-1}) \tag{3.2}$$

for all $w_i \in \partial_i F(u,v)$ $(i \in \{1,2\})$ with $|u| + |v| < \delta$.

On the other hand, if $|u| + |v| \geq \delta$, we have

$$\frac{(|u| + |v|)^{q-p}}{\delta^{q-p}} \geq 1.$$

From (F_1), we obtain

$$|w_1| + |w_2| \le c_1(2^{q-p}\delta^{p-q} + 1)(|u|^{q-1} + |v|^{q-1}).$$

Combining the above estimation with (3.2), we have

$$|w_1| + |w_2| \le \varepsilon(|u|^{p-1} + |v|^{p-1}) + c_\varepsilon(|u|^{q-1} + |v|^{q-1}), \tag{3.3}$$

where $c_\varepsilon = c_1(2^{q-p}\delta^{p-q} + 1)$. By Lemma 2.1(iv), we are led to (3.1).

Lemma 3.1 *Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function which satisfies (F^1)–(F^2). Let E be a closed subspace of $W^{1,p}(\mathbb{R}^N)$ which is compactly embedding in $L^p(\mathbb{R}^N)$, and denote by $\mathcal{I}_E$ the restriction of $\mathcal{I}$ to $E \times E$. Then $\mathcal{I}$ satisfies $(\mathrm{PS})_c$ for all $c > 0$.*

Proof Let $\{(u_n, v_n)\}$ be a sequence from $W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$, such that

$$\mathcal{I}(u_n, v_n) \to c > 0, \tag{3.4}$$

$$\lambda_{\mathcal{I}}(u_n, v_n) \to 0, \tag{3.5}$$

as $n \to \infty$. Now we prove that $\{(u_n, v_n)\}$ is bounded in $W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$.

For every $n \in N$, there exists $z_n^* \in \partial\mathcal{I}(u_n, v_n)$ such that $\|z_n^*\| + \lambda_{\mathcal{I}}(u_n, v_n)$. Clearly, (3.5) implies that

$$\mathcal{I}^\circ(u_n, v_n; u_n, v_n) \ge \langle z_n^*, (u_n, v_n)\rangle \ge -\|z_n^*\|\|(u_n, v_n)\| \ge -\alpha\|(u_n, v_n)\|$$

for n large enough.

Using Lemma 2.3, the above estimation, (3.4) and (F^4), we get that for n large enough,

$$\begin{aligned}
&c + 1 + \|(u_n, v_n)\|_E \\
&\ge \mathcal{I}_E(u_n, v_n) - \frac{1}{\alpha}\mathcal{I}_E^\circ(u_n, v_n; u_n, v_n) \\
&\ge \frac{1}{p}(\|u_n\|_E^p + \|v_n\|_E^p) - \mathcal{F}_E(u_n, v_n) - \frac{1}{\alpha}[\|u_n\|_E^p + \|v_n\|_E^p + (-\mathcal{I}_E^\circ(u_n, v_n; u_n, v_n))] \\
&= \Big(\frac{1}{p} - \frac{1}{\alpha}\Big)\|(u_n, v_n)\|_{E\times E}^p - \mathcal{F}_E(u_n, v_n) - \frac{1}{\alpha}\mathcal{F}_E^\circ(u_n, v_n; -u_n, -v_n) \\
&\ge \Big(\frac{1}{p} - \frac{1}{\alpha}\Big)\|(u_n, v_n)\|_{E\times E}^p - \int_{\mathbb{R}^N}\Big[F(u_n(x), v_n(x)) + \frac{1}{\alpha}F^\circ(u_n(x), v_n(x); -u_n(x), -v_n(x))\Big]\mathrm{d}x \\
&\ge \Big(\frac{1}{p} - \frac{1}{\alpha}\Big)\|(u_n, v_n)\|_{E\times E}^p - \int_{\mathbb{R}^N}\Big[F(u_n(x), v_n(x)) + \frac{1}{\alpha}(F_1^\circ(u_n(x), v_n(x); -u_n(x)) \\
&\quad + F_2^\circ(u_n(x), v_n(x); -v_n(x))\Big]\mathrm{d}x \\
&\ge \Big(\frac{1}{p} - \frac{1}{\alpha}\Big)\|(u_n, v_n)\|_{E\times E}^p.
\end{aligned}$$

This shows that (u_n, v_n) is bounded in E. So we obtain

$$(u_n, v_n) \rightharpoonup (u, v), \quad \text{weakly in } E \times E, \tag{3.6}$$

$$u_n \to u, \quad \text{strongly in } L^p(\mathbb{R}^N), \tag{3.7}$$

$$v_n \to v, \quad \text{strongly in } L^p(\mathbb{R}^N) \tag{3.8}$$

and

$$\mathcal{I}_E^\circ(u_n, v_n; u_n - u_m, 0) = \langle J'(u_n, v_n), (u_n - u_m, 0)\rangle + \mathcal{F}_E^\circ(u_n, v_n; u_m - u_n, 0), \tag{3.9}$$

where $J(u,v) = \frac{1}{p}\|\nabla\|_E^p + \frac{1}{p}\|\nabla v\|_E^p$, and

$$\mathcal{I}_E^\circ(u_m, v_m; -u_n + u_m, 0) = \langle J'(u_m, v_m), (u_m - u_n, 0)\rangle + \mathcal{F}_E^\circ(u_m, v_m; u_n - u_m, 0). \tag{3.10}$$

Combining (3.9) with (3.10), we have

$$\begin{aligned} &\mathcal{I}_E^\circ(u_n, v_n; u_n - u_m, 0) + \mathcal{I}_E^\circ(u_m, v_m; -u_n + u_m, 0) \\ &= \langle (J'(u_n, v_n) - J'(u_m, v_m)), (u_n - u_m, 0)\rangle \\ &\quad + \mathcal{F}_E^\circ(u_n, v_n; u_m - u_n, 0) + \mathcal{F}_E^\circ(u_m, v_m; u_n - u_m, 0). \end{aligned}$$

According to the elementary inequalities

$$(|b|^{p-2}b + |a|^{p-2}a)(b-a) \geq c_p \begin{cases} |b-a|^p, & \text{if } p \geq 2, \\ (1+|b|+|a|)^{p-2}|b-a|^2, & \text{if } 1 < p < 2, \end{cases}$$

where $c_p > 0$ is a constant, and $a, b \in \mathbb{R}^N$, substituting b and a by ∇u_n and ∇u_m respectively, and integrating over $\mathbb{R}^N$, we obtain

$$\|u_n - u_m\|_E^p \leq (J'(u_n, v_n) - J'(u_m, v_m))(u_n - u_m, 0) = p_n^1 - p_n^2 - p_n^3,$$

where

$$\begin{aligned} p_n^1 &= \mathcal{F}_E^\circ(u_n, v_n; u_m - u_n, 0) + \mathcal{F}_E^\circ(u_m, v_m; u_n - u_m, 0), \\ p_n^2 &= \mathcal{I}_E^\circ(u_n, v_n; u_m - u_n, 0), \\ p_n^3 &= \mathcal{I}_E^\circ(u_m, v_m; -u_n + u_m, 0). \end{aligned} \tag{3.11}$$

Now we estimate p^i $(i \in \{1,2,3\})$. Using Lemma 2.1(ii) and (3.5), we obtain

$$\begin{aligned} p_n^1 &\leq \int_{\mathbb{R}^n} F^\circ(u_n(x), v_n(x); u_n(x) - u_m(x), 0) + F^\circ(u_m(x), v_m(x); u_m(x) - u_n(x), 0)\mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} [|F_1^\circ(u_n(x), v_n(x); u_n(x) - u_m(x))| + 0 + |F_1^\circ(u_m(x), v_m(x), u_m(x) - u_n(x))| + 0]\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |\max\{w_n^1(u_n(x) - u_m(x)) : w_n^1 \in \partial_1 F(u_n(x), v_n(x))\}|\mathrm{d}x + 0 \\ &\quad + \int_{\mathbb{R}^N} |\max\{w_m^1(u_m(x) - u_n(x)) : w_m^1 \in \partial_1 F(u_m(x), v_m(x))\}|\mathrm{d}x + 0 \\ &\leq \int_{\mathbb{R}^N} [\varepsilon(|u_n|^{p-1} + |v_n|^{p-1} + |u_m|^{p-1} + |v_m|^{p-1}) \\ &\quad + C_\varepsilon(|u_n|^{q-1} + |v_n|^{q-1} + |u_m|^{q-1} + |v_m|^{q-1})](|u_m(x) - u_n(x)|)\mathrm{d}x \\ &\leq [\varepsilon(\|u_n\|_{L^p}^{p-1} + \|v_n\|_{L^p}^{p-1} + \|u_m\|_{L^p}^{p-1} + \|v_m\|_{L^p}^{p-1}) \\ &\quad + C_\varepsilon(\|u_n\|_{L^p}^{q-1} + \|v_n\|_{L^p}^{q-1} + \|u_m\|_{L^p}^{q-1} + \|v_m\|_{L^p}^{q-1})](\|u_n - u_m\|_E + \|v_n - v_m\|_E). \end{aligned}$$

Since the sequence u_n and v_n are bounded in $W^{1,p}(\mathbb{R}^N)$ and the relations (3.9) and (3.10), from the arbitrariness of $\varepsilon > 0$, we obtain

$$\limsup_{n\to\infty} p_n^1 \leq 0. \tag{3.12}$$

Since

$$p_n^2 = \mathcal{I}^\circ(u_n, v_n; -u_n + u_m, 0) \geq \langle z_n^*, (u_m - u_n), 0\rangle \geq -\|z_n^*\|\|u_m - u_n\|,$$

due to (3.8), we have

$$\liminf_{n\to\infty} p_n^2 \geq 0. \tag{3.13}$$

Finally, in the same way, we obtain

$$\liminf_{n\to\infty} p_n^3 \geq 0. \tag{3.14}$$

Thus, from relations (3.11)–(3.13), we have

$$\|u_n - u_m\|_{E\times E}^p \to 0, \quad \text{as } n \to \infty.$$

In the same way, we prove that v_n converges in $W^{1,p}(\mathbb{R}^N)$. So $\{(u_n, v_n)\}$ converges strongly to (u, v) in $E \times E$.

Now we verify the geometic conditions of Mountain Pass Theorem.

Lemma 3.2 *Suppose that a locally Lipschitz function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies (F^1)–(F^5). Then there exist $\eta > 0$, $\rho > 0$ and $e \in E$ such that*

$$\mathcal{I}_E(u, v) \geq \eta \quad \textit{for all } \|(u, v)\|_{E\times E} = \rho \tag{3.15}$$

and

$$\|(u, v)\|_{E\times E} > \rho, \quad \mathcal{I}_E(u, v) \leq 0. \tag{3.16}$$

Proof We have $F(0, 0) = 0$. To prove (3.15), we need to obtain the following estimation:

$$F(u, v) \leq \Big[\frac{1}{4p}(|u|^{p-1} + |v|^{p-1}) + c_4(|u|^{q-1} + |v|^{q-1})\Big](|u| + |v|).$$

Form (3.1), we have

$$\begin{aligned}|F(u_1, u_2) - F(u_3, u_4)| &\leq (|w_1| + |w_2|)(|u_1 - u_3| + |u_2 - u_4|)\\ &\leq \Big(\varepsilon \sum |u_i|^{p-1} + c_\varepsilon \sum |u_i|^{q-1}\Big)(|u_1 - u_3| + |u_2 - u_4|).\end{aligned}$$

Let $u_1 = u$, $u_2 = v$ and $u_3 = u_4 = 0$. Since $F(0, 0) = 0$, we have

$$F(u, v) \leq [\varepsilon(|u|^{p-1} + |v|^{p-1}) + c_\varepsilon(|u|^{q-1} + |v|^{q-1})](|u| + |v|).$$

Let $\varepsilon = \frac{1}{4p}$ and $c_\varepsilon = c_4$. We obtain

$$F(u, v) \leq \Big[\frac{1}{4p}(|u|^{p-1} + |v|^{p-1}) + c_4(|u|^{q-1} + |v|^{q-1})\Big](|u| + |v|).$$

Then

$$\begin{aligned}\mathcal{I}_E(u, v) &= \frac{1}{p}\|(u, v)\|_{E\times E}^p - \int_{\mathbb{R}^N} F(u, v)\mathrm{d}x\\ &\geq \frac{1}{p}\|(u, v)\|_{E\times E}^p - \int_{\mathbb{R}^N} \Big[\frac{1}{4p}(|u|^{p-1} + |v|^{p-1}) + c_4(|u|^{q-1} + |v|^{q-1})\Big](|u| + |v|)\mathrm{d}x\\ &\geq \frac{1}{p}\|(u, v)\|_{E\times E}^p - \frac{1}{2p}(\|u\|_p^p + \|v\|_p^p) - 2c_4(\|u\|_q^q + \|v\|_q^q)\\ &\geq \frac{1}{p}\|(u, v)\|_{E\times E}^p - \frac{1}{2p}\|(u, v)\|_E^p - 2c_4 c_{1,q}^q(\|u\|_{1,q}^q + \|v\|_{1,q}^q)\\ &= \frac{1}{2p}\|(u, v)\|_{E\times E}^p - 2c_4 c_{1,q}^q(\|u\|_{1,q}^q + \|v\|_{1,q}^q)\\ &\geq \frac{1}{2p}\|(u, v)\|_{E\times E}^p - 2c_4 c_{1,q}^q\|(u, v)\|_{E\times E}^q\\ &= \Big(\frac{1}{2p} - 2c_4 c_{1,q}^q\|(u, v)\|_{E\times E}^{q-p}\Big)\|(u, v)\|_{E\times E}^p.\end{aligned}$$

Choosing $\|(u,v)\|_E = \rho > 0$ small enough, the number $\eta = \big(\frac{1}{2p} - 2c_4 c_{1,q}^q \rho^{q-p}\big)\rho^p$ will be strictly positive, due to the fact $q > p$. Thus (3.15) holds.

Now to prove (3.16), we first show that

$$t^\alpha F(u,v) \le F(tu,tv) \quad \text{for all } t > 1 \text{ and } (u,v) \in \mathbb{R}^2. \tag{3.17}$$

Fix an arbitrarily $(u,v) \in \mathbb{R}^2$. From the Second Chain Rule and Lemma 2.2(i), we have

$$\partial_t F(tu,tv) \subseteq \partial F(tu,tv) \circ (u,v) \subseteq \partial_1 F(tu,tv)u + \partial_2 F(tu,tv)v$$

for all $t > 0$.

Since $t \mapsto t^{-\alpha}F(tu,tv)$ $(t > 0)$ is locally Lipschitz, we have for all $t > 0$,

$$\partial_t(t^{-\alpha}F(tu,tv)) = -\alpha t^{-\alpha-1}F(tu,tv) + t^{-\alpha}\partial_t F(tu,tv).$$

Therefore,

$$\partial_t(t^{-\alpha}F(tu,tv)) \subseteq t^{-\alpha-1}[-\alpha F(tu,tv) + t\partial_1 F(tu,tv)u + t\partial_2 F(tu,tv)v] \tag{3.18}$$

for all $t > 0$.

Now, we fix $t > 1$. Due to the Lebourg's mean value theorem and (3.18), there exists a $\tau \in (1,t)$, such that

$$\begin{aligned} &t^{-\alpha}F(tu,tv) - F(u,v) \\ &\in \partial_t(\tau^{-\alpha}F(\tau u,\tau v))(t-1) \\ &\subseteq \tau^{-\alpha-1}[-\alpha F(\tau u,\tau v) + \tau\partial_1 F(\tau u,\tau v)u + \tau\partial_2 F(\tau u,\tau v)v](t-1). \end{aligned}$$

Thus there exist $\omega_i^\tau \in \partial_i F(\tau u,\tau v)$ $(i \in \{1,2\})$, such that

$$t^{-\alpha}F(tu,tv) - F(u,v) = -\tau^{-\alpha-1}[\alpha F(\tau u,\tau v) + \omega_1^\tau(-\tau u) + \omega_2^\tau(-\tau v)](t-1).$$

Using (F^4), we have

$$\begin{aligned} t^{-\alpha}F(tu,tv) - F(u,v) &\ge -\tau^{-\alpha-1}[\alpha F(\tau u,\tau v) + F_1^0(\tau u,\tau v;-\tau u) \\ &\qquad + F_1^0(\tau u,\tau v;-\tau v)](t-1) \\ &\ge 0. \end{aligned}$$

This leads exactly to (3.17).

Now, we choose an element $u_0 \in W^{1,p}(\mathbb{R}^N)$ such that $\|u_0\|_E = 1$. Due to (F^5), (3.17) and $\int_{\mathbb{R}^N} F(u_0,u_0)\mathrm{d}x > 0$, we get

$$I(tu_0,tu_0) = \frac{2}{p}t^p - \int_{\mathbb{R}^N} F(tu_0,tu_0)\mathrm{d}x \le \frac{2}{p}t^p - t^\alpha \int_{\mathbb{R}^N} F(u_0,u_0)\mathrm{d}x \to -\infty,$$

as $t \to +\infty$. Because $\alpha > p$, by choosing $t_0 > \frac{\rho}{\sqrt[p]{2}}$ large enough and denoting by $e = t_0 u_0 \in W^{1,p}(\mathbb{R}^N)$, we are led to (3.17).

Proof of Theorem 2.1 It is clear that $E = W_G^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : \ gu = u, \ \forall g \in G\}$ is a closed subspace of $W^{1,p}(\mathbb{R}^N)$, which is compactly embedded in $L^p(\mathbb{R}^N)$. Choosing $X = E \times E$ and $h = \mathcal{I}_E$, the geometric conditions (i) and (ii) in Theorem 2.2 are verified for $\mathcal{I}_E$, due to Proposition 3.1. Let $\eta > 0$ and $e \in E$ be the corresponding elements from (3.15) and (3.16). Defining $c \in \mathbb{R}$ as in Theorem 2.2 for the element $(e,e) \in E \times E$, we have $c \ge \eta$. By Proposition 3.1, $\mathcal{I}_E$ satisfies $(\mathrm{PS})_c$. Hence, there exists at least one critical point

$(u_1, v_1) \in E \times E$ of $\mathcal{I}_E$, such that the critical value $c = \mathcal{I}_E(u_1, v_1)$ is strictly positive, which means that $(u_1, v_1) \neq (0, 0)$. Since $E \times E$ is the subspace of $W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$, we may conclude that (u_1, v_1) will be a critical point of $\mathcal{I}$ on the whole space $W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$. Consequently, by Lemma 2.4, this element will be a weak solution to (S).

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