

Fractional Moments of Automorphic L -Functions on $GL(m)^*$

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Abstract Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$, $m \geq 2$. Assume that π is self-contragredient. The author gets upper and lower bounds of the same order for fractional moments of automorphic L -function $L(s, \pi)$ on the critical line under Generalized Ramanujan Conjecture; the upper bound being conditionally subject to the truth of Generalized Riemann Hypothesis.

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1 Introduction

Understanding moments of families of L -functions on the critical line has long been an important subject in number theory. According to conjectures of Langlands, the general L -functions should be expressed as products of primitive L -functions $L(s, \pi)$ attached to cuspidal automorphic representations of $GL_m(\mathbb{A}_{\mathbb{Q}})$. For $m = 1$, these are the Riemann zeta function $\zeta(s)$ and Dirichlet L -functions $L(s, \chi)$ with χ primitive. It is conjectured that

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \sim C_k T (\log T)^{k^2}, \quad (1.1)$$

where $k \geq 0$ and $C_k > 0$ is a constant. For $k = 1$ and $k = 2$, (1.1) was proved by Hardy and Littlewood [1] in 1918 and Ingham [2] in 1926, respectively. However, no unconditional asymptotic formula has yet been proved for any other k .

It is of interest therefore to ask for the weaker result

$$T(\log T)^{k^2} \ll \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \ll T (\log T)^{k^2}. \quad (1.2)$$

In this direction, Ramachandra [3–4] and Heath-Brown [5–6] proved that the lower bound in (1.2) holds for all rational $k \geq 0$, and the upper bound holds for $k = \frac{1}{v}$, where v is a positive integer. Moreover, under Riemann Hypothesis (RH in brief), they showed that the lower bound holds for all real $k \geq 0$, and the upper bound holds for $0 \leq k \leq 2$. Recently, Soundararajan [7] showed that, under RH,

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \ll T (\log T)^{k^2 + \varepsilon}$$

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for any $\varepsilon > 0$ and any $k > 0$. For Dirichlet L -functions $L(s, \chi)$, we refer to [8]. For $m = 2$, the asymptotic formula for the second moment of automorphic L -functions $L(s, f)$ attached to a holomorphic cusp form f for $\mathrm{SL}_2(\mathbb{Z})$ was obtained by Good [9]. Recently, using Heath-Brown's method [5], Laurinćikas and Steuding [10] showed that

$$\begin{aligned} \int_0^T \left| L\left(\frac{1}{2} + it, f\right) \right|^{k^2} dt &\gg T (\log T)^{k^2} \quad \text{for } k = \frac{1}{v} \text{ with } v \in \mathbb{N}, \\ \int_0^T \left| L\left(\frac{1}{2} + it, f\right) \right|^{k^2} dt &\ll T (\log T)^{k^2} \quad \text{for } k = \frac{1}{v} \text{ with } 2 \mid v \text{ and } v \in \mathbb{N}, \end{aligned}$$

where the upper bound was proved under Generalized Riemann Hypothesis (GRH in brief). Lü and Sun [11] further improved the lower bound by extending the range of k to $k = \frac{u}{v} \leq \frac{1}{2}$ with $u, v \in \mathbb{N}$.

In this paper, we are concerned with the fractional moments of automorphic L -functions $L(s, \pi)$, where $\pi = \otimes \pi_p$ is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$, where, throughout the paper, $m \geq 2$. To this end, we recall some background on automorphic L -functions.

Let $s = \sigma + it$. For $\sigma > 1$, $L(s, \pi)$ is defined by the products of local factors,

$$L(s, \pi) = \prod_{p < \infty} L_p(s, \pi_p), \quad L_p(s, \pi_p) = \prod_{j=1}^m \left(1 - \frac{\alpha_{\pi}(p, j)}{p^s} \right)^{-1}.$$

The complete L -function attached to π is defined by

$$\Phi(s, \pi) = L_{\infty}(s, \pi_{\infty}) L(s, \pi), \quad (1.3)$$

where $L_{\infty}(s, \pi_{\infty}) = \prod_{j=1}^m \Gamma_{\mathbb{R}}(s + \mu_{\pi}(j))$. Here $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$, and $\alpha_{\pi}(p, j)$ and $\mu_{\pi}(j)$ ($j = 1, \dots, m$) are complex numbers associated with π_p and π_{∞} , respectively. It is well-known that all the non-trivial zeros of $L(s, \pi)$ are in the critical strip $0 < \sigma < 1$, while GRH predicts that they lie on the vertical line $\sigma = \frac{1}{2}$. For $m \geq 2$, $\Phi(s, \pi)$ is entire and satisfies a functional equation

$$\Phi(s, \pi) = \varepsilon(s, \pi) \Phi(1 - s, \tilde{\pi}), \quad (1.4)$$

with $\tilde{\pi}$ the representation contragredient to π and $\varepsilon(s, \pi) = \varepsilon_{\pi} N_{\pi}^{\frac{1}{2} - s}$, where ε_{π} is the root number and $N_{\pi} > 1$ is the conductor. For any $p \leq \infty$, $\tilde{\pi}_p$ is equivalent to the complex conjugate $\overline{\pi}_p$, and thus

$$\{\alpha_{\tilde{\pi}}(j, p)\} = \{\overline{\alpha_{\pi}(j', p)}\}, \quad \{\mu_{\tilde{\pi}}(j)\} = \{\overline{\mu_{\pi}(j')}\}. \quad (1.5)$$

Denote

$$a_{\pi}(p^{\ell}) = \sum_{1 \leq j \leq m} \alpha_{\pi}(p, j)^{\ell}. \quad (1.6)$$

Then for $\sigma > 1$, we have

$$\frac{d}{ds} \log L(s, \pi) = - \sum_{n=1}^{\infty} \frac{\Lambda(n) a_{\pi}(n)}{n^s},$$

where $\Lambda(n) = \log p$ is $n = p^\ell$ and if 0 otherwise.

In this paper, we assume the Generalized Ramanujan Conjecture (GRC in brief) which states that for any unramified p ,

$$|\alpha_\pi(p, j)| = 1 \quad \text{and} \quad \operatorname{Re} \mu_\pi(j) = 0, \quad j = 1, \dots, m. \quad (1.7)$$

An important consequence of GRC is Selberg's orthogonality conjecture proposed by Selberg [12] in 1989, which states as follows. Let π and π' be automorphic irreducible cuspidal representations of the groups $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_{m'}(\mathbb{A}_\mathbb{Q})$, respectively. Then

$$\sum_{p \leq x} \frac{a_\pi(p) \bar{a}_{\pi'}(p)}{p} = \begin{cases} \log \log x + O(1), & \text{if } \pi \cong \pi', \\ O(1), & \text{if } \pi \not\cong \pi'. \end{cases} \quad (1.8)$$

(1.8) was proved by Rudnick and Sarnak [13] under the hypothesis

$$\sum_p \frac{|a_\pi(p^\ell)|^2 (\log p)^2}{p^\ell} < \infty,$$

which is an easy consequence of GRC. Under GRC, Liu and Ye [14] proved (1.8) in a more precise form. Precisely, they proved the following result.

Proposition 1.1 *Let π and π' be automorphic irreducible cuspidal representations of the groups $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_{m'}(\mathbb{A}_\mathbb{Q})$, respectively, such that at least one of π and π' is self contragredient: $\pi \cong \tilde{\pi}$ or $\pi' \cong \tilde{\pi}'$. Assume GRC for both π and π' . Then*

$$\sum_{p \leq x} \frac{a_\pi(p) \bar{a}_{\pi'}(p)}{p} = \begin{cases} \log \log x + C_1 + O(\exp\{-c\sqrt{\log x}\}), & \pi' \cong \pi, \\ C_2 + \operatorname{Ei}(i\tau_0 \log x) + O(\exp\{-c\sqrt{\log x}\}), & \pi' \cong \pi \otimes \alpha^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}^\times, \\ C_3 + O(\exp\{-c\sqrt{\log x}\}), & \pi' \not\cong \pi \otimes \alpha^{i\tau} \text{ for any } \tau \in \mathbb{R}, \end{cases}$$

where Ei is the exponential integral, and C_1, C_2, C_3 are positive constants.

For $k \geq 0$ and $\sigma \geq \frac{1}{2}$, we define $I_k(\sigma, T, \pi) = \int_0^T |L(\sigma + it, \pi)|^{2k} dt$. For brevity, we denote

$$I_k\left(\frac{1}{2}, T\right) := I_k\left(\frac{1}{2}, T, \pi\right) = \int_0^T \left|L\left(\frac{1}{2} + it, \pi\right)\right|^{2k} dt.$$

Theorem 1.1 *Assume GRC. Let π be an automorphic irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ such that π is self-contragredient. Let $k \in \mathbb{Q}$, $k \geq 0$. Then as $T \rightarrow \infty$, we have $I_k(\frac{1}{2}, T) \gg T(\log T)^{k^2}$. Under GRH, the range of k can be extended to $k \in \mathbb{R}$, $k \geq 0$.*

Theorem 1.2 *Assume GRC and GRH. Let π be as in Theorem 1.1. Let $0 \leq k \leq \frac{2}{m} - \varepsilon$ for any $\varepsilon > 0$. Then as $T \rightarrow \infty$, we have $I_k(\frac{1}{2}, T) \ll T(\log T)^{k^2}$.*

Theorems 1.1 and 1.2 are proved by Heath-Brown [5]. Since GRC was proved by Deligne [15] for π being representations corresponding to holomorphic cusp forms, Theorems 1.1 and 1.2 hold without assuming GRC for $m = 2$. Thus Theorems 1.1 and 1.2 improve Laurinćikas and Steuding's result and Lü and Sun's result. We also note that Theorems 1.1 and 1.2 generalize the recent results of Fomenko [16].

In the sequel, we will use $c_1, c_2, \dots$ to denote positive constants and the implied constants in " $\ll$ " and " $\gg$ " depending on m, k and π .

2 Proof of Theorems 1.1 and 1.2

Let $w(t, T) = \int_T^{2T} e^{-(t-\tau)^2} d\tau$. Then $w(t, T)$ has the following properties:

$$\begin{cases} w(t, T) \ll 1 & \text{for all } t, \\ w(t, T) \gg 1 & \text{for } \frac{4T}{3} \leq t \leq \frac{5T}{3}. \end{cases} \quad (2.1)$$

Define

$$J(\sigma, T) := J_k(\sigma, T, \pi) = \int_{-\infty}^{\infty} |L(\sigma + it, \pi)|^{2k} w(t, T) dt.$$

By (2.1), we have $J(\frac{1}{2}, T) \ll I_k(\frac{1}{2}, 3T)$. Thus Theorem 1.1 follows if under GRC,

$$J\left(\frac{1}{2}, T\right) \gg T(\log T)^{k^2} \quad \text{for } k \in \mathbb{Q}, k \geq 0, \quad (2.2)$$

and under GRH, it holds for $k \geq 0$.

On the other hand, if we can establish the upper bound

$$J\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2} \quad \text{for } 0 \leq k \leq \frac{2}{m} - \varepsilon \text{ for any } \varepsilon > 0, \quad (2.3)$$

under GRC and GRH, then by (2.1), we also have

$$I_k\left(\frac{1}{2}, \frac{5T}{3}\right) - I_k\left(\frac{1}{2}, \frac{4T}{3}\right) \ll J\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2}.$$

Replacing T by $\left(\frac{4}{5}\right)^j T$ and summing up over $j = 1, 2, \dots$, we obtain Theorem 1.2.

The following three sections will be devoted to the proof of (2.2) and (2.3).

3 The Coefficients of $L(s, \pi)^k$

Let $s = \sigma + it$. For $\sigma > 1$, we define a branch of the multi-valued function $L(s, \pi)^k$ by

$$L(s, \pi)^k = \exp\{k \log L(s, \pi)\} = \exp\left\{k \sum_p \sum_{j=1}^m \sum_{\ell=1}^{\infty} \frac{\alpha_{\pi}(p, j)^{\ell}}{\ell p^{\ell s}}\right\} = \prod_p \prod_{j=1}^m \left(1 - \frac{\alpha_{\pi}(p, j)}{p^s}\right)^{-k}.$$

For $|z| < 1$, we have $(1 - z)^{-k} = \sum_{\ell=0}^{\infty} \frac{\Gamma(k+\ell)}{\Gamma(k)\ell!} z^{\ell}$. For positive integers ℓ , define

$$d_k(p^{\ell}) = \frac{\Gamma(k+\ell)}{\Gamma(k)\ell!} = \frac{k(k+1) \cdots (k+\ell-1)}{\ell!}.$$

Then for $\sigma > 1$, we have

$$L(s, \pi)^k = \prod_p \sum_{\ell=0}^{\infty} \frac{h_k(p^{\ell})}{p^{\ell s}} = \sum_{n=1}^{\infty} \frac{h_k(n)}{n^s},$$

where $h_k(n)$ is the multiplicative function given by

$$h_k(p^{\ell}) = \sum_{\substack{\ell_1 + \cdots + \ell_m = \ell \\ \ell_j \geq 0}} d_k(p^{\ell_1}) \alpha_{\pi}(p, 1)^{\ell_1} \cdots d_k(p^{\ell_m}) \alpha_{\pi}(p, m)^{\ell_m} \quad \text{for } \ell \in \mathbb{N}. \quad (3.1)$$

Lemma 3.1 (see [5]) *The multiplicative function $d_k(n)$ satisfies the following properties:*

- (1) *For $k \geq 0$ and $n \geq 1$, we have $d_k(n) \geq 0$;*
- (2) *For fixed $k \geq 0$ and $\varepsilon > 0$, we have $d_k(n) \ll n^\varepsilon$;*
- (3) *If ℓ is an integer, then $d_{k\ell}(n) = \sum_{n_1 \cdots n_\ell = n} d_k(n_1) \cdots d_k(n_\ell)$.*

Lemma 3.2 (see [17, 18]) *Let $f(n) \geq 0$ be a multiplicative function satisfying*

- (i) $\sum_{p \leq x} \frac{\log p}{p} f(p) \sim \tau \log x$ *for some $\tau > 0$;*
- (ii) $f(p) \ll 1$;
- (iii) $\sum_{p, \ell \geq 2} \frac{f(p^\ell)}{p^\ell} < \infty$;
- (iv) $\sum_{\substack{p, \ell \geq 2 \\ p^\ell \leq x}} f(p^\ell) \ll \frac{x}{\log x}$.

Then

$$\sum_{n \leq x} f(n) \sim \frac{e^{-\gamma_0 \tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \leq x} \left(1 + \sum_{\ell=1}^{\infty} \frac{f(p^\ell)}{p^\ell} \right),$$

where γ_0 is Euler's constant.

Lemma 3.3 *Assume GRC. Let π be an automorphic irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$, such that π is self-contragredient. We have*

$$\sum_{n \leq x} |h_k(n)|^2 \sim \frac{e^{-\gamma_0 k^2}}{\Gamma(k^2)} \frac{x}{\log x} \prod_{p \leq x} \left(1 + \sum_{\ell=1}^{\infty} \frac{|h_k(p^\ell)|^2}{p^\ell} \right),$$

where γ_0 is Euler's constant.

Proof Note that $d_k(p) = k$. By (3.1) and (1.6), we have $h_k(p) = d_k(p)(\alpha_\pi(p, 1) + \cdots + \alpha_\pi(p, m)) = ka_\pi(p)$. By Proposition 1.1,

$$\sum_{p \leq x} \frac{|a_\pi(p)|^2}{p} = \log \log x + C_1 + R(x),$$

where $R(x) \ll \exp\{-c\sqrt{\log x}\}$. Integrating by parts, we get

$$\begin{aligned} \sum_{p \leq x} \frac{\log p}{p} |h_k(p)|^2 &= k^2 \int_2^x \log u \, d \sum_{p \leq u} \frac{|a_\pi(p)|^2}{p} \\ &= k^2 \int_2^x \log u \, d \log \log u + k^2 \int_2^x \log u \, dR(u) \\ &= k^2 \log x + O(1) \\ &\sim k^2 \log x, \quad \text{as } x \rightarrow \infty. \end{aligned} \tag{3.2}$$

Next, by (1.7), we have

$$|h_k(p)|^2 = k^2 |a_\pi(p)|^2 \leq k^2 m^2 \ll_{k,m} 1. \tag{3.3}$$

Moreover, by (1.7), (3.1) and Lemma 3.1(3), we have for any $\varepsilon > 0$,

$$|h_k(p^\ell)| \leq \sum_{\substack{\ell_1 + \cdots + \ell_m = \ell \\ \ell_j \geq 0}} d_k(p^{\ell_1}) \cdots d_k(p^{\ell_m}) = d_{km}(p^\ell) \ll p^{\varepsilon \ell}. \tag{3.4}$$

Thus, for $\varepsilon > 0$ sufficiently small,

$$\sum_{p, \ell \geq 2} \frac{|h_k(p^\ell)|^2}{p^\ell} \ll \sum_{p, \ell \geq 2} \frac{1}{p^{\ell(1-\varepsilon)}} = \sum_{p=2}^{\infty} \frac{1}{p^{1-\varepsilon}(p^{1-\varepsilon}-1)} < \infty. \quad (3.5)$$

Finally,

$$\sum_{\substack{p, \ell \geq 2 \\ p^\ell \leq x}} |h_k(p^\ell)|^2 \ll \sum_{\substack{p, \ell \geq 2 \\ p^\ell \leq x}} p^{2\varepsilon\ell} \ll \sum_{p \leq \sqrt{x}} \sum_{\ell \leq \frac{\log x}{\log p}} p^{2\varepsilon\ell} \ll x^{2\varepsilon} \log x \sum_{p \leq \sqrt{x}} 1 \ll x^{\frac{1}{2}+3\varepsilon} \ll \frac{x}{\log x}.$$

Combined with this estimate, Lemma 3.3 follows from (3.2), (3.4), (3.5) and Lemma 3.2.

Lemma 3.4 *Assume GRC. For any fixed $k > 0$, there exists a constant $C > 0$, such that*

$$\begin{aligned} \left(\sigma - \frac{1}{2}\right)^{-k^2} &\ll \sum_{n \leq N} \frac{|h_k(n)|^2}{n^{2\sigma}} \ll \left(\sigma - \frac{1}{2}\right)^{-k^2}, \quad \text{uniformly for } \frac{1}{2} + \frac{C}{\log N} \leq \sigma \leq 1, \\ (\log N)^{k^2} &\ll \sum_{n \leq N} \frac{|h_k(n)|^2}{n} \ll (\log N)^{k^2}. \end{aligned}$$

Proof By Lemma 3.3, we have the asymptotic formula

$$\sum_{n \leq x} |h_k(n)|^2 \sim \frac{e^{-\gamma_0 k^2}}{\Gamma(k^2)} \frac{x}{\log x} \prod_{p \leq x} \left(1 + \sum_{\ell=1}^{\infty} \frac{|h_k(p^\ell)|^2}{p^\ell}\right).$$

By (3.3), (3.4) and Proposition 1.1, we have

$$\begin{aligned} \prod_{p \leq x} \left(1 + \sum_{\ell=1}^{\infty} \frac{|h_k(p^\ell)|^2}{p^\ell}\right) &= \exp \left\{ \sum_{p \leq x} \log \left(1 + \frac{|h_k(p)|^2}{p} + O\left(\frac{1}{p^{2(1-\varepsilon)}}\right)\right) \right\} \\ &= \exp \left\{ \sum_{p \leq x} \frac{|h_k(p)|^2}{p} + O(1) \right\} \\ &= \exp \left\{ k^2 \sum_{p \leq x} \frac{|a_\pi(p)|^2}{p} + O(1) \right\} \\ &= e^{O(1)} (\log x)^{k^2}. \end{aligned}$$

Therefore, there exist positive constants $c_1 < c_2$, such that

$$c_1 x (\log x)^{k^2-1} \leq \sum_{n \leq x} |h_k(n)|^2 \leq c_2 x (\log x)^{k^2-1}.$$

By partial summation, the first assertion of the lemma follows. The second assertion follows from the first one since, for $\sigma = \frac{1}{2} + \frac{C}{\log N}$ and $1 \leq n \leq N$, we have $n^{-1} \ll n^{2\sigma} \ll n^{-1}$.

Define $S_N(s) = \sum_{n \leq N} \frac{h_k(n)}{n^s}$ and $H(\sigma, T) = \int_{-\infty}^{\infty} |S_N(\sigma + it)|^2 w(t, T) dt$.

Lemma 3.5 *Assume GRC. Let $N \ll T$ and $\log N \gg \log T$. Then for $\frac{1}{2} + \frac{C}{\log N} \leq \sigma \leq \frac{3}{4}$, we have*

$$\begin{aligned} T \left(\sigma - \frac{1}{2}\right)^{-k^2} &\ll H(\sigma, T) \ll T \left(\sigma - \frac{1}{2}\right)^{-k^2}, \\ T (\log T)^{k^2} &\ll H\left(\frac{1}{2}, T\right) \ll T (\log T)^{k^2}. \end{aligned}$$

Proof Note that $w(t, T) \ll e^{-c_3(t^2+T^2)}$ for $t \leq 0$ and $t \geq 3T$. By (3.4), we have $S_N(s) \ll N^{\frac{1}{2}+\varepsilon}$ for $\sigma \geq \frac{1}{2}$ and any $\varepsilon > 0$. Thus, by (2.1) and the Montgomery-Vaughan Theorem (see [19]), we find that

$$H(\sigma, T) \ll \int_0^{3T} |S_N(\sigma + it)|^2 dt \ll (T + N) \sum_{n \leq N} \frac{|h_k(n)|^2}{n^{2\sigma}}.$$

On the other hand, by (2.1), we have

$$H(\sigma, T) \gg \int_{\frac{4T}{3}}^{\frac{5T}{3}} |S_N(\sigma + it)|^2 dt \gg T \sum_{n \leq N} \frac{|h_k(n)|^2}{n^{2\sigma}}.$$

Therefore, Lemma 3.5 follows from Lemma 3.4.

4 Applications of Gabriel's Inequality

We need the following Gabriel's inequality.

Lemma 4.1 (see [20] or [5]) *Let $G(s)$ be regular in the strip $\{s \in \mathbb{C} : \alpha < \sigma < \beta\}$ and continuous in the closed strip $\{s \in \mathbb{C} : \alpha \leq \sigma \leq \beta\}$. Moreover, assume that $G(s) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $\{s \in \mathbb{C} : \alpha \leq \sigma \leq \beta\}$. Then, for $\alpha \leq \gamma \leq \beta$ and any $\theta > 0$,*

$$\int_{-\infty}^{\infty} |G(\gamma + it)|^\theta dt \leq \left(\int_{-\infty}^{\infty} |G(\alpha + it)|^\theta dt \right)^{\frac{\beta-\gamma}{\beta-\alpha}} \left(\int_{-\infty}^{\infty} |G(\beta + it)|^\theta dt \right)^{\frac{\gamma-\alpha}{\beta-\alpha}}.$$

Lemma 4.2 *Let $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ and $T \geq 2$. Then for all $k > 0$,*

$$J\left(\frac{1}{2}, T\right) \ll J(\sigma, T) T^{km(\sigma-\frac{1}{2})} + e^{-c_4 T^2}.$$

Proof Applying Lemma 4.1 with $\gamma = \frac{1}{2}$, $\alpha = 1 - \sigma$, $\beta = \sigma$, $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $\theta = 2k$, and

$$G(s) = L(s, \pi) e^{\frac{(s-i\tau)^2}{2k}}, \quad (4.1)$$

we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, \pi\right) \right|^{2k} e^{-(t-\tau)^2} dt \\ & \ll \left(\int_{-\infty}^{\infty} |L(1 - \sigma + it, \pi)|^{2k} e^{-(t-\tau)^2} dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |L(\sigma + it, \pi)|^{2k} e^{-(t-\tau)^2} dt \right)^{\frac{1}{2}}. \end{aligned}$$

By the functional equation (1.3)–(1.5) and the Stirling's formula, we have

$$|L(1 - \sigma + it, \pi)| \ll |L(\sigma + it, \pi)| (1 + |t|)^{m(\sigma-\frac{1}{2})},$$

where the implied constant depends on $\mathrm{Im}\mu_\pi(j)$, $j = 1, \dots, m$. It follows that

$$\begin{aligned} & \int_{-\infty}^{\infty} |L(1 - \sigma + it, \pi)|^{2k} e^{-(t-\tau)^2} dt \\ & \ll \left\{ \int_{-\infty}^{\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} + \int_{\frac{3\tau}{2}}^{\infty} \right\} |L(\sigma + it, \pi)|^{2k} (1 + |t|)^{km(2\sigma-1)} e^{-(t-\tau)^2} dt \\ & \ll \tau^{km(2\sigma-1)} \int_{-\infty}^{\infty} |L(\sigma + it, \pi)|^{2k} e^{-(t-\tau)^2} dt + e^{-c_5 \tau^2}, \end{aligned}$$

where we have used the bound $L(\sigma + it, \pi) \ll |t|^{\frac{m}{4}}$ for $\sigma \geq \frac{1}{2}$. Therefore,

$$\int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, \pi\right) \right|^{2k} e^{-(t-\tau)^2} dt \ll \tau^{km(\sigma-\frac{1}{2})} \int_{-\infty}^{\infty} |L(\sigma + it, \pi)|^{2k} e^{-(t-\tau)^2} dt + e^{-c_6\tau^2}.$$

Now integration respect to τ on $[T, 2T]$ completes the proof of Lemma 4.2.

Lemma 4.3 *Let $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ and $T \geq 2$. Then for all $k > 0$, $J(\sigma, T) \ll J(\frac{1}{2}, T)^{\frac{3}{2}-\sigma} T^{\sigma-\frac{1}{2}}$.*

Proof Applying Lemma 4.1 with $\gamma = \sigma$, $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\theta = 2k$ and $G(s)$ as in (4.1), we obtain

$$\int_{-\infty}^{\infty} |L(\sigma + it, \pi)|^{2k} e^{-(t-\tau)^2} dt \ll \left(\int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, \pi\right) \right|^{2k} e^{-(t-\tau)^2} dt \right)^{\frac{3}{2}-\sigma}.$$

Here we have used the fact that $\int_{-\infty}^{\infty} |L(\frac{3}{2} + it, \pi)|^{2k} e^{-(t-\tau)^2} dt \ll 1$. Now integration respect to τ on $[T, 2T]$ and the Jensen's inequality give the assertion of Lemma 4.3.

For the proof of Theorem 1.1, in what follows, we will take $k = \frac{u}{v}$ with $u, v \in \mathbb{N}$ and $(u, v) = 1$; under GRH, we can take $k = \frac{u}{v}$ with $v = 1$ and $u = k \geq 0$. For the proof of Theorem 1.2, we will take $k = \frac{u}{v}$ with $0 \leq u = k \leq \frac{2}{m} - \varepsilon$ for any $\varepsilon > 0$, and $v = 1$.

Define $g(s, \pi) = L(s, \pi)^u - S_N(s)^v$ and $K(\sigma, T) = \int_{-\infty}^{\infty} |g(\sigma + it, \pi)|^{\frac{2}{v}} w(t, T) dt$.

Lemma 4.4 *Assume GRC. Let $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $N \ll T$ and $T \geq 2$. Then for any $\varepsilon > 0$,*

$$K(\sigma, T) \ll K\left(\frac{1}{2}, T\right)^{\frac{5-4\sigma}{3}} (TN^{-\frac{1}{v}(\frac{3}{2}-\varepsilon)})^{\frac{4\sigma-2}{3}}.$$

Proof Applying Lemma 4.1 with $\gamma = \sigma$, $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $\alpha = \frac{1}{2}$, $\beta = \frac{5}{4}$, $\theta = \frac{2}{v}$ and $G(s) = g(s, \pi)e^{\frac{v(s-i\tau)^2}{2}}$, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} |g(\sigma + it, \pi)|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \\ & \ll \left(\int_{-\infty}^{\infty} \left| g\left(\frac{1}{2} + it, \pi\right) \right|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \right)^{\frac{5-4\sigma}{3}} \left(\int_{-\infty}^{\infty} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \right)^{\frac{4\sigma-2}{3}}. \end{aligned}$$

Recall that $S_N(s) \ll N^{\frac{1}{2}+\varepsilon}$ for $\sigma \geq \frac{1}{2}$ and any $\varepsilon > 0$. Thus $g(s, \pi) \ll N^{(\frac{1}{2}+\varepsilon)v} + |t|^{\frac{mu}{4}}$ for $\sigma \geq \frac{1}{2}$. This gives

$$\int_{-\infty}^{\infty} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \ll \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^{\frac{2}{v}} e^{-(t-\tau)^2} dt + N^{1+2\varepsilon} e^{-c_7\tau^2}.$$

Therefore,

$$\begin{aligned} & \int_{-\infty}^{\infty} |g(\sigma + it, \pi)|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \\ & \ll \left(\int_{-\infty}^{\infty} \left| g\left(\frac{1}{2} + it, \pi\right) \right|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \right)^{\frac{5-4\sigma}{3}} \left(\int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \right)^{\frac{4\sigma-2}{3}} \\ & \quad + \left(\int_{-\infty}^{\infty} \left| g\left(\frac{1}{2} + it, \pi\right) \right|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \right)^{\frac{5-4\sigma}{3}} (N^{1+2\varepsilon} e^{-c_7\tau^2})^{\frac{4\sigma-2}{3}}. \end{aligned}$$

Now integration respect to τ over $[T, 2T]$ and the Hölder's inequality give us

$$\begin{aligned} K(\sigma, T) &\ll K\left(\frac{1}{2}, T\right)^{\frac{5-4\sigma}{3}} \left(\int_T^{2T} \int_{\frac{T}{2}}^{\frac{3T}{2}} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^{\frac{2}{v}} e^{-(t-\tau)^2} dt d\tau \right)^{\frac{4\sigma-2}{3}} \\ &\quad + e^{-c_8 T^2} N^{\frac{4\sigma-2+\varepsilon}{3}} K\left(\frac{1}{2}, T\right)^{\frac{5-4\sigma}{3}} \\ &\ll K\left(\frac{1}{2}, T\right)^{\frac{5-4\sigma}{3}} \left(\int_T^{2T} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^{\frac{2}{v}} dt \right)^{\frac{4\sigma-2}{3}} + e^{-c_9 T^2} K\left(\frac{1}{2}, T\right)^{\frac{5-4\sigma}{3}}. \end{aligned}$$

It remains to establish the bound

$$\int_T^{2T} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^{\frac{2}{v}} dt \ll TN^{-k(\frac{3}{2}-\varepsilon)}. \quad (4.2)$$

The function $g\left(\frac{5}{4} + it, \pi\right)$ has a representation as an absolutely convergent Dirichlet series. In view of the identity

$$h_{k\ell}(n) = \sum_{n=n_1 \cdots n_\ell} h_k(n_1) \cdots h_k(n_\ell), \quad \ell \in \mathbb{N},$$

we find that

$$g\left(\frac{5}{4} + it, \pi\right) = L\left(\frac{5}{4} + it, \pi\right)^u - S_N\left(\frac{5}{4} + it\right)^v = \sum_{n=N}^{\infty} \frac{a(n)}{n^{\frac{5}{4}+it}},$$

where

$$\begin{aligned} a(n) &= h_u(n) - \sum_{\substack{n=n_1 \cdots n_v \\ n_j \leq N, j=1, \dots, v}} h_k(n_1) \cdots h_k(n_v) \\ &= \sum_{\substack{n=n_1 \cdots n_v \\ \exists n_j > N}} h_k(n_1) \cdots h_k(n_v) \\ &\ll \sum_{n=n_1 \cdots n_v} |h_k(n_1)| \cdots |h_k(n_v)| \\ &\ll \sum_{n=n_1 \cdots n_v} \hat{h}_k(n_1) \cdots \hat{h}_k(n_v) = \hat{h}_u(n), \end{aligned}$$

where $\hat{h}_u(n)$ is the n th coefficient of the Dirichlet series expansion of the function

$$\hat{L}(s, \pi)^u = \left(\prod_p \prod_{1 \leq j \leq m} \left(1 - \frac{|\alpha_\pi(p, j)|}{p^s} \right)^{-1} \right)^u, \quad \sigma > \frac{1}{2}.$$

By (1.7), we have $\hat{h}_u(n) = d_u(n)$. Thus $a(n) \ll d_u(n) \ll n^\varepsilon$ for any $\varepsilon > 0$. By Montgomery-Vaughan Theorem (see [19]), we have for $N \ll T$,

$$\int_{\frac{T}{2}}^{\frac{3T}{2}} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^2 dt \ll T \sum_{n \geq N} \frac{|a(n)|^2}{n^{\frac{5}{2}}} \ll TN^{-\frac{3}{2}+\varepsilon}.$$

By the Jensen's inequality,

$$\int_{\frac{T}{2}}^{\frac{3T}{2}} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^{\frac{2}{v}} dt \ll T \left(\frac{1}{T} \int_0^{3T} \left| g\left(\frac{5}{4} + it, \pi\right) \right|^2 dt \right)^{\frac{1}{v}} \ll TN^{-\frac{1}{v}(\frac{3}{2}-\varepsilon)}.$$

This proves (4.2) and thus completes the proof of Lemma 4.4.

5 Proof of (2.2) and (2.3)

First, we prove (2.2). By the definition of $g(s, \pi)$,

$$|S_N(s)|^2 = |S_N(s)^v|^{\frac{2}{v}} = |L(s, \pi)^u - g(s, \pi)|^{\frac{2}{v}} \ll |L(s, \pi)|^{2k} + |g(s, \pi)|^{\frac{2}{v}}.$$

Hence

$$H(\sigma, T) \ll J(\sigma, T) + K(\sigma, T). \quad (5.1)$$

Similarly,

$$K(\sigma, T) \ll H(\sigma, T) + J(\sigma, T). \quad (5.2)$$

If $K(\frac{1}{2}, T) \leq T$, then by Lemma 3.5 and (5.1) with $\sigma = \frac{1}{2}$, we have

$$T(\log T)^{k^2} \ll H\left(\frac{1}{2}, T\right) \ll J\left(\frac{1}{2}, T\right) + K\left(\frac{1}{2}, T\right) \ll J\left(\frac{1}{2}, T\right) + T,$$

i.e., $J(\frac{1}{2}, T) \gg T(\log T)^{k^2}$. If $K(\frac{1}{2}, T) > T$, then by Lemma 4.4, we see that

$$K(\sigma, T) \ll K\left(\frac{1}{2}, T\right) N^{-\frac{1}{v} \frac{4\sigma-2}{3} (\frac{3}{2}-\varepsilon)} \ll K\left(\frac{1}{2}, T\right) N^{\frac{1}{v} (1-\varepsilon)(1-2\sigma)}. \quad (5.3)$$

Now we choose $N = T^{\frac{1}{2}}$ and $\varepsilon = \frac{1}{2}$. By (5.1)–(5.3), we have

$$H(\sigma, T) \ll J(\sigma, T) + K\left(\frac{1}{2}, T\right) T^{\frac{1}{2v}(\frac{1}{2}-\sigma)} \ll J(\sigma, T) + \left[H\left(\frac{1}{2}, T\right) + J\left(\frac{1}{2}, T\right)\right] T^{\frac{1}{2v}(\frac{1}{2}-\sigma)}.$$

Hence, either

$$H(\sigma, T) \ll H\left(\frac{1}{2}, T\right) T^{\frac{1}{2v}(\frac{1}{2}-\sigma)} \quad (5.4)$$

or

$$H(\sigma, T) \ll J(\sigma, T) + J\left(\frac{1}{2}, T\right) T^{\frac{1}{2v}(\frac{1}{2}-\sigma)}. \quad (5.5)$$

Take $\sigma = \frac{1}{2} + \frac{C}{\log T}$. By Lemma 3.5 and (5.4), we have

$$C^{-k^2} T(\log T)^{k^2} = T\left(\sigma - \frac{1}{2}\right)^{-k^2} \ll H(\sigma, T) \ll H\left(\frac{1}{2}, T\right) T^{\frac{1}{2v}(\frac{1}{2}-\sigma)} \ll T(\log T)^{k^2} e^{-\frac{C}{2v}}.$$

Thus, $e^{\frac{C}{2v}} \leq c_{10}^{k^2}$ for some $c_{10} > 0$, which is impossible when C is sufficiently large. Therefore, (5.5) is valid. Now (5.5) and Lemma 4.3 imply

$$H(\sigma, T) \ll J\left(\frac{1}{2}, T\right)^{\frac{3}{2}-\sigma} T^{\sigma-\frac{1}{2}} + J\left(\frac{1}{2}, T\right) T^{\frac{1}{2v}(\frac{1}{2}-\sigma)}. \quad (5.6)$$

Taking $\sigma = \frac{1}{2} + \frac{C}{\log T}$ in (5.6) and applying Lemma 3.5, we get

$$T(\log T)^{k^2} \ll H\left(\frac{1}{2}, T\right) \ll J\left(\frac{1}{2}, T\right)^{1-\frac{C}{\log T}} + J\left(\frac{1}{2}, T\right) e^{-\frac{C}{2v}} \ll J\left(\frac{1}{2}, T\right).$$

This completes the proof of (2.2).

Next we prove (2.3). Note that in this case $k = \frac{u}{v}$ with $0 \leq u = k \leq \frac{2}{m} - \varepsilon$ for any $\varepsilon > 0$, and $v = 1$. By the definition of $g(s, \pi)$, we have

$$J(\sigma, T) \ll H(\sigma, T) + K(\sigma, T). \quad (5.7)$$

If $K(\frac{1}{2}, T) \leq T$, then by Lemma 3.5 and (5.7) with $\sigma = \frac{1}{2}$, we have

$$J\left(\frac{1}{2}, T\right) \ll H\left(\frac{1}{2}, T\right) + K\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2}.$$

If $K(\frac{1}{2}, T) > T$, then by Lemma 4.4, we see that

$$K(\sigma, T) \ll K\left(\frac{1}{2}, T\right) N^{-\frac{4\sigma-2}{3}(\frac{3}{2}-\varepsilon)} \ll K\left(\frac{1}{2}, T\right) N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)}.$$

This estimate combined with (5.7) and (5.2) gives

$$\begin{aligned} J(\sigma, T) &\ll H(\sigma, T) + K\left(\frac{1}{2}, T\right) N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)} \\ &\ll H(\sigma, T) + \left[H\left(\frac{1}{2}, T\right) + J\left(\frac{1}{2}, T\right)\right] N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)}. \end{aligned}$$

Hence, either

$$J(\sigma, T) \ll J\left(\frac{1}{2}, T\right) N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)} \quad (5.8)$$

or

$$J(\sigma, T) \ll H(\sigma, T) + H\left(\frac{1}{2}, T\right) N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)}. \quad (5.9)$$

Now we take $N = T$ and $\sigma = \frac{1}{2} + \frac{C}{\log T}$. Recall that $k \leq \frac{2}{m} - \varepsilon$. Then by (5.8) and Lemma 4.2, we have

$$J\left(\frac{1}{2}, T\right) \ll J(\sigma, T) T^{mk(\sigma-\frac{1}{2})} \ll J\left(\frac{1}{2}, T\right) T^{\varepsilon(m-\frac{4}{3})(\frac{1}{2}-\sigma)},$$

i.e., $e^{\varepsilon(m-\frac{4}{3})C} \leq C(m, k)$, which is false for C sufficiently large. Therefore, (5.9) holds. By Lemma 3.5 and (5.9) with $\sigma = \frac{1}{2} + \frac{C}{\log T}$, we have $J(\frac{1}{2}, T) \ll H(\frac{1}{2}, T) \ll T(\log T)^{k^2}$. This completes the proof of (2.3).

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