

A Criterion of Normality Concerning Holomorphic Functions Whose Derivative Omits a Function*

Xiaojun LIU¹ Yasheng YE¹

Abstract The authors discuss the normality concerning holomorphic functions and get the following result. Let \mathcal{F} be a family of holomorphic functions on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k , where $k \geq 2$ is an integer. And let $h(z) \not\equiv 0$ be a holomorphic function on D . Assume also that the following two conditions hold for every $f \in \mathcal{F}$: (a) $f(z) = 0 \implies |f^{(k)}(z)| < |h(z)|$; (b) $f^{(k)}(z) \neq h(z)$. Then \mathcal{F} is normal on D .

Keywords Normal family, Holomorphic functions, Omitted function

2000 MR Subject Classification 30D35

1 Introduction

In [5], X. C. Pang, D. G. Yang and L. Zalcman proved the following theorem.

Theorem 1.1 (see [5]) *Let \mathcal{F} be a family of meromorphic functions on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least $k + 3$, where $k \geq 1$ is an integer, and let $h(z) (\not\equiv 0)$ be a holomorphic function on D . Suppose that for every $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, $z \in D$, then \mathcal{F} is a normal family on D .*

Also in [5], they considered reducing the multiplicity for the zeros of f and proved the following result.

Theorem 1.2 (see [5]) *Let \mathcal{F} be a family of meromorphic functions on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least $k + 2$, where $k \geq 1$ is an integer. Let $h(z) (\not\equiv 0)$ be a holomorphic function on D , all of whose zeros have multiplicity at least 2. Suppose that for every $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, $z \in D$, then \mathcal{F} is a normal family on D .*

The question is that can the restriction for the zeros of $f(z)$ with multiplicity at least $k + 2$ be reduced to k ? In this paper, we continue to study the above problem and get the confirmed result.

Theorem 1.3 *Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k , where $k \geq 2$ is an integer. And let $h(z) \not\equiv 0$ be a holomorphic*

Manuscript received February 28, 2011. Revised June 4, 2011.

¹Department of Mathematics, College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China. E-mail: xiaojunliu2007@hotmail.com yashengye@yahoo.com.cn

*Project supported by the National Natural Science Foundation of China (No. 11071074) and the Outstanding Youth Foundation of Shanghai (No. slg10015).

function on D . Assume also that the following two conditions hold for every $f \in \mathcal{F}$:

- (a) $f(z) = 0 \implies |f^{(k)}(z)| < |h(z)|$;
- (b) $f^{(k)}(z) \neq h(z)$.

Then \mathcal{F} is normal on D .

The following counterexample shows that Theorem 1.3 does not hold for meromorphic functions when $k = 2$.

Example 1.1 Let $D = \Delta = \{z : |z| < 1\}$ be a unit disc,

$$f_n(z) = \frac{(z + \frac{1}{n})^2(z + \frac{2}{n})^2}{6(z + \frac{6}{n})} \quad \text{and} \quad h(z) = z.$$

It is easy to check that f_n are meromorphic on Δ and have only two zeros $z_1^{(n)} = -\frac{1}{n}$ and $z_2^{(n)} = -\frac{2}{n}$ with multiplicity 2. By calculation, we have

$$f_n''(z) = z + \frac{400}{3n^4(z + \frac{6}{n})^3}.$$

So

$$\begin{aligned} f_n = 0 &\implies z_1^{(n)} = -\frac{1}{n}, \quad z_2^{(n)} = -\frac{2}{n} \\ &\implies |f_n''(z_{1,2}^{(n)})| = |z_{1,2}^{(n)}| \left| 1 + \frac{400}{3n^4 z_{1,2}^{(n)} (z_{1,2}^{(n)} + \frac{6}{n})^3} \right| < |z_{1,2}^{(n)}| = |h(z_{1,2}^{(n)})| \\ &\implies |f_n''| < |h| \quad \text{and} \quad f_n''(z) = z + \frac{400}{3n^4(z + \frac{6}{n})^3} \neq z = h(z). \end{aligned}$$

But, $\mathcal{F} = \{f_n\}$ is not normal on Δ .

So, the question is what about the case $k \geq 3$?

Question 1.1 Let \mathcal{F} be a family of functions meromorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k , where $k \geq 3$ is an integer. And let $h(z) \not\equiv 0$ be a holomorphic function on D . Assume also that the following two conditions hold for every $f \in \mathcal{F}$:

- (a) $f(z) = 0 \implies |f^{(k)}(z)| < |h(z)|$;
- (b) $f^{(k)}(z) \neq h(z)$.

Then is \mathcal{F} normal on D ?

Let us set some notations. Throughout this paper, D is a domain in \mathbb{C} . For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z : |z - z_0| < r\}$ and $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$. The unit disc is denoted by Δ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We write $f_n(z) \xrightarrow{x} f(z)$ on D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric, uniformly on compact subsets of D , and $f_n \implies f$ on D if the convergence is in the Euclidean metric. The spherical derivative of the meromorphic function f at the point z is denoted by $f^\sharp(z)$.

Frequently, given a sequence $\{f_n\}_1^\infty$ of functions, we need to extract an appropriate subsequence. This necessity may recur within a single proof. To avoid the awkwardness of multiple indices, we again denote the extracted subsequence by $\{f_n\}$ (rather than, say, $\{f_{n_k}\}$) and signal

this operation by writing “taking a subsequence and renumbering” or simply “renumbering”. The same convention applies to the sequences of constants.

The plan of this paper is as follows. In Section 2, we state a number of preliminary results. Then, in Section 3, we prove Theorem 1.3.

2 Preliminary Results

The following lemma is taken from [2, p. 145], [5, p. 259] and [10, pp. 216–217].

Lemma 2.1 *Let \mathcal{F} be a family of functions meromorphic on a domain D , all of whose zeros have multiplicity at least k , and suppose that there exists an $A \geq 1$, such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if \mathcal{F} is not normal at $z_0 \in D$, for each $0 \leq \alpha \leq k$, there exist*

- (a) *points $z_n \rightarrow z_0$;*
- (b) *functions $f_n \in \mathcal{F}$;*
- (c) *positive numbers $\rho_n \rightarrow 0^+$,*

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + f_n \zeta) \xrightarrow{\chi} g(\zeta)$ on \mathbb{C} , where g is a nonconstant meromorphic function on \mathbb{C} , such that for every $\zeta \in \mathbb{C}$, $g^\#(\zeta) \leq g^\#(0) = kA + 1$.

Lemma 2.2 (see [1, pp. 118–119, 122–123]) *Let f be a meromorphic function on \mathbb{C} . If $f^\#$ is uniformly bounded on \mathbb{C} , then the order of f is at most 2. If f is an entire function, then the order of f is at most 1.*

Lemma 2.3 *Let f be an entire function of finite order $\rho(f)$ on \mathbb{C} , all of whose zeros have multiplicity at least k , where $k \geq 2$ is an integer and $a \neq 0$ is a constant. Suppose that $\rho(f) \leq 1$ and $f(z)$ satisfies the following two conditions:*

- (a) $f(z) = 0 \implies |f^{(k)}(z)| < |a|$;
- (b) $f^{(k)}(z) \neq a$.

Then

$$f(z) = \frac{b(z - z_0)^k}{k!},$$

where $b \neq a$ and z_0 are constants.

Proof We separate it into two cases.

Case 1 f is a transcendental entire function on \mathbb{C} .

By $\rho(f^{(k)}) = \rho(f) \leq 1$ and $f^{(k)} \neq a$, we have $f^{(k)}(z) = a + B \exp(A\zeta)$, where $A, B \in \mathbb{C}^*$ are two constants.

By calculation,

$$f(z) = \frac{az^k}{k!} + a_{k-1}z^{k-1} + \cdots + a_0 + BA^{-k} \exp(A\zeta),$$

where a_{k-1}, \dots, a_0 are constants.

So there exist $z_m, z_m \rightarrow \infty$, such that $f(z_m) = 0, m = 1, 2, \dots$. By the condition that all zeros of f have multiplicity at least k (≥ 2), we have $f'(z_m) = 0$. Set

$$P(z) = A^{-1}f'(z) - f(z).$$

It is obvious to see that P is a polynomial and $P(z_m) = 0$, $m = 1, 2, \dots$. Then we have $P(z) \equiv 0$, $f(z) = C \exp(Az)$, where $C \neq 0$ is a constant, a contradiction.

Case 2 f is a polynomial.

Then by $f^{(k)} \neq a$, we have $f^{(k)}(z) = b$, where $b \neq a$ is a constant. Since all zeros of f have multiplicity at least k (≥ 2),

$$f(z) = \frac{b(z - z_0)^k}{k!},$$

where z_0 is a constant.

Lemma 2.4 *Let $\{f_n\}$ be a sequence of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k and $\{h_n\}$ be a sequence of functions analytic on D such that $h_n(z) \Rightarrow h(z)$ on D , where $h(z) \neq 0$ for $z \in D$ and $k \geq 2$ is an integer. Suppose that, for each n , $f_n(z) = 0 \Rightarrow |f_n^{(k)}(z)| < |h_n(z)|$ and $f_n^{(k)}(z) \neq h_n(z)$. Then $\{f_n\}$ is normal on D .*

Proof Suppose to the contrary that there exists a $z_0 \in D$ such that $\{f_n\}$ is not normal at z_0 . The convergence of $\{h_n\}$ to h implies that, in some neighborhood of z_0 , we have $f_n(z) = 0 \Rightarrow |f_n^{(k)}(z)| \leq |h(z_0)| + 1$ (for large enough n). Thus we can apply Lemma 2.1 with $\alpha = k$ and $A = |h(z_0)| + 1$. So we can take an appropriate subsequence of $\{f_n\}$ (denoted also by $\{f_n\}$ after renumbering), together with points $z_n \rightarrow z_0$ and positive numbers $\rho_n \rightarrow 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k} \xrightarrow{\chi} g(\zeta), \quad \text{on } \mathbb{C},$$

where g is a nonconstant entire function, all of whose zeros have multiplicity at least k and $g^\sharp(\zeta) \leq g^\sharp(0) = k(|h(z_0)| + 1) + 1$.

We claim that $g = 0 \Rightarrow |g^{(k)}| \leq |h(z_0)|$ and $g^{(k)} \neq h(z_0)$.

In fact, if there exists a $\zeta_0 \in \mathbb{C}$, such that $g(\zeta_0) = 0$, then since $g(\zeta) \not\equiv 0$, there exist ζ_n , $\zeta_n \rightarrow \zeta_0$, such that if n is sufficiently large,

$$g_n(\zeta_n) = \frac{f_n(z_n + \rho_n \zeta_n)}{\rho_n^k} = 0.$$

Thus $f_n(z_n + \rho_n \zeta_n) = 0$, so that $|f_n^{(k)}(z_n + \rho_n \zeta_n)| < |h_n(z_n + \rho_n \zeta_n)|$, i.e., $|g_n^{(k)}(\zeta_n)| < |h_n(z_n + \rho_n \zeta_n)|$. Since $|g^{(k)}(\zeta_0)| = \lim_{n \rightarrow \infty} |g_n^{(k)}(\zeta_n)| \leq |h(z_0)|$, we have established the first part of the claim.

Now, suppose that there exists a $\zeta_0 \in \mathbb{C}$, such that $g^{(k)}(\zeta_0) = h(z_0)$. If $g^{(k)}(\zeta) \equiv h(z_0)$, then we have $g^\sharp(0) \leq k|h(z_0)|$, which contradicts $g^\sharp(0) = k(|h(z_0)| + 1) + 1$. Thus $g^{(k)}$ is not constant. So by Hurwitz's theorem, there exist ζ_n , $\zeta_n \rightarrow \zeta_0$, such that

$$f_n^{(k)}(z_n + \rho_n \zeta_n) - h_n(z_n + \rho_n \zeta_n) = g_n^{(k)}(\zeta_n) - h_n(z_n + \rho_n \zeta_n) = 0,$$

which contradicts $f_n^{(k)} \neq h_n$. This completes the proof of the claim.

By Lemma 2.3,

$$g(\zeta) = \frac{b}{k!}(\zeta - \zeta_0)^k,$$

where $\zeta_0 \in \mathbb{C}$ and $b \neq h(z_0)$ are constants. Since $g(\zeta_0) = 0$, $|g^{(k)}(\zeta_0)| = |b| \leq |h(z_0)|$. We have $g^\sharp(0) \leq k|b| \leq k|h(z_0)|$, a contradiction. The lemma is proved.

Lemma 2.5 *Let h be a holomorphic function on D with a zero of order ℓ (≥ 1) at $z_0 \in D$, $\{f_n\}_1^\infty$ be a sequence of functions such that $\{f_n\}$ and h satisfy conditions (a) and (b) of Theorem 1.3. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of nonzero numbers such that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. Then*

(a) $\left\{ \frac{f_n(z_0 + \alpha_n \zeta)}{\alpha_n^{k+\ell}} \right\}_{n=1}^\infty$ is normal in \mathbb{C}^* .

In addition, if

$$\frac{f_n(z_0 + \alpha_n \zeta)}{\alpha_n^{k+\ell}} \Rightarrow G(\zeta), \quad \text{on } \mathbb{C}^* \text{ (or on } \mathbb{C}),$$

where $G(\zeta) \neq 0$, then

(b)

(i) $G(\zeta_0) = 0 \Rightarrow |G^{(k)}(\zeta_0)| \leq |\zeta_0^\ell|$ for every $\zeta_0 \in \mathbb{C}^*$ (or for every $\zeta_0 \in \mathbb{C}$);

(ii) If $G^{(k)}(\zeta) \neq \zeta^\ell$, then $G^{(k)}(\zeta) \neq \zeta^\ell$.

Proof Without loss of generality, we may assume that $z_0 = 0$. In a neighborhood of the origin, we have $h(z) = z^\ell b(z)$, where $b(z)$ is analytic, $b(0) \neq 0$. Without loss of generality, we can assume that $b(0) = 1$. Define $r_n(\zeta) = \zeta^\ell b(\alpha_n \zeta)$. We will show that the assumptions of Lemma 2.4 hold in \mathbb{C}^* for the sequence $\{G_n(\zeta)\}_{n=1}^\infty$, $G_n(\zeta) := \frac{f_n(\alpha_n \zeta)}{\alpha_n^{k+\ell}}$ and $\{r_n(\zeta)\}_{n=1}^\infty$. First, we have that $r_n(\zeta) \Rightarrow \zeta^\ell$ on \mathbb{C} and $\zeta^\ell \neq 0$ in \mathbb{C}^* . Assume that $G_n(\zeta) = 0$. Hence $f_n(\alpha_n \zeta) = 0$ and $|f_n^{(k)}(\alpha_n \zeta)| < |(\alpha_n \zeta)^\ell b(\alpha_n \zeta)|$, and we get $|G_n^{(k)}(\zeta)| < |r_n(\zeta)|$. Obviously, we have

$$G_n^{(k)}(\zeta) = \frac{f_n^{(k)}(\alpha_n \zeta)}{\alpha_n^k} \neq \frac{h(\alpha_n \zeta)}{\alpha_n^\ell} = r_n(\zeta),$$

which means that the assumptions of Lemma 2.4 hold. Hence we deduce that $\{G_n(\zeta)\}$ is normal in \mathbb{C}^* , and (a) is proved.

Suppose now that $G(\zeta_0) = 0$. Then there exist $\zeta_n \rightarrow \zeta_0$ such that $G_n(\zeta_n) = 0$, i.e., $f_n(\alpha_n \zeta_n) = 0$. It then follows that $|f_n^{(k)}(\alpha_n \zeta_n)| < |\alpha_n^\ell \zeta_n^\ell b(\alpha_n \zeta_n)|$, and this implies $|G_n^{(k)}(\zeta_n)| < |\zeta_n^\ell b(\alpha_n \zeta_n)|$. Letting $n \rightarrow \infty$, $|G^{(k)}(\zeta_0)| \leq |\zeta_0^\ell|$, so (i) of (b) is proved.

For the proof of (ii), observe first that

$$\frac{f_n^{(k)}(\alpha_n \zeta)}{\alpha_n^k b(\alpha_n \zeta)} = G_n^{(k)}(\alpha_n \zeta) b(\alpha_n \zeta) \xrightarrow{\chi} G^{(k)}(\zeta), \quad \text{on } \mathbb{C}. \quad (2.1)$$

If $G^{(k)}(\zeta_0) = \zeta_0^\ell$, then by (2.1) we have $\zeta_n \rightarrow \zeta_0$ such that

$$f_n^{(k)}(\alpha_n \zeta_n) = [\alpha_n^\ell b(\alpha_n \zeta_n)] \zeta_n^\ell = h(\alpha_n \zeta_n),$$

which contradicts the condition (b) of Theorem 1.3. This completes the proof of the lemma.

3 Proof of Theorem 1.3

By Lemma 2.4, \mathcal{F} is normal at every point $z_0 \in D$ at which $h(z_0) \neq 0$ (so that \mathcal{F} is quasinormal in D). Consider $z_0 \in D$ such that $h(z_0) = 0$. Without loss of generality, we can assume $z_0 = 0$. Then

$$h(z) = z^\ell b(z), \quad (3.1)$$

where $\ell (\geq 1)$ is an integer, $b(z) \neq 0$ is an analytic function in $\Delta(0, \delta)$ and we can assume also that $b(0) = 1$. We take a subsequence $\{f_n\}_1^\infty \subset \mathcal{F}$, and we want to prove that $\{f_n\}$ is normal at $z = 0$. Suppose by negation that $\{f_n\}$ is not normal at $z = 0$. Since $\{f_n\}$ is normal in $\Delta'(0, \delta)$, we can assume (after renumbering) that $f_n \Rightarrow F$ on $\Delta'(0, \delta)$. If $F(z) \neq \infty$, then it is a holomorphic function. Hence by the maximum principle, F extends to be analytic also at $z = 0$. So $f_n \Rightarrow F$ on $\Delta(0, \delta)$, and we are done. Hence we assume that

$$f_n(z) \Rightarrow \infty, \quad \text{on } \Delta'(0, \delta). \quad (3.2)$$

Define $\mathcal{F}_1 = \{F_n = \frac{f_n}{h} : n \in \mathbb{N}\}$. It is enough to prove that \mathcal{F}_1 is normal in $\Delta(0, \delta)$. Indeed, if (after renumbering) $\frac{f_n(z)}{h} \Rightarrow H(z)$ on $\Delta(0, \delta)$, then since $h \neq 0$ in $\Delta'(0, \delta)$, it follows from (3.2) that $H(z) \equiv \infty$ in $\Delta'(0, \delta)$, and thus $H(z) \equiv \infty$ also in $\Delta(0, \delta)$. In particular, $\frac{f_n}{h}(z) \neq 0$ on each compact subset of $\Delta(0, \delta)$ for large enough n . Since $h \neq 0$ on $\Delta'(0, \delta)$ and $f_n(0) \neq 0$ for every $n \geq 1$, by the assumptions of the theorem, we obtain $f_n(z) \neq 0$ on each compact subset of $\Delta(0, \delta)$ for large enough n . Then by the minimum principle, it follows from (3.2) that $f_n(z) \Rightarrow \infty$ on $\Delta(0, \delta)$, and this implies the normality of \mathcal{F} . So suppose to the contrary that \mathcal{F}_1 is not normal at $z = 0$. By Lemma 2.1 and the assumptions of Theorem 1.3, there exist (after renumbering) points $z_n \rightarrow 0$, $\rho_n \rightarrow 0^+$ and a nonconstant meromorphic function on \mathbb{C} , $g(\zeta)$ such that

$$g_n(\zeta) = \frac{F_n(z_n + \rho_n \zeta)}{\rho_n^k} = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k h(z_n + \rho_n \zeta)} \xrightarrow{\chi} g(\zeta), \quad \text{on } \mathbb{C}, \quad (3.3)$$

all of whose zeros have multiplicity at least k and

$$\text{for every } \zeta \in \mathbb{C}, \quad g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1, \quad (3.4)$$

where $A > 1$ is a constant. Here we have used Lemma 2.1 with $\alpha = k$. Observe that $g_n(\zeta) = 0$ implies $|g_n^{(k)}(\zeta)| < 1$ and so A can be chosen to be any number such that $A \geq 1$. After renumbering, we can assume that $\{\frac{z_n}{\rho_n}\}_{n=1}^\infty$ converges. We separate it now into two cases.

Case 1

$$\frac{z_n}{\rho_n} \rightarrow \infty. \quad (3.5)$$

Claim (1) $g(\zeta) = 0 \Rightarrow |g^{(k)}(\zeta)| \leq 1$; (2) $g^{(k)}(\zeta) \neq 1$.

Proof of the Claim From (3.3) and the fact that $h(z) \neq 0$ in $\Delta'(0, \delta)$, we have that g is an entire function. Suppose $g(\zeta_0) = 0$. Since $g(\zeta) \not\equiv 0$, there exist $\zeta_n \rightarrow \zeta_0$, such that $g_n(\zeta_n) = 0$. Thus $f_n(z_n + \rho_n \zeta_n) = 0$. By assumption, we then have $f_n^{(j)}(z_n + \rho_n \zeta_n) = 0$ and $|f_n^{(k)}(z_n + \rho_n \zeta_n)| < |h(z_n + \rho_n \zeta_n)|$, where $j = 2, 3, \dots, k-1$. Thus $|g_n^{(k)}(\zeta_n)| < 1$. Letting $n \rightarrow \infty$, we obtain $|g^{(k)}(\zeta_0)| \leq 1$.

If there exists a $\zeta_0 \in \mathbb{C}$ such that $g^{(k)}(\zeta_0) = 1$, then there exists a neighborhood $U = U(\zeta_0)$ of ζ_0 , such that the functions $g_n^{(j)}$ are analytic on U for sufficiently large n , $j = 0, 1, \dots, k+1$.

Obviously,

$$\begin{aligned} g_n^{(k)}(\zeta) &= F_n^{(k)}(z_n + \rho_n \zeta) = \left(\frac{f_n(z)}{h(z)} \right)^{(k)} \Big|_{z=z_n+\rho_n \zeta} \\ &= \left[\frac{f_n^{(k)}(z)}{h(z)} + \sum_{j=1}^k \binom{k}{j} f_n^{(k-j)}(z) \left(\frac{1}{h(z)} \right)^{(j)} \right] \Big|_{z=z_n+\rho_n \zeta}. \end{aligned}$$

By Leibniz's formula, we have that

$$f_n^{(k-j)}(z) = [F_n(z)h(z)]^{(k-j)} = \sum_{s=0}^{k-j} \binom{k-j}{s} \rho_n^{j+s} g_n^{(k-j-s)} \left(\frac{z-z_n}{\rho_n} \right) h^{(s)}(z)$$

and

$$\left(\frac{1}{h(z)} \right)^{(j)} = [z^{-\ell} \tilde{b}(z)]^{(j)} = z^{-\ell-j} [(-1)^j \ell(\ell+1) \cdots (\ell+j-1) \tilde{b}(z) + P(z)],$$

where $\tilde{b}(z) = \frac{1}{b(z)}$, and $P(z)$ is holomorphic on $\Delta(0, \delta)$ with $P(0) = 0$.

Since

$$\begin{aligned} \rho_n^{j+s} h^{(s)}(z) z^{-\ell-j} \Big|_{z=z_n+\rho_n \zeta} &= \rho_n^{j+s} z^{\ell-s} Q(z) z^{-\ell-j} \Big|_{z=z_n+\rho_n \zeta} \\ &= \frac{\rho_n^{j+s}}{(z_n + \rho_n \zeta)^{j+s}} Q(z_n + \rho_n \zeta) \Rightarrow 0 \end{aligned}$$

on \mathbb{C} , where $Q(z)$ is holomorphic on $\Delta(0, \delta)$ and $Q(0) = \ell(\ell-1) \cdots (\ell-s+1) \neq 0$, we have

$$\begin{aligned} f_n^{(k-j)}(z) \left(\frac{1}{h(z)} \right)^{(j)} \Big|_{z=z_n+\rho_n \zeta} &= \sum_{s=0}^{k-j} \binom{k-j}{s} \rho_n^{j+s} g_n^{(k-j-s)} \left(\frac{z-z_n}{\rho_n} \right) h^{(s)}(z) \\ &\quad \times z^{-\ell-j} [(-1)^j \ell(\ell+1) \cdots (\ell+j-1) \tilde{b}(z) + P(z)] \Big|_{z=z_n+\rho_n \zeta} \\ &\Rightarrow 0, \quad \text{on } \mathbb{C} \setminus \{\text{the poles of } g\}. \end{aligned} \quad (3.6)$$

Now

$$\frac{f_n^{(k)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)} \Rightarrow g^{(k)}(\zeta), \quad \text{on } \mathbb{C} \setminus \{\text{the poles of } g\}.$$

So $\frac{f_n^{(k)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)}$ converges locally uniformly to $g^{(k)}(\zeta)$ on U . By (3.4) we deduce that $g^{(k)}(\zeta) \neq 1$. Thus there exist $\zeta_n \rightarrow \zeta_0$, such that $\frac{f_n^{(k)}(z_n + \rho_n \zeta_n)}{h(z_n + \rho_n \zeta_n)} = 1$. So

$$f_n^{(k)}(z_n + \rho_n \zeta_n) = h(z_n + \rho_n \zeta_n), \quad (3.7)$$

which contradicts the condition (b) of Theorem 1.3. Then the claim is proved.

Also by Lemma 2.3, we have

$$g(\zeta) = \frac{b}{k!} (\zeta - \zeta_0)^k,$$

where $\zeta_0 \in \mathbb{C}$ and $b \neq 1$ are constants. Since $g(\zeta_0) = 0$, $|g^{(k)}(\zeta_0)| = |b| \leq 1$. We have $g^\#(0) \leq k|b| \leq k$, a contradiction.

Case 2

$$\frac{z_n}{\rho_n} \rightarrow \alpha \in \mathbb{C}. \quad (3.8)$$

As before, we have $g(\zeta_0) = 0 \implies |g^{(k)}(\zeta_0)| \leq 1$. Now set

$$G_n(\zeta) = \frac{f_n(\rho_n \zeta)}{\rho_n^{k+\ell}}.$$

From (3.3) and (3.8) we have

$$G_n(\zeta) \implies G(\zeta) = g(\zeta - \alpha)\zeta^\ell, \quad \text{on } \mathbb{C}.$$

Indeed,

$$\frac{f_n(\rho_n \zeta)}{\rho_n^{k+\ell}} = \frac{f_n(\rho_n \zeta)}{\rho_n^k h(\rho_n \zeta)} \cdot \frac{h(\rho_n \zeta)}{\rho_n^\ell} = \frac{f_n(z_n + \rho_n(\zeta - \frac{z_n}{\rho_n}))}{\rho_n^k h(z_n + \rho_n(\zeta - \frac{z_n}{\rho_n}))} \frac{(\rho_n \zeta)^\ell b(\rho_n \zeta)}{\rho_n^\ell}$$

(see [10, p. 7]). Since g has a pole of order ℓ at $\zeta = -\alpha$,

$$G(0) \neq 0, \infty. \quad (3.9)$$

We now consider several subcases, depending on the nature of G .

Case 2.1 G is a polynomial.

Since $\{f_n\}$ is not normal at $z = 0$, there exists (after renumbering) a sequence $z_n^* \rightarrow 0$ such that

$$f_n(z_n^*) = 0. \quad (3.10)$$

Otherwise, there is some δ' , $0 < \delta' < \delta$ such that (before renumbering) $f_n(z) \neq 0$ in $\Delta(0, \delta')$. Since $f_n(z) \implies \infty$ on $\Delta'(0, \delta)$, by the minimum principle, we would have that $f_n(z) \implies \infty$ on $\Delta(0, \delta)$, a contradiction to the non-normality of $\{f_n\}$ at $z = 0$. If G is a polynomial of degree $\ell \geq 1$, then by Lemma 2.5 and (3.9), all zeros of $G(\zeta)$ have multiplicity exactly k . We consider now two kinds of possibilities.

Case 2.1.1 $G^{(k)} \equiv \zeta^\ell$.

Since $k \geq 2$, we have $G^{(k-1)}(\zeta) = \frac{\zeta^{\ell+1}}{\ell+1} + C$ and $G^{(k-2)}(\zeta) = \frac{\zeta^{\ell+2}}{(\ell+1)(\ell+2)} + C\zeta + D$, where C and D are two constants. Since all zeros of G have multiplicity at least k , for any zero ζ_j of G , we have $G^{(k-2)}(\zeta_j) = G^{(k-1)}(\zeta_j) = 0$. So

$$\frac{\zeta_j^{\ell+1}}{\ell+1} + C = 0 \quad \text{and} \quad \frac{\zeta_j^{\ell+2}}{(\ell+1)(\ell+2)} + C\zeta_j + D = 0. \quad (3.11)$$

By calculation, we have $\frac{(\ell+1)C}{\ell+2}\zeta_j = -D$. If $CD = 0$, then by (3.11), $\zeta_j = 0$, a contradiction. So $CD \neq 0$ and $\zeta_j = -\frac{(\ell+2)D}{(\ell+1)C}$, which implies that G has only one zero ζ_0 . Thus

$$G = \frac{\ell!(\zeta - \zeta_0)^{k+\ell}}{(k+\ell)!}. \quad (3.12)$$

Since $G^{(k)} \equiv \zeta^\ell$, $\zeta_0 = 0$, a contradiction.

Case 2.1.2 $G^{(k)} \neq \zeta^\ell$.

By Lemma 2.5, we have $G(\zeta) = 0 \implies |G^{(k)}(\zeta)| \leq |\zeta^\ell|$ and $G^{(k)} \neq \zeta^\ell$. So G is a nonconstant polynomial and $G^{(k)} = \zeta^\ell + B$, where $B \neq 0$ is a constant. Since all zeros of G have multiplicity

at least k , for any zero ζ_j of G , we have $G^{(k-2)}(\zeta_j) = G^{(k-1)}(\zeta_j) = 0$. So

$$\frac{\zeta_j^{\ell+1}}{\ell+1} + B\zeta_j + C = 0 \quad \text{and} \quad \frac{\zeta_j^{\ell+2}}{(\ell+1)(\ell+2)} + \frac{B\zeta_j^2}{2} + C\zeta_j + D = 0. \quad (3.13)$$

By calculation, we have $\frac{\ell B}{2(\ell+2)}\zeta_j^2 + \frac{C(\ell+1)}{\ell+2}\zeta_j + D = 0$, which implies that G has at most two zeros ζ_1, ζ_2 . Then we divide it into two subcases.

Case 2.1.2(a) G has only one zero ζ_1 .

Set

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta - \zeta_1)^{k+\ell}. \quad (3.14)$$

Since $G^{(k)} = \zeta^\ell + B$, we have $\ell = 1$ and $\zeta_1 = -B$. So

$$G(\zeta) = \frac{(\zeta + B)^{k+1}}{(k+1)!}. \quad (3.15)$$

By Hurwitz's theorem, there exists a sequence $\zeta_{n,0} \rightarrow -B$, such that $G_n(\zeta_{n,0}) = 0$. If there exists a δ' , $0 < \delta' < \delta$, such that for every n (after renumbering), $f_n(z)$ has only one zero $z_{n,0} = \rho_n \zeta_{n,0}$ in $\Delta(0, \delta')$.

Set

$$H_n(z) = \frac{f_n(z)}{(z - z_{n,0})^{k+1}}.$$

Since $H_n(z)$ is a nonvanishing holomorphic function in $\Delta(0, \delta')$ and $H_n(z) \rightarrow \infty$ on $\Delta'(0, \delta)$, we can deduce as before by the minimum principle that $H_n(z) \rightarrow \infty$ on $\Delta(0, \delta')$. But

$$H_n(2z_{n,0}) = \frac{f_n(2z_{n,0})}{z_{n,0}^{k+1}} = \frac{G_n(2\zeta_{n,0})}{\zeta_{n,0}^{k+1}} \rightarrow \frac{1}{(k+1)!}, \quad (3.16)$$

a contradiction. Thus, we can assume, after renumbering, that for every $\delta' > 0$, f_n has at least two zeros in $\Delta(0, \delta')$ for large enough n . Thus, there exists another sequence of points $z_{n,1} = \rho_n \zeta_{n,1}$, tending to zero, where $z_{n,1}$ is also a zero of $f_n(z)$ and $\zeta_{n,1} \rightarrow \infty$, as $n \rightarrow \infty$. We can also assume that $z_{n,1}$ is the closest zero to the origin of f_n , except $z_{n,0}$. Now set $c_n = \frac{z_{n,0}}{z_{n,1}}$ and define $K_n(\zeta) = \frac{f_n(z_{n,1}\zeta)}{z_{n,1}^{k+1}}$. By Lemma 2.5, $\{K_n(\zeta)\}$ is normal in \mathbb{C}^* . Now, if $\{K_n\}$ is normal at $\zeta = 0$, then after renumbering we can assume that

$$K_n(\zeta) \rightarrow K(\zeta), \quad \text{on } \mathbb{C}.$$

Since $K_n(c_n) = 0$ and $c_n \rightarrow \infty$, letting $n \rightarrow \infty$, we obtain $K(0) = 0$. Also we have $K^{(k)}(\zeta) \equiv \zeta$ or $K^{(k)}(\zeta) \neq \zeta$, by $K_n^{(k)}(\zeta) = \frac{f_n^{(k)}(z_{n,1}\zeta)}{z_{n,1}^{k+1}} \neq \zeta b(z_{n,1}\zeta)$.

If $K^{(k)}(\zeta) \equiv \zeta$, by $K(0) = 0$, we have $K(\zeta) = \frac{\zeta^{k+1}}{(k+1)!}$, which contradicts $K(1) = 0$.

If $K^{(k)}(\zeta) \neq \zeta$, by Lemma 2.5, we have $K(\zeta) = 0 \Rightarrow |K^{(k)}(\zeta)| \leq |\zeta|$ and then $K^{(k)}(0) = 0$, a contradiction.

Hence we can deduce that $\{K_n\}$ is not normal at $\zeta = 0$. Since $K_n(\zeta)$ is holomorphic in Δ , we have

$$K_n(\zeta) \rightarrow \infty, \quad \text{on } \mathbb{C}^*.$$

But $K_n(1) = 0$, a contradiction.

Case 2.1.2(b) G has exactly two distinct zeros ζ_1, ζ_2 .

By $G^{(k+1)} = \ell\zeta^{\ell-1}$, we have that none of the two zeros of G has multiplicity at least $k+2$.

If both of the two zeros of G has multiplicity exactly $k+1$, then we may assume that

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta - \zeta_1)^{k+1}(\zeta - \zeta_2)^{k+1}. \quad (3.17)$$

Since $G^{(k)}(\zeta) = \zeta^\ell + B$, by calculation, we have $\ell = k+2$ and $\zeta_1 + \zeta_2 = 0$, $\zeta_1\zeta_2 = 0$, a contradiction.

If only one of the two zeros of G have multiplicity exactly $k+1$, then we may assume that

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta - \zeta_1)^{k+1}(\zeta - \zeta_2)^k. \quad (3.18)$$

By (3.18),

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta - \zeta_1) \left[\zeta^{2k} - k(\zeta_1 + \zeta_2)\zeta^{2k-1} + \left(k\zeta_1\zeta_2 + \binom{k}{2}(\zeta_1 + \zeta_2)^2 \right) \zeta^{2k-2} + \dots \right].$$

Since $G^{(k)}(\zeta) = \zeta^\ell + B$, by calculation, we have $\ell = k+1$ and

$$k(\zeta_1 + \zeta_2) + \zeta_1 = 0, \quad k(\zeta_1 + \zeta_2)\zeta_1 + k\zeta_1\zeta_2 + \binom{k}{2}(\zeta_1 + \zeta_2)^2 = 0, \quad (3.19)$$

which means $\zeta_1 = 0$, a contradiction.

If both of the two zeros of G have multiplicity exactly k , then we may assume that

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta - \zeta_1)^k(\zeta - \zeta_2)^k. \quad (3.20)$$

Since $G^{(k)}(\zeta) = \zeta^\ell + B$, by calculation, we have $\ell = k$ and $\zeta_1 + \zeta_2 = 0$.

For $k \geq 3$, we also have $\zeta_1\zeta_2 = 0$, a contradiction.

For $k = 2$, we have

$$G(\zeta) = \frac{1}{12}(\zeta - \zeta_1)^2(\zeta + \zeta_1)^2. \quad (3.21)$$

By Hurwitz's theorem, there exist sequences $\zeta_{n,1} \rightarrow \zeta_1$, $\zeta_{n,2} \rightarrow -\zeta_1$, such that $G_n(\zeta_{n,j}) = 0$, $j = 1, 2$. If there exists a δ' , $0 < \delta' < \delta$, such that for every n (after renumbering), $f_n(z)$ has only two zeros $z_{n,j} = \rho_n\zeta_{n,j}$, $j = 1, 2$ in $\Delta(0, \delta')$.

Set

$$H_n(z) = \frac{f_n(z)}{(z - z_{n,1})^2(z - z_{n,2})^2}.$$

Since $H_n(z)$ is a nonvanishing holomorphic function in $\Delta(0, \delta')$ and $H_n(z) \Rightarrow \infty$ on $\Delta'(0, \delta)$, we can deduce as before by the minimum principle that $H_n(z) \Rightarrow \infty$ on $\Delta(0, \delta')$. But

$$H_n(2z_{n,1}) = \frac{f_n(2z_{n,1})}{z_{n,1}^2(2z_{n,1} - z_{n,2})^2} = \frac{G_n(2\zeta_{n,1})}{\zeta_{n,1}^2(2\zeta_{n,1} - \zeta_{n,2})^2} \rightarrow \frac{1}{12}, \quad (3.22)$$

a contradiction. Thus, we can assume, after renumbering, that for every $\delta' > 0$, f_n has at least two zeros in $\Delta(0, \delta')$ for large enough n . Thus, there exists another sequence of points

$z_{n,3} = \rho_n \zeta_{n,3}$ tending to zero, where $z_{n,3}$ is also a zero of $f_n(z)$ and $\zeta_{n,3} \rightarrow \infty$, as $n \rightarrow \infty$. We can also assume that $z_{n,3}$ is the closest zero to the origin of f_n , except $z_{n,j}$, $j = 1, 2$. Now set $c_{n,j} = \frac{z_{n,j}}{z_{n,3}}$, $j = 1, 2$ and define $K_n(\zeta) = \frac{f_n(z_{n,3}\zeta)}{z_{n,3}^4}$. By Lemma 2.5, $\{K_n(\zeta)\}$ is normal in \mathbb{C}^* . Now, if $\{K_n\}$ is normal at $\zeta = 0$, then after renumbering we can assume that

$$K_n(\zeta) \Rightarrow K(\zeta), \quad \text{on } \mathbb{C}.$$

Since $K_n(c_{n,j}) = 0$ and $c_{n,j} \rightarrow \infty$, $j = 1, 2$, letting $n \rightarrow \infty$, we obtain $K(0) = 0$. Also we have $K''(\zeta) \equiv \zeta^2$ or $K''(\zeta) \neq \zeta^2$, by $K_n''(\zeta) = \frac{f_n''(z_{n,3}\zeta)}{z_{n,3}^2} \neq \zeta^2 b(z_{n,3}\zeta)$.

If $K''(\zeta) \equiv \zeta^2$, by $K(0) = 0$, we have $K(\zeta) = \frac{\zeta^4}{12}$, which contradicts $K(1) = 0$.

If $K''(\zeta) \neq \zeta^2$, by Lemma 2.5, we have $K(\zeta) = 0 \Rightarrow |K''(\zeta)| \leq |\zeta^2|$ and then $K''(0) = 0$, a contradiction.

Hence we can deduce that $\{K_n\}$ is not normal at $\zeta = 0$. Since $K_n(\zeta)$ is holomorphic in Δ , we have

$$K_n(\zeta) \Rightarrow \infty, \quad \text{on } \mathbb{C}^*.$$

But $K_n(1) = 0$, a contradiction.

Case 2.2 $G(\zeta)$ is a transcendental entire function.

By Lemma 2.5, we have

$$G(\zeta) = 0 \Rightarrow |G^{(k)}(\zeta)| \leq |\zeta^\ell| \quad \text{and} \quad G^{(k)}(\zeta) \neq \zeta^\ell. \quad (3.23)$$

Since G is a transcendental entire function with order at most 1, we have

$$G^{(k)}(\zeta) = \zeta^\ell + B \exp(A\zeta), \quad (3.24)$$

where $A \neq 0$, $B \neq 0$ are two constants. By calculation,

$$G(\zeta) = \frac{\ell!}{(k+\ell)!} \zeta^{k+\ell} + a_{k-1} \zeta^{k-1} + \cdots + a_0 + BA^{-k} \exp(A\zeta). \quad (3.25)$$

Obviously, G has infinitely many zeros ζ_m on \mathbb{C} , and $\zeta_m \rightarrow \infty$, $m \rightarrow \infty$. By (3.23), $|G^{(k)}(\zeta_m)| = |\zeta_m^\ell + B \exp(A\zeta_m)| \leq |\zeta_m^\ell|$, there exists an $M > 0$, such that, for every m ,

$$\left| \frac{\exp(A\zeta_m)}{\zeta_m^\ell} \right| \leq M.$$

But

$$\left| \frac{G(\zeta_m)}{\zeta_m^\ell} \right| = \left| \frac{\ell!}{(k+\ell)!} \zeta_m^k + a_{k-1} \zeta_m^{k-1-\ell} + \cdots + a_0 \zeta_m^{-\ell} + \frac{BA^{-k} \exp(A\zeta_m)}{\zeta_m^\ell} \right| \rightarrow \infty,$$

a contradiction. The theorem is proved.

Acknowledgement The authors are very grateful to the referee for his or her many valuable suggestions.

References

- [1] Clunie, J. and Hayman, W. K., The spherical derivative of integral and meromorphic functions, *Comment. Math. Helv.*, **40**, 1966, 117–148.
- [2] Liu, X. J. and Nevo, S., A criterion of normality based on a single holomorphic function, *Acta Math. Sinica*, **27**, 2011, 141–154.
- [3] Pang, X. C., Bloch's principle and normal criterion, *Sci. China Ser. A*, **32**, 1989, 782–791.
- [4] Pang, X. C., Shared values and normal families, *Analysis*, **22**, 2002, 175–182.
- [5] Pang, X. C., Yang, D. G. and Zalcman, L., Normal families of meromorphic functions whose derivative omit a function, *Comput. Methods Funct. Theory*, **2**, 2002, 257–265.
- [6] Pang, X. C. and Zalcman, L., Normal families and shared values, *Bull. London Math. Soc.*, **32**, 2000, 325–331.
- [7] Pang, X. C. and Zalcman, L., Normal families of meromorphic functions with multiple zeros and poles, *Israel J. Math.*, **136**, 2003, 1–9.
- [8] Wang, Y. F. and Fang, M. L., Picard values and normal families of meromorphic functions with multiple zeros, *Acta Math. Sinica (N. S.)*, **14**, 1998, 17–26.
- [9] Zalcman, L., A heuristic principle in complex function theory, *Amer. Math. Monthly*, **82**, 1975, 813–817.
- [10] Zalcman, L., Normal families: new perspectives, *Bull. Amer. Math. Soc. (N. S.)*, **35**, 1998, 215–230.