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Free Boundary Value Problems for Abstract Elliptic Equations and Applications

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Abstract The free boundary value problems for elliptic differential-operator equations are studied. Several conditions for the uniform maximal regularity with respect to boundary parameters and the Fredholmness in abstract L_p-spaces are given. In application, the nonlocal free boundary problems for finite or infinite systems of elliptic and anisotropic type equations are studied.

Keywords Free boundary value problems, Sobolev-Lions type spaces, Differential-operator equations, Maximal L_p regularity, Banach spaces, Operator-valued multipliers, Interpolation of Banach spaces
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1 Introduction, Notations and Background

In the last years, the maximal regularity properties of boundary value problems (BVPs) for differential-operator equations (DOEs) have been applied to PDE, psedo DE and the different physical processes (see [3, 4, 10, 12, 15, 20, 22–26, 30, 33–39] and the references therein). In these works, BVPs were essentially considered in Hilbert space valued class of functions defined in fixed domains. The L_p-maximal regularity properties of differential operators in fixed domains were studied e.g. in [11, 26–28, 32, 37–39]. Maximal regularity properties for PDE in moving domain were studied e.g. in [21], and free BVPs were investigated e.g. in [11, 14] and the references therein. The main objective of the present paper is to discuss nonlocal BVPs in moving domains for ordinary DOE and free BVP for partial DOE in Banach-valued L_p spaces. More precisely,

(1) the boundaries depend on the perturbation parameter or on space variable;

(2) the boundary conditions are nonlocal;

(3) the operators appearing in the equations and in the boundary conditions are unbounded.

In the present work, the maximal L_p-regularity and the Fredholmness of this problem uniformly with respect to the boundary parameter are established. These results are applied to nonlocal BVPs for elliptic, quasi-elliptic partial differential equations and finite or infinite systems of PDEs in moving domains. In application part, we establish well-posedness of free BVPs for anisotropic elliptic equations in $L_{\mathbf{p}}$, $\mathbf{p}=(p_1,p)$ (i.e., Lebesgue spaces with mixed norm) and L_p separability for infinite systems of elliptic equations.

Let E be a Banach space, and $L_p(\Omega,E)$ denote the space of strongly measurable E-valued

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functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm given by

$$\|f\|_{L_p} = \|f\|_{L_p(\Omega;E)} = \Big(\int_\Omega \|f(x)\|_E^p \mathrm{d}x \Big)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

The Banach space E is said to be a UMD-space (see [6–7]) if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \mathrm{d}y$$

is bounded in $L_p(R, E)$, $p \in (1, \infty)$. UMD spaces include e.g. L_p, l_p and Lorentz spaces L_{pq}, $p, q \in (1, \infty)$.

Let $\mathbb{C}$ be the set of the complex numbers and

$$S_\varphi = \{\lambda; \lambda \in \mathbb{C}, |\arg \lambda| \le \varphi\} \cup \{0\}, \quad 0 \le \varphi < \pi.$$

A linear operator A is said to be φ-positive (or positive) in a Banach space E if $D(A)$ is dense on E and

$$\|(A + \lambda I)^{-1}\|_{B(E)} \le M(1 + |\lambda|)^{-1}$$

for any $\lambda \in S_\varphi$, where $\varphi \in [0, \pi)$, I is the identity operator in E, and $B(E)$ is the space of bounded linear operators in E. Sometimes $A + \lambda I$ is written as $A + \lambda$ and denoted by A_λ. It is known that there exist the fractional powers A^θ of a positive operator A (see [31, Section 1.15.1]).

The operator $A(s)$ is said to be φ-positive (or positive) in E uniformly with respect to s if $D(A(s))$ is independent of s, $D(A(s))$ is dense in E and

$$\|(A(s) + \lambda)^{-1}\| \le \frac{M}{1 + |\lambda|}$$

for all $\lambda \in S_\varphi$, $0 \le \varphi < \pi$, where M does not depend on s and λ. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with norm

$$\|u\|_{E(A^\theta)} = (\|u\|^p + \|A^\theta u\|^p)^{\frac{1}{p}}, \quad 1 \le p < \infty,\ 0 < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces. By $(E_1, E_2)_{\theta,p}$ $(0 < \theta < 1,\ 1 \le p \le \infty)$, we will denote the interpolation spaces obtained from $\{E_1, E_2\}$ by the K-method (see [31, Section 1.3.1]).

Let $S = S(\mathbb{R}^n; E)$ denote a Schwartz class, i.e., the space of E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$, equipped with its usual topology generated by seminorms. Let $S'(\mathbb{R}^n; E)$ denote the space of all continuous linear operators $L : S \to E$, equipped with the bounded convergence topology. Recall that $S(\mathbb{R}^n; E)$ is norm dense in $L_p(\mathbb{R}^n; E)$ when $1 \le p < \infty$. Let F denote the Fourier transformation. A function $\Psi \in C(\mathbb{R}^n; L(E_1, E_2))$ is called a Fourier multiplier from $L_p(\mathbb{R}^n; E_1)$ to $L_q(\mathbb{R}^n; E_2)$ if the map $u \to \Phi u = F^{-1}\Psi(\xi)Fu$, $u \in S(\mathbb{R}^n; E_1)$ is well defined and extends to a bounded linear operator

$$\Phi : \ L_p(\mathbb{R}^n; E_1) \to \ L_q(\mathbb{R}^n; E_2).$$

The set of all multipliers from $L_p(\mathbb{R}^n; E_1)$ to $L_q(\mathbb{R}^n; E_2)$ will be denoted by $M_p^q(E_1, E_2)$. For $E_1 = E_2 = E$, it will be denoted by $M_p^q(E)$. Most important facts on Fourier multipliers and some related reference can be found e.g. in [8, 10, 19, 32] and [31, Section 2.2.1]).

Let $W_s = \{\Psi_s \in M_p^q(E_1, E_2),\ s \in Q\}$ be a collection of multipliers in $M_p^q(E_1, E_2)$. We say that W_s is a uniform collection of multipliers if there exists a constant $M > 0$ independent of $s \in Q$, such that

$$\|F^{-1}\Psi_s F u\|_{L_q(\mathbb{R}^n;E_2)} \leq M\|u\|_{L_p(\mathbb{R}^n;E_1)}$$

for all $s \in Q$ and $u \in S(\mathbb{R}^n; E_1)$.

Let $\mathbb{N}$ denote the set of natural numbers. A set $W \subset B(E_1, E_2)$ is called R-bounded (see, e.g., [11]) if there is a positive constant C, such that for all $T_1, T_2, \cdots, T_m \in W$ and $u_1, u_2, \cdots, u_m \in E_1$, $m \in \mathbb{N}$,

$$\int_0^1 \Big\|\sum_{j=1}^m r_j(y) T_j u_j\Big\|_{E_2} \mathrm{d}y \leq C \int_0^1 \Big\|\sum_{j=1}^m r_j(y) u_j\Big\|_{E_1} \mathrm{d}y,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$ (see [8]). The smallest C for which the above estimate holds is called an R-bound of the collection W and denoted by $R(W)$.

A set $W_s \subset B(E_1, E_2)$ depending on parameters $s \in Q$ is called uniformly R-bounded with respect to s if there is a constant C independent of $s \in Q$, such that for all $T_1(s), T_2(s), \cdots$, $T_m(s) \in W_s$ and $u_1, u_2, \cdots$, $u_m \in E_1$, $m \in \mathbb{N}$,

$$\int_0^1 \Big\|\sum_{j=1}^m r_j(y) T_j(h) u_j\Big\|_{E_2} \mathrm{d}y \leq C \int_0^1 \Big\|\sum_{j=1}^m r_j(y) u_j\Big\|_{E_1} \mathrm{d}y.$$

It implies that $R(W_s) \leq C$ for all $s \in Q$. Let

$$\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{N}^n, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n),$$
$$\xi^\beta = \xi_1^{\beta_1}\xi_2^{\beta_2}\cdots\xi_n^{\beta_n}, \quad |\beta| = \sum_{k=1}^n \beta_k, \quad D_j = \frac{\partial}{\partial \xi_j},$$
$$D^\beta = D_1^{\beta_1} D_2^{\beta_2} \cdots D_n^{\beta_n}, \quad U_n = \{\beta, \beta_k \in \{0, 1\}\}.$$

Definition 1.1 *A Banach space E is said to be a space satisfying a multiplier condition if, for any $\Psi \in \mathbb{C}^{(n)}(\mathbb{R}^n; B(E))$, the R-boundedness of the set $\{\xi^{|\beta|} D^\beta \Psi(\xi) : \xi \in \mathbb{R}^n \backslash 0, \beta \in U_n\}$ implies that Ψ is a Fourier multiplier, i.e., $\Psi \in M_p^p(E)$ for any $p \in (1, \infty)$.*

The uniform R-boundedness of the set $\{\xi^{|\beta|} D^\beta \Psi_s(\xi) : \xi \in \mathbb{R}^n \backslash 0, \beta \in U\}$, i.e.,

$$\sup_{s \in Q} R(\{\xi^{|\beta|} D^\beta \Psi_s(\xi) \colon \xi \in \mathbb{R}^n \backslash 0, \beta \in U\}) \leq K$$

implies that Ψ_s is a uniform collection of Fourier multipliers.

The φ-positive operator A is said to be R-positive in a Banach space E if the set

$$L_A = \{\lambda(A + \lambda)^{-1} \colon \lambda \in S_\varphi\}, \quad 0 \leq \varphi < \pi$$

is R-bounded.

A positive operator $A(s)$ is said to be uniformly R-positive in a Banach space E if there exists a $\varphi \in [0, \pi)$ such that the set

$$L_A = \{\lambda(A(s) + \lambda)^{-1} : \lambda \in S_\varphi\}$$

is uniformly R-bounded. Let $\sigma_\infty(E)$ denote the space of all compact operators in E. Let E_0 and E be two Banach spaces and E_0 continuously and densely embedded into E; let Ω be a measurable subset in $\mathbb{R}^n$ and m be a natural number. Let $W_p^m(\Omega; E_0, E)$ denote the collection of all functions $u \in L_p(\Omega; E_0)$ that have the generalized derivatives $D_k^m u = \frac{\partial^m}{\partial x_k^m} u \in L_p(\Omega; E)$ with the norm given by

$$\|u\|_{W_p^m} = \|u\|_{W_p^m(\Omega;E_0,E)} = \|u\|_{L_p(\Omega;E_0)} + \sum_{k=1}^{n} \|D_k^m u\|_{L_p(\Omega;E)} < \infty.$$

We will called it Sobolev-Lions type space. For $n = 1$, $\Omega = (a, b)$, $a, b \in \mathbb{R}$, the space $W_p^m(\Omega; E_0, E)$ will be denoted by $W_p^m(a, b; E_0, E)$, and for $E_0 = E$ it will be denoted by $W_p^m(\Omega; E)$. It is clear to see that

$$W_p^m(\Omega; E_0, E) = W_p^m(\Omega; E) \cap L_p(\Omega; E_0).$$

Let $\Omega_t \subset \mathbb{R}^n$ be a domain dependent on the parameter t.

Condition 1.1 *For $p \in (1, \infty)$, there is a bounded linear extension operator from $W_p^m(\Omega_t; E(A), E)$ to $W_p^m(\mathbb{R}^n; E(A), E)$ independent of t.*

Remark 1.1 If $\Omega \subset \mathbb{R}^n$ is a region with smooth boundary, $E = R$, $A = I$, then for $p \in (1, \infty)$ there exists a bounded linear extension operator from $W_p^m(\Omega) = W_p^m(\Omega; R, R)$ to $W_p^m(\mathbb{R}^n) = W_p^m(\mathbb{R}^n; R, R)$ (see [5, Section 7]). If G_s is a bounded domain with the uniform cone property for all s (see [1, p. 66]), or if $s_i \in C^\gamma[0, T]$, $0 < \gamma < 1$ and there is a $b > 0$ so that $\nu(s) \geq b > 0$, then in a similar way as in [1, pp. 83–94] or [5, Section 7] it can be shown that Condition 1.1 holds.

Remark 1.2 By virtue of [10, Lemma 2.3], there is a positive constant C such that

$$|\lambda + \mu| \geq C(|\lambda| + |\mu|)$$

for $|\arg \lambda| \leq \varphi_1$, $|\arg \mu| \leq \varphi_2$ and $0 \leq \varphi_1 + \varphi_2 < \pi$.

By using a similar technique as in [24–27], we obtain the following result.

Theorem 1.1 *Suppose that Condition* 1.1 *holds. Let the following conditions be satisfied:*

(1) *E is a Banach space satisfying the uniform multiplier condition, $p \in (1, \infty)$, $0 < h \leq h_0$ is a parameter;*

(2) *$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ are n-tuples of nonnegative integer numbers such that*

$$\varkappa = \sum_{k=1}^{n} \frac{|\alpha|}{m} \leq 1, \quad 0 \leq \mu \leq 1 - \varkappa;$$

(3) *A is an R-positive operator in E with $0 \leq \varphi < \pi$.*

Then the embedding $D^\alpha W_p^m(\Omega_t; E(A), E) \subset L_p(\Omega_t; E(A^{1-\varkappa-\mu}))$ is continuous, and there exists a positive constant C_μ such that

$$\|D^\alpha u\|_{L_p(\Omega_t;E(A^{1-\varkappa-\mu}))} \leq C_\mu[h^\mu \|u\|_{W_p^m(\Omega_t;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega_t;E)}]$$

for all $u \in W_p^m(\Omega_t; E(A), E)$, t and h.

Theorem 1.2 *Suppose that all conditions of Theorem* 1.1 *are satisfied, and Ω_t is a bounded region in $\mathbb{R}^n$, $A^{-1} \in \sigma_\infty(E)$. Then for $0 < \mu \leq 1- \varkappa$, the embedding $D^\alpha W_p^m(\Omega_t; E(A), E) \subset L_p(\Omega_t; E(A^{1-\varkappa-\mu}))$ is compact.*

Theorem 1.3 *Suppose that all conditions of Theorem* 1.1 *are satisfied for* $\varphi \in (\frac{\pi}{2}, \pi)$. *Then the embedding*

$$D^\alpha W_p^m(\Omega_t; E(A), E) \subset L_p(\Omega_t; (E(A), E)_{\varkappa,p})$$

is continuous, and there exists a positive constant C_μ, *such that*

$$\|D^\alpha u\|_{L_p(\Omega_t;(E(A),E)_{\varkappa,p})} \le C_\mu(h^\mu \|u\|_{W_p^m(\Omega_t;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_p(\Omega_t;E)})$$

for all $u \in W_p^m(\Omega_t; E(A), E)$, t *and* $h > 0$.

Theorem 1.4 (see [26]) *Let* E *be a Banach space,* A *be a positive operator in* E *of type* φ, m *be a positive integer,* $1 \le p < \infty$ *and* $\frac{1}{2p} < \alpha < m + \frac{1}{2p}$, *and let* $0 \le \gamma < 2p\alpha - 1$. *Then for* $\lambda \in S(\varphi)$ *the operator* $-A_\lambda^{\frac{1}{2}}$ *generates a semigroup* $\mathrm{e}^{-A_\lambda^{\frac{1}{2}}x}$ *which is holomorphic for* $x > 0$ *and strongly continuous for* $x \ge 0$. *Moreover, there exists a constant* $C > 0$, *such that*

$$\int_0^\infty \|A_\lambda^\alpha \mathrm{e}^{-xA_\lambda^{\frac{1}{2}}} u\|_E^p x^\gamma \mathrm{d}x \le C(\|u\|^p_{(E,E(A^m))_{\frac{\alpha}{m}-\frac{1+\gamma}{2mp},p}} + |\lambda|^{p\alpha-\frac{1+\gamma}{2}}\|u\|_E^p)$$

for every $u \in (E, E(A^m))_{\frac{\alpha}{m}-\frac{1+\gamma}{2pm},p}$ *and* $\lambda \in S(\varphi)$.

From [31, Section 1.8.2] and [5, Section 10.1], we have the theorem below.

Theorem 1.5 *Let the following conditions be satisfied:*
(1) m *and* j *are integer numbers, and* $0 \le j \le m-1$;
(2) $\theta_j = \frac{pj+1}{pm}$, $0 < t \le T < \infty$, $h > 0$, $x_0 \in [0, b]$.
Then, for $u \in W_p^m(0, b; E_0, E)$, *the transformations* $u \to u^{(j)}(x_0)$ *are bounded linear from* $W_p^m(0, b; E_0, E)$ *onto* $(E_0, E)_{\theta_j,p}$, *and the following inequalities hold:*

$$t^{\theta_j}\|u^{(j)}(x_0)\|_{(E_0,E)_{\theta_j,p}} \le C(\|tu^{(m)}\|_{L_p(0,b;E)} + \|u\|_{L_p\ (0,b;E_0)}),$$
$$t^{\theta_j}\|u^{(j)}(x_0)\|_E \le C(h^{1-\theta_j}\|tu^{(m)}\|_{L_p(0,b;E)} + h^{-\theta_j}\|u\|_{L_p\ (0,b;E)}).$$

Consider the following differential-operator equation:

$$Lu = u^{(m)}(x) + \sum_{k=1}^m a_k A^k u^{(m-k)}(x) = 0, \quad x \in (a, b). \tag{1.1}$$

Let $\omega_1, \omega_2, \cdots, \omega_m$ be the roots of the equation

$$\omega^m + a_1\omega^{m-1} + \cdots + a_m = 0 \tag{1.2}$$

and

$$\omega_{\min} = \min\{\arg \omega_j,\ j = 1, \cdots, q;\ \arg \omega_j + \pi,\ j = q+1, \cdots, m\},$$
$$\omega_{\max} = \max\{\arg \omega_j,\ j = 1, \cdots, q;\ \arg \omega_j + \pi,\ j = q+1, \cdots, m\},$$

where q is some integer number from $(1, m)$.

A system of complex numbers $\omega_1, \omega_2, \cdots, \omega_m$ is called q-separated if there exists a straight line P passing through 0, such that no value of the numbers ω_j lies on it, and $\omega_1, \omega_2, \cdots, \omega_q$ are on one side of P while $\omega_{q+1}, \cdots, \omega_m$ are on the other side.

By reasoning as in [39, p. 263], we obtain the following lemma.

Lemma 1.1 *Let the following conditions be satisfied:*

(1) $a_m \neq 0$ *and the roots of equation* (2.2), ω_j, $j = 1, \cdots, m$, *are q-separated*;

(2) A *is a closed operator in Banach space* E *with a dense domain* $D(A)$ *and* $\|(A-\lambda)^{-1}\| \leq C|\lambda|^{-1}$, $-\frac{\pi}{2} - \omega_{\max} \leq \arg\lambda \leq \frac{\pi}{2} - \omega_{\min}$, $|\lambda| \to \infty$.

Then for a function $u(x)$ to be a solution of equation (2.1), which belongs to the space $W_p^m(a, b; D(A^m), E)$, it is necessary that $u = \sum\limits_{k=1}^{q} \mathrm{e}^{-(x-a)\omega_k A} g_k + \sum\limits_{k=q+1}^{m} \mathrm{e}^{-(b-x)\omega_k A} g_k$, where

$$g_k \in (D(A^m), E)_{\frac{1}{mp}, p}, \quad k = 1, 2, \cdots, m.$$

2 Statement of the Problem

Consider the following nonlocal free boundary value problem:

$$\begin{aligned} & -D_x^2 u(x,y) - D_y^2 u(x,y) + (A(s)+\lambda)u(x,y) \\ & + A_1(s,x,y)D_x u(x,y) + A_2(s,x,y)D_x u(x,y) = f(x,y), \end{aligned} \tag{2.1}$$

$$\begin{aligned} L_{1k}u &= \Big[\sum_{i=0}^{m_{1k}} \nu^{\delta_i}[\alpha_{1ki}u^{(i)}(0,y) + \beta_{1ki}u^{(i)}(T,y)] + \sum_{j=1}^{N_k} \delta_{1kj}u^{(i)}(x_{kj},y)\Big] = 0, \\ L_{2k}u &= \Big[\sum_{i=0}^{m_{2k}} \nu^{\delta_i}[\alpha_{2ki}u^{(i)}x, s_0(x) + \beta_{2ki}u^{(i)}(x, s_1(x))] + \sum_{j=1}^{N_k} \delta_{2kj}u^{(i)}(x, y_{kj})\Big] \\ &= 0, \quad k = 1, 2 \end{aligned} \tag{2.2}$$

on the moving domain $G_s = \{(x,y) \in \mathbb{R}^2,\ 0 < x < T,\ y \in \sigma(s)\}$, where

$$\begin{aligned} & s = (s_0, s_1), \quad s_0 = s_0(x), \quad s_1 = s_1(x), \quad \sigma(s) = (s_0(x), s_1(x)), \\ & \nu = \nu(s) = s_1(x) - s_0(x), \quad x \in [0,T], \quad \delta_i = i - \frac{1}{p}, \\ & u = u_s, \quad m_{ik} \in \{0,1\}, \end{aligned}$$

α_{ik}, β_{ik}, δ_{ikj} are complex numbers,

$$\begin{aligned} & x_{kj} \in (0,T), \quad y_{kj} \in (s_1(x), s_2(x)), \quad \alpha = (\alpha_1,\ \alpha_2), \\ & D^\alpha = D_x^{\alpha_1} D_y^{\alpha_2}, \quad D_x^{\alpha_1} = \Big(\frac{\partial}{\partial x}\Big)^{\alpha_1}, \quad D_y^{\alpha_2} = \Big(\frac{\partial}{\partial y}\Big)^{\alpha_2}, \end{aligned}$$

and $A(s)$, $A_1(s,x,y)$, $A_2(s,x,y)$ for $x, y \in G_s$ are possibly unbounded operators in E. Let

$$\alpha_{jk} = \alpha_{jkm_{jk}}, \quad \beta_{jk} = \beta_{jkm_{jk}}, \quad j, k = 1, 2.$$

The function $u \in W_p^2(G_s; E(A), E)$ satisfying equation (2.1) a.e. on G_s is said to be the solution to equation $(2,1)$. Consider at first, the following BVP for ordinary DOE:

$$Lu = -u^{(2)}(y) + A(s)u(y) + B_1(s,y)u^{(1)}(y) + B_2(s,y)u(y) = f(y), \quad y \in \sigma(s), \tag{2.3}$$

$$L_k u = \sum_{i=0}^{m_k} \nu^{\delta_i}\Big[\alpha_{ki}u^{(i)}(s_0) + \beta_{ki}u^{(i)}(s_1) + \sum_{j=1}^{N_k} \delta_{kji}u^{(i)}(y_{kj})\Big] = f_k, \quad k = 1, 2, \tag{2.4}$$

where $f_k \in E_k = (E(A), E)_{\theta_k, p}$, $\delta_i = i - \frac{1}{p}$, $\theta_k = \frac{m_k}{2} + \frac{1}{2p}$, $p \in (1, \infty)$, $m_k \in \{0, 1\}$; N_k and M_k are integer numbers, α_k, β_k, δ_{kj} are complex numbers; $s_0(t) < s_1(t)$, $y_{kj} \in (s_0, s_1)$;

$s = (s_0, s_1), s_0 = s_0(t), s_1 = s_1(t)$, $\nu = \nu(t) = s_1(t) - s_0(t)$, $\sigma(s) = (s_0(t), s_1(t)), t \in [0, T]$; $A(s)$ and $B_k(s, y)$ for $y \in [s_0, s_1]$ are possibly unbounded operators in E, $u = u_s$ is the solution to the equation (2.3) on $\sigma(s)$.

3 Homogeneous Equations

Let us first consider the following nonlocal BVP on the moving domain $\sigma(s)$:

$$(L_0 + \lambda)u = -u^{(2)}(y) + (A(s) + \lambda)u(y) = 0, \quad y \in \sigma(s), \tag{3.1}$$

$$L_k u = \sum_{i=0}^{m_k} \nu^{\delta_i} \Big[\alpha_{ki} u^{(i)}(s_0) + \beta_{ki} u^{(i)}(s_1) + \sum_{j=1}^{N_k} \delta_{kji} u^{(i)}(y_{kj})\Big] = f_k, \quad k = 1, 2, \tag{3.2}$$

where λ is a complex parameter, $m_k \in \{0, 1\}$; $\alpha_{ki}, \beta_{ki}, \delta_{kji}$ are complex numbers; $A(s) = A(\nu(s))$ is a possibly unbounded operator in E. Let $\alpha_k = \alpha_{km_k}$, $\beta_k = \beta_{km_k}$.

Theorem 3.1 *Suppose that*

(1) *E is a Banach space satisfying the uniform multiplier condition*;

(2) *$A(s)$ is an uniformly R-positive operator in E, $\eta = (-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$, $p \in (1, \infty)$*;

(3) *$s_i \in C^\gamma[0, T]$, $0 < \gamma < 1$, and there is a $b > 0$ so that $\nu(s) \geq b > 0$.*

Then, problem (3.1)–(3.2) *for $f_k \in E_k$, $\theta_k = \frac{m_k}{2} + \frac{1}{2p}$, $p \in (1, \infty)$ and $|\arg \lambda| \leq \varphi$, with sufficiently large $|\lambda|$, has a unique solution u belonging to $W_p^2(\sigma(s); E(A), E)$, and the following coercive uniform estimate*

$$\sum_{j=0}^{2} \lambda^{1-\frac{j}{2}} \|u^{(j)}\|_{L_p(\sigma(s);E)} + \|Au\|_{L_p(\sigma(s);E)} \leq M \sum_{k=1}^{2} (\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|) \tag{3.3}$$

holds with respect to parameters s and λ.

Proof By the substitution of variable $y = s_0 + \nu(s)x$, $x \in (0, 1)$ in problem (3.1)–(3.2) and dividing both sides of the equation by ν^{-2}, we obtain the following equivalent BVP on the fixed domain $(0, 1)$:

$$(L_0 + \lambda)u = -u^{(2)}(x) + \nu^2[A(s) + \lambda]u(x) = 0, \tag{3.4}$$

$$L_{k0} u = \sum_{i=0}^{m_k} \nu^{-\sigma_i} \Big[\alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(1) + \sum_{j=1}^{N_k} \delta_{kji} u^{(i)}(x_{kj})\Big] = f_k, \quad k = 1, 2, \tag{3.5}$$

where $\sigma_i = i + \frac{1}{p}$. Due to uniform positivity of A, by using Remark 1.2 and by the condition (2.1), we have the following estimate:

$$\|[(A(s) + \lambda)\nu^2 + \mu]^{-1}\| \leq \frac{M_0}{|\mu|} \tag{3.6}$$

with M_0 independent of s and depending on φ only. In view of (3.6) and by [10, Lemma 2.6], there exists a semigroup $\mathrm{e}^{-x\nu(A(s)+\lambda)^{\frac{1}{2}}}$ which is holomorphic for $x > 0$ and strongly continuous for $x \geq 0$. By virtue of Lemma 1.1, an arbitrary solution to equation (3.4) belonging to $W_p^2(0, 1; E(A), E)$ has the form

$$u(x) = U_{1s}(x)g_1 + U_{2s}(x)g_2, \tag{3.7}$$

where

$$U_{1s}(x) = \mathrm{e}^{-x\nu(A(s)+\lambda)^{\frac{1}{2}}}, \quad U_{2s}(x) = \mathrm{e}^{-\nu(1-x)A_\lambda^{\frac{1}{2}}(s)},$$
$$A_\lambda^{\frac{1}{2}}(s) = [A(s)+\lambda]^{\frac{1}{2}}, \qquad g_k \in (E(A), E)_{\frac{1}{2p},p}, \quad k = 1,2.$$

Now taking into account boundary conditions (3.5), we obtain the algebraic linear equations with respect to g_1, g_2:

$$L_{k0}u = \nu^{-\frac{1}{p}} \sum_{i=0}^{m_k} \Big[(-1)^i A_\lambda^{\frac{i}{2}}(s)\Big(\alpha_{ki} + \beta_{ki}U_{1s}(1) + \sum_{j=1}^{N_k} \delta_{kji}U_{1s}(x_{kj})\Big)\Big] g_1$$
$$+ \sum_{i=0}^{m_k} \Big[A_\lambda^{\frac{i}{2}}(s)\Big(\alpha_{ki}U_{2s}(0) + \beta_{ki} + \sum_{j=1}^{N_k} \delta_{kji}U_{2s}(x_{kj})\Big)\Big] g_2 = f_k, \quad k = 1,2. \tag{3.8}$$

Therefore, it is easy to see that the matrix-operator of system (3.8) for sufficiently large $|\lambda|$ is invertible in E^2 and has a unique solution expressed in the form

$$g_k = \nu^{\frac{1}{p}}[C_{1k} + \widetilde{d}_{1k}(\lambda,s)]A_\lambda^{-\frac{m_1}{2}} f_1 + \nu^{\frac{1}{p}}[C_{2k} + \widetilde{d}_{2k}(\lambda,s)]A_\lambda^{-\frac{m_2}{2}} f_2, \quad k = 1,2, \tag{3.9}$$

where C_{ik} are some complex numbers and $\widetilde{d}_{jk}$ like d_{jk} go to 0 in $B(E)$ and $B(E(A))$ as $|\lambda| \to \infty$, $\lambda \in S_\varphi$. Substituting (3.9) into (3.7), we obtain the representation of the solution to problem (3.4)–(3.5):

$$u(x) = \nu^{\frac{1}{p}}\{U_{1s}(x)[C_{11} + d_{11}(\lambda,s)] + U_{2s}(x)[C_{1k} + d_{12}(\lambda,s)]A_\lambda^{-\frac{m_1}{2}}(s)\}f_1$$
$$+ \{U_{1s}(x)[C_{21} + d_{21}(\lambda,s)] + U_{2s}(x)[C_{2k} + d_{22}(\lambda,s)]A_\lambda^{-\frac{m_2}{2}}(s)\}f_2. \tag{3.10}$$

By virtue of [10, Lemma 2.6], we have

$$\|\mathrm{e}^{-\nu x A_\lambda^{\frac{1}{2}}}\| \le C\mathrm{e}^{-\varkappa\nu(s)|\lambda|^{\frac{1}{2}}x}, \quad \varkappa > 0, \ x \in (0,1), \ \lambda \in S(\varphi).$$

So, in view of uniform boundedness of the operator $d_{kj}(\lambda,s)$, for $|\arg\lambda| \le \varphi$ and $|\lambda|$ from (3.10), we obtain

$$\sum_{i=0}^{2} \nu^{-i}|\lambda|^{1-\frac{i}{2}}\|u^{(i)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)}$$
$$\le C\nu^{\frac{1}{p}} \sum_{k=1}^{2}\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \Big[\Big(\int_0^1 \|A_\lambda^{1-\frac{m_k}{2}} U_{ks}(x)f_k\|^p \mathrm{d}x\Big)^{\frac{1}{p}}$$
$$+ \Big(\int_0^1 \|AA_\lambda^{-\frac{m_k}{2}} U_{ks}(x)f_k\|^p \mathrm{d}x\Big)^{\frac{1}{p}}\Big]. \tag{3.11}$$

By the substitution of variable $\nu x = \xi$ and in view of Theorem 1.4, we obtain

$$\nu^{\frac{1}{p}}\Big(\int_0^1 \|A_\lambda^{1-\frac{m_k}{2}} U_{1s}(x)f_k\|^p \mathrm{d}x\Big)^{\frac{1}{p}} \le M_1 \sum_{k=1}^{2}(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|). \tag{3.12}$$

Moreover, due to positivity of operator A, for $u_k = U_{1s}(x)f_k$ (or $u_k = U_{2s}(x)f_k$), we have

$$\|AA_\lambda^{-\frac{m_k}{2}} u_k\| = \|A_\lambda^{1-\frac{m_k}{2}} u_k - \lambda A_\lambda^{-\frac{m_k}{2}} u_k\| \le (1 + \|\lambda A_\lambda^{-1}\|)\|A_\lambda^{1-\frac{m_k}{2}} u_k\|.$$

By using the above estimate, in view of (3.11) and (3.12), by the substitution of variable and by virtue of Theorem 1.4, we get the following uniform estimate for the solution to problem (3.4)–(3.5):

$$\sum_{k=1}^{2}\nu^{\frac{1}{p}}\Big(\int_0^1\|AA_\lambda^{-\frac{m_k}{2}}\|[\|e^{-\nu xA_\lambda^{\frac{1}{2}}}f_k\|^p+\|e^{-\nu(1-x)A^{\frac{1}{2}}_\lambda}f_k\|^p]dx\Big)^{\frac{1}{p}}$$
$$\leq M_1\sum_{k=1}^{2}(\|f_k\|_{E_k}+|\lambda|^{1-\theta_k}\|f_k\|). \tag{3.13}$$

Then by taking into account the substitution of variable, we obtain from (3.11) and (3.13) the estimate (3.3).

4 Non-homogenous Equations

Now consider non-homogenous BVPs on the moving intervals $\sigma(s)=(s_0,s_1)$:

$$(L_0+\lambda)u=-u^{(2)}(y)+(A+\lambda)u(y)=f(y),\quad y\in\sigma(s), \tag{4.1}$$
$$L_ku=\sum_{i=0}^{m_k}\nu^{\delta_i}\Big[\alpha_{ki}u^{(i)}(0)+\beta_{ki}u^{(i)}(1)+\sum_{j=1}^{N_k}\delta_{kji}u^{(i)}(y_{kj})\Big]=f_k,\quad k=1,2, \tag{4.2}$$

where $y_{kj}\in\sigma(s),\ s=(s_0(t),s_1(t)),\ 0<t\leq T<\infty;\ \alpha_k,\beta_k,\delta_{kj}$ are complex numbers; λ is a complex parameter, and $\delta_i=i-\frac{1}{p}$.

Remark 4.1 If functions s_i satisfy the Hölder's condition (i.e., $s_i\in C^\gamma[0,T]$, $0<\gamma\leq 1$), then by a similar way as in [1, pp. 83–94] or [5, Section 7], it can be shown that there is a bounded linear extension operator from $W_p^2(\sigma(s);E(A),E)$ to $W_p^2(R;E(A),E)$ for $p\in(1,\infty)$ independent of s.

Theorem 4.1 *Suppose that all conditions of Theorem* 3.1 *are satisfied. Then, the operator* $u\to\{(L_0+\lambda)u,L_1u,L_2u\}$ *for* $|\arg\lambda|\leq\varphi$, $0\leq\varphi<\pi$ *and sufficiently large* $|\lambda|$, *is an isomorphism from* $W_p^2(\sigma(s);E(A),E)$ *onto* $L_p(\sigma(s);E)\times E_1\times E_2$. *Moreover, the following uniform coercive estimate holds:*

$$\sum_{j=0}^{2}|\lambda|^{1-\frac{j}{2}}\|u^{(j)}\|_{L_p(\sigma(s);E)}+\|Au\|_{L_p(\sigma(s);E)}$$
$$\leq C\Big[\|f\|_{L_p(\sigma(s);E)}+\sum_{k=1}^{2}(\|f_k\|_{E_k}+|\lambda|^{1-\theta_k}\|f_k\|_E)\Big]. \tag{4.3}$$

Proof By the substitution of variable $y=s_0+\nu(s)x$, $x\in(0,1)$, problem (4.1)–(4.2) is transformed to BVP with parameters on fixed domain

$$(L_0+\lambda)u=-\nu^{-2}u^{(2)}(x)+(A+\lambda)u(x)=f(x),\quad x\in(0,1), \tag{4.4}$$
$$L_ku=\sum_{i=0}^{m_k}\nu^{-\sigma_i}\Big[\alpha_{ki}u^{(i)}(0)+\beta_{ki}u^{(i)}(1)+\sum_{j=1}^{N_k}\delta_{kji}u^{(i)}(x_{kj})\Big]$$
$$=f_k,\quad k=1,2,\ \sigma_i=i+\frac{1}{p}. \tag{4.5}$$

We proved the uniqueness of the solution to problem (4.4)–(4.5) in Theorem 3.1. Let us define

$$\overline{f}(x) = \begin{cases} f(x), & \text{if } x \in [0,1], \\ 0, & \text{if } x \notin [0,1]. \end{cases}$$

We now show that problem (4.4)–(4.5) has a solution $u \in W_p^2(0,1;E(A),E)$ for all $f \in L_p(0,1;E)$, $f_k \in E_k$ and $u = u_1 + u_2$, where u_1 is the restriction on $[0,1]$ of the solution to equation

$$(L_0 + \lambda)u = \overline{f}(x), \quad x \in R = (-\infty, \infty), \tag{4.6}$$

and u_2 is a solution to problem

$$(L_0 + \lambda)u = 0, \quad L_k u = f_k - L_k u_1. \tag{4.7}$$

A solution to equation (4.6) is given by the formula

$$u(x) = F^{-1} L_0^{-1}(\lambda, s, \xi) F\overline{f} = \frac{1}{2\pi} \int_{\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} L_0^{-1}(\lambda, s, \xi)(F\overline{f})(\xi) \mathrm{d}\xi,$$

where $L_0(\lambda, s, \xi) = \nu^{-2}\xi^2 + \lambda + A$. So, it is sufficient to show that operator-functions

$$\Psi_{s,\lambda}(\xi) = AL_0^{-1}(\lambda, s, \xi), \quad \Psi_{s,\lambda,j}(\xi) = \nu^j |\lambda|^{1-\frac{j}{2}} \xi^j L_0^{-1}(\lambda, s, \xi), \quad j = 0,1,2$$

are Fourier multipliers in $L_p(R;E)$ uniformly with respect to s and λ. Really, due to positivity of A and by virtue of Remark 1.2, we have the following uniform estimates:

$$\|L_0^{-1}(\lambda, s, \mathrm{i}\xi)\| \le M[1 + \nu^{-2}\xi^2 + |\lambda|]^{-1},$$
$$\|\Psi_{s,\lambda}(\xi)\| = \|A[A + \lambda + \nu^{-2}\xi^2]^{-1}\| \le C_1,$$
$$\|\Psi_{s,\lambda,j}(\xi)\| = \|\nu^{-j} |\lambda|^{1-\frac{j}{2}} \xi^j L_0^{-1}(\lambda, s, \xi)\| \le C_2.$$

Since $A(s)$ is uniformly R-positive, in view of equality $AR(\lambda) = I + \lambda R(\lambda)$ and by virtue of Kahane's contraction principle for collection of R-bounded operators (see [9, Lemma 3.5]), we get that the set $\{AL_0^{-1}(\lambda, s, \xi), \xi \in R\backslash\{0\}\}$ is uniformly R-bounded. Moreover, it is clear to see that

$$\xi \frac{\mathrm{d}}{\mathrm{d}\xi} \Psi_{s,\lambda}(\xi) = -2\xi^2 \nu^{-2} AL_0^{-2}(\lambda, s, \xi) = [-2\nu^{-2}\xi^2 L_0^{-1}(\lambda, s, \xi)] AL_0^{-1}(\lambda, s, \xi).$$

In view of Kahane's contraction principle, from additional and product properties of the collection of R-bounded operators (see [9, Lemma 3.5, Proposition 3.4]) and the uniform R-positivity of operator $A(s)$, we obtain

$$\sup_{s,\lambda} R\Big\{ |\xi|^k \frac{\mathrm{d}^k}{\mathrm{d}\xi^k} \Psi_{s,\lambda}(\xi) \colon \xi \in R\backslash\{0\} \Big\} \le C, \quad k = 0,1.$$

Namely, the R-bound of the set $\{|\xi|^k \frac{\mathrm{d}^k}{\mathrm{d}\xi^k} \Psi_{s,\lambda}(\xi) \colon \xi \in R\backslash\{0\}\}$ is independent of s and λ. In a similar way, we have the uniform estimate

$$\sup_{s,\lambda} R\Big(\Big\{ |\xi|^k \frac{\mathrm{d}^k}{\mathrm{d}\xi^k} \Psi_{s,\lambda,j}(\xi) : \xi \in R\backslash\{0\} \Big\} \Big) \le C, \quad k = 0,1.$$

Then, in view of Definition 1.1, it follows that $\Psi_{s,\lambda}(\xi)$ and $\Psi_{s,\lambda,j}(\xi)$ are uniform collections of multipliers in $L_p(R;E)$. Then, we obtain that problem (4.6) has the solution $u \in W_p^2(R;E(A),E)$ and the following uniform estimate

$$\sum_{j=0}^{2} \nu^{-j}|\lambda|^{1-\frac{j}{2}}\|u^{(j)}\|_{L_p(R;E)} + \|Au\|_{L_p(R;E)} \leq C\|\overline{f}\|_{L_p(R;E)} \tag{4.8}$$

holds with respect to s and λ. Let u_1 be the restriction of u on $(0,1)$. Then it implies that $u_1 \in W_p^2(0,1;E(A),E)$. By virtue of Theorem 1.5, we get

$$u_1^{(m_k)}(\,\cdot\,) \in (E(A);E)_{\theta_k,p}, \quad k = 1,2.$$

Hence, $L_{0k}u_1 \in E_k$. Thus by virtue of Theorem 3.1, problem (4.7) has a unique solution $u_2(x)$ that belongs to $W_p^2(0,1;E(A),E)$ and for sufficiently large $|\lambda|$, we have

$$\begin{aligned}&\sum_{j=0}^{2} \nu^{-j}|\lambda|^{1-\frac{j}{2}}\|u_2^{(j)}\|_{L_p(0,1;E)} + \|Au_2\|_{L_p(0,1;E)}\\ &\leq C\sum_{k=1}^{2}(\|f_k - L_k u_1\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k - L_k u_1\|_E).\end{aligned} \tag{4.9}$$

From (4.9), for $|\arg\lambda| \leq \varphi$, we obtain

$$\sum_{j=0}^{2} \nu^{-j}|\lambda|^{1-\frac{j}{2}}\|u_1^{(j)}\|_{L_p(0,1;E)} + \|Au_1\|_{L_p(0,1;E)} \leq C\|f\|_{L_p(0,1;E)}. \tag{4.10}$$

Therefore, in view of Theorem 1.5 and by estimate (4.10) we have

$$\nu^{-(j+\frac{1}{p})}\|u_1^{(j)}(\,\cdot\,)\|_{E_k} \leq C_1[\|\nu^{-2}u_1^{(2)}\|_{L_p(0,1;E)} + \|Au_1\|_{L_p(0,1;E)}] \leq C\|f\|_{L_p(0,1;E)}. \tag{4.11}$$

In virtue of Theorem 1.5 for $\lambda = \mu^2$, $u \in W_p^2(0,1;E)$, we obtain

$$|\mu|^{2-j}\nu^{-(j+\frac{1}{p})}\|u^{(j)}(\,\cdot\,)\|_E \leq C[|\mu|^{\frac{1}{p}}\|\nu^{-2}u^{(2)}\|_{L_p(0,1;E)} + |\mu|^{2+\frac{1}{p}}\|u\|_{L_p(0,1;E)}]. \tag{4.12}$$

Hence, from estimates (4.9)–(4.12), we have

$$\begin{aligned}&\sum_{j=0}^{2} \nu^{-j}|\lambda|^{1-\frac{j}{2}}\|u_2^{(j)}\|_{L_p(0,1;E)} + \|Au_2\|_{L_p(0,1;E)}\\ &\leq C\Big(\|f\|_{L_p(0,1;E)} + \sum_{k=1}^{2}(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E)\Big).\end{aligned} \tag{4.13}$$

Then, from estimates (4.8) and (4.13), we obtain that the operator generated by problem (4.4)–(4.5) for $|\arg\lambda| \leq \varphi$, $0 \leq \varphi < \pi$ and sufficiently large $|\lambda|$, is an isomorphism from $W_p^2(0,1;E(A),E)$ onto $L_p(0,1;E) \times E_1 \times E_2$. Moreover, for these s, λ, the following uniform coercive estimate holds:

$$\sum_{j=0}^{2} \nu^{-j}|\lambda|^{1-\frac{j}{2}}\|u^{(j)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)}$$

$$\leq C\Big[\|f\|_{L_p(0,1;E)} + \sum_{k=1}^{2}(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E)\Big]. \tag{4.14}$$

Finally, by changing of variable from (4.14), we obtain the assertion.

Consider the following problem:

$$\begin{aligned} &(L_0+\lambda)u = -u^{(2)}(y) + (A(s)+\lambda)u(y) = f(y), \quad y \in \sigma(s),\\ &L_k u = \sum_{i=0}^{m_k} \nu^{\delta_i}\Big[\alpha_{ki}u^{(i)}(s_0) + \beta_{ki}u^{(i)}(s_1) + \sum_{j=1}^{N_k}\delta_{kji}u^{(i)}(y_{kj})\Big] = 0, \quad k=1,2. \end{aligned} \tag{4.15}$$

Let $B(s)$ denote the operator in $L_p(\sigma(s);E)$ generated by problem (4.15), i.e.,

$$D(B(s)) = W_p^2(\sigma(s);E(A),E,L_k), \quad B(s)u = -u^{(2)} + Au.$$

By changing of variable $y = s_0 + \nu(s)x$, $x \in (0,1)$, problem (4.15) is transformed to the following BVP with parameter on fixed domain:

$$\begin{aligned} &(\widetilde{L}_0+\lambda)u = -\nu^{-2}u^{(2)}(x) + (A+\lambda)u(x) = f(x), \quad x \in (0,1),\\ &\widetilde{L}_k u = \sum_{i=0}^{m_k} \nu^{-\sigma_i}\Big[\alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(1) + \sum_{j=1}^{N_k}\delta_{kji}u^{(i)}(x_{kj})\Big] = 0, \quad k=1,2. \end{aligned} \tag{4.16}$$

Let $B(s)$ denote the operator in $F = L_p(0,1;E)$ generated by problem (4.16), i.e.,

$$D(B(s)) = W_p^2(0,1;E(A),E,\widetilde{L}_k), \quad B(s)u = -\nu^{-2}u'' + A(s)u.$$

Theorem 4.1 implies the following result.

Result 4.1 The operator $B(s)$ is uniformly positive in F and for $\lambda \in S(\varphi)$ the following estimate holds:

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}}\|[B(s)+\lambda]^{-1}\|_F + \|A[B(s)+\lambda]^{-1}\|_F \leq M.$$

Theorem 4.2 *Let all conditions of Theorem* 3.1 *be satisfied. Then the operator* $G(s)$ *is uniformly* R*-positive in* F.

Proof The estimate (4.3) implies that $G(s)$ is uniformly positive in F. The equation (4.16) can be express as

$$-u^{(2)}(x) + \nu^2 A_\lambda u(x) = -\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \nu A_\lambda^{\frac{1}{2}}\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x} + \nu A_\lambda^{\frac{1}{2}}\Big) = \nu^2 f(x).$$

Then, by using a similar technique as in [35], we obtain that for $f \in D(0,1;E(A))$ the solution to equation (4.16) is represented as

$$u(x) = U_{1\lambda\nu}(x)g_1 + U_{2\lambda\nu}(x)g_2 + \nu^2\int_0^1 U_{0\lambda\nu}(x-y)f(y)\mathrm{d}y, \quad g_k \in E, \tag{4.17}$$

where

$$U_{0\lambda\nu}(x-y) = \begin{cases} -\dfrac{1}{\nu}A_\lambda^{-\frac{1}{2}}\mathrm{e}^{-\nu(x-y)A_\lambda^{\frac{1}{2}}}, & x \geq y,\\ \dfrac{1}{\nu}A_\lambda^{-\frac{1}{2}}\mathrm{e}^{-\nu(y-x)A_\lambda^{\frac{1}{2}}}, & x \leq y, \end{cases}$$

and $U_{j\lambda\nu}(x)$, $j=1,2$, are analytic semigroups defined by

$$U_{1\lambda\nu}(x)=\mathrm{e}^{-\nu x A_\lambda^{\frac{1}{2}}},\quad U_{2\lambda\nu}(x)=\mathrm{e}^{-\nu(1-x)A_\lambda^{\frac{1}{2}}}.$$

By taking into account the boundary conditions, we obtain the following equation with respect to g_1 and g_2:

$$L_k(U_{1\lambda\nu})g_1+L_k(U_{2\lambda\nu})g_2=L_k(\Phi_{\lambda\nu}),\quad k=1,2,$$
$$\Phi_{\lambda\nu}=\nu^2\int_0^1 U_{0\lambda\nu}(x-y)f(y)\mathrm{d}y.$$

By solving the above system, substituting it into (4.17), and calculating $L_k(\Phi_{\lambda\nu})$, we obtain from the above as in Theorem 3.1, the representation of the solution to problem (4.16):

$$\begin{aligned}u(x)&=[B(s)+\lambda]^{-1}f=\int_0^1 K_\nu(\lambda,x,y)f(y)\mathrm{d}y,\\ K_\nu(\lambda,x,y)&=\nu\sum_{j=1}^2\sum_{k=1}^2\sum_{\mu=0}^{m_k}A_\lambda^{-\frac{(m_k-\mu)}{2}}(s)B_{kj\mu\nu}(\lambda)U_{j\lambda\nu}(x)\widetilde{U}_{kj\lambda\nu}(x-y)+U_{0\lambda\nu}(x-y),\end{aligned}\tag{4.18}$$

where $B_{kj\nu}(\lambda)$ are like d_{jk}, so are uniformly bounded operators in E and

$$\widetilde{U}_{kj\mu\lambda\nu}(x-y)=\begin{cases}b_{kj\mu}\mathrm{e}^{-\nu(x-y)A_\lambda^{\frac{1}{2}}}, & x\geq y,\\ \delta_{kj\mu}\mathrm{e}^{-\nu(y-x)A_\lambda^{\frac{1}{2}}}, & x\leq y,\end{cases}\quad b_{kj\mu},\delta_{kj\mu}\in\mathbb{C}.$$

Let us at first show that the set $\{K_\nu(\lambda,x,y);\lambda\in S(\varphi)\}$ is uniformly R-bounded. Really, by using the generalized Minkowcki's, Young inequalities by semigroups estimates and by the substitution of variable $\nu\xi=\eta$, we have the uniform estimate

$$\begin{aligned}\|K_\nu(\lambda,x,y)f\|_F&\leq C\sum_{j=1}^2\sum_{k=1}^2\sum_{\mu=0}^{m_k}\nu\{\|A_\lambda^{-\frac{1}{2}}\|\|B_{kj\mu\nu}(\lambda)\|\|\widetilde{U}_{kj\lambda\nu}(x)f\|_F+\|U_{0\lambda\nu}(x)f\|_F\}\\ &\leq C\sum_{k=1}^2\nu\|f\|_F+\|f\|_F\leq C\|f\|_F.\end{aligned}$$

Due to uniform R-positivity of $A(s)$, and uniform boundedness of operators $B_{kj\mu\nu}(\lambda)$, and by using the Kahane's contraction principle, we get that the following sets

$$\begin{aligned}b_{kj\mu\nu}(\lambda,x,y)&=\{\nu B_{kj\mu\nu}(\lambda)A_\lambda^{-\frac{1}{2}}U_{j\lambda\nu}(x)[U_{1\lambda\nu}(1-y)+U_{2\lambda\nu}(y)]:\lambda\in S_\varphi\},\\ b_{0\nu}(\lambda,x,y)&=\{U_{0\lambda\nu}(x-y):\lambda\in S_\varphi\}\end{aligned}$$

are uniformly R-bounded. Then by using the Kahane's contraction principle, product and additional properties of the collection of R-bounded operators and R-boundedness of the sets $b_{kj\nu}$, $d_{0\nu}$, for all $u_1,u_2,\cdots,u_m\in F$, $\lambda_1,\lambda_2,\cdots,\lambda_m\in S(\varphi)$, and independent symmetric $\{-1,1\}$-valued random variables $r_i(y)$, $i=1,2,\cdots,m$, $m\in\mathbb{N}$, we have the estimate

$$\int_\Omega\Big\|\sum_{i=1}^m r_i(y)K_\nu(\lambda_i,x,y)u_i\Big\|_F\mathrm{d}\tau$$

$$\le C\Big\{\sum_{k,j=1}^{2}\sum_{\mu=0}^{m_k}\int_\Omega\Big\|\sum_{i=1}^{m} r_i(y)b_{kj\mu\nu}(\lambda_i,x,y)u_i\Big\|_F \mathrm{d}\tau+\int_\Omega\Big\|\sum_{i=1}^{m} r_i(y)d_{0\nu}(\lambda_i,x,y)u_i\Big\|_F \mathrm{d}\tau$$

$$\le C\mathrm{e}^{\beta|\lambda|^{\frac12}|x-y|}\int_\Omega\Big\|\sum_{i=1}^{m} r_i(y)u_i\Big\|_F \mathrm{d}\tau,\quad \beta<0,$$

uniformly in x, y and t. This implies that

$$R\{K_\nu(\lambda,x,y):\lambda\in S_\varphi\}\le C\mathrm{e}^{\beta|\lambda|^{\frac12}|x-y|},\quad \beta<0,\ x,y\in(0,b).$$

By applying the R-bondedness property of kernel operators (see [9, Proposition 4.12]) and due to density of $D(0,1;E(A))$ in $L_p(0,1;E)$ (see, e.g., [22, Section 2.2]) and by the substitution of variable, we obtain the assertion.

5 Coerciveness on the Space Variable and Fredholmness

Consider problem (2.3)–(2.4).

Theorem 5.1 *Suppose that all conditions of Theorem* 3.1 *hold. Moreover, the function* $B_1(x)u$ *for* $u\in D(A^{\frac12})$ *and the function* $B_2(x)u$ *for* $u\in D(A)$ *are measurable on* (s_0,s_1), *and for any* $\varepsilon>0$ *there is* $C(\varepsilon)>0$ *such that for almost all* $x\in[s_0,s_1]$,

$$\|B_1(s,x)u\|\le\varepsilon\|A^{\frac12}u\|+C(\varepsilon)\|u\|,\quad u\in D(A^{\frac12}),$$
$$\|B_2(s,x)u\|\le\varepsilon\|Au\|+C(\varepsilon)\|u\|,\quad u\in D(A).$$

Then

(a) *for solution* $u\in W_p^2(\sigma(s);E(A),E)$ *the following uniform coercive estimate holds:*

$$\sum_{j=0}^{2}\|u^{(j)}\|_{L_p(\sigma(s);E)}+\|Au\|_{L_p(\sigma(s);E)}$$
$$\le C\Big[\|Lu\|_{L_p(\sigma(s);E)}+\sum_{k=1}^{2}(\|L_ku\|_{E_k}+\|u\|_{L_p(\sigma(s);E)})\Big];\tag{5.1}$$

(b) *if* $A^{-1}\in\sigma_\infty(E)$, *then the operator* $u\to O(s)u=\{Lu,L_1u,L_2u\}$ *from* $W_p^2(\sigma;E(A),E)$ *into* $L_p(\sigma;E)\times E_1\times E_2$ *is Fredholm.*

Proof The substitution of variable $y=s_0+(s_1-s_0)x,\ x\in(0,1)$ in problem (2.3)–(2.4) leads to the following BVP on the fixed domain $(0,1)$:

$$\begin{aligned}(L+\lambda)u&=-\nu^{-2}u^{(2)}(x)+(A+\lambda)u(x)+\nu^{-1}B_1(s,x)u^{(1)}(x)+B_2(s,x)u(x)\\&=f(x),\quad x\in(0,1),\\ L_ku&=\sum_{i=0}^{m_k}\nu^{-\delta_i}\Big[\alpha_{ki}u^{(i)}(0)+\beta_ku^{(i)}(1)+\sum_{j=1}^{N_k}\delta_{kji}u^{(i)}(x_{kj})\Big]=f_k,\quad k=1,2,\end{aligned}\tag{5.2}$$

where $x_{kj}\in(0,1)$. Let $u\in W_p^2(0,1;E(A),E)$ be a solution to problem (5.2) and d be a positive number. Then $u(x)$ is a solution to the problem

$$-\nu^{-2}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+(A+d)u=f(x)+\mathrm{d}u-\nu^{-1}B_1\frac{\mathrm{d}u}{\mathrm{d}x}-B_2u,$$

$$L_k u = f_k, \quad k = 1, 2.$$

By Theorem 4.1 for sufficiently large d, we have the following uniform estimate:

$$\sum_{j=0}^{2} \nu^{-j} \|u^{(j)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)}$$

$$\leq C\Big[\|f + \mathrm{d}u - \nu^{-1} B_1 u^{(1)} - B_2 u\|_{L_p(0,1;E)} + \sum_{k=1}^{2} \|f_k\|_{E_k}\Big]. \tag{5.3}$$

By virtue of the condition (1.2), it follows that

$$\|B_1(s,x) u^{(1)}(x)\|_E \leq \varepsilon \|A^{\frac{1}{2}} u^{(1)}(x)\|_E + C(\varepsilon) \|u^{(1)}(x)\|_E,$$
$$\|B_2(s,x) u(x)\|_E \leq \varepsilon \|Au(x)\|_E + C(\varepsilon) \|u(x)\|_E, \quad x \in (0,1).$$

Hence

$$\begin{aligned} \|B_1 u^{(1)}\|_{L_p(0,1;E)} &\leq \varepsilon \|A^{\frac{1}{2}} u^{(1)}\|_{L_p(0,1;E)} + C(\varepsilon) \|u^{(1)}\|_{L_p(0,1;E)}, \\ \|B_2 u\|_{L_p(0,1;E)} &\leq \varepsilon \|Au\|_{L_p(0,1;E)} + C(\varepsilon) \|u\|_{L_p(0,1;E)}. \end{aligned} \tag{5.4}$$

By virtue of Theorem 1.1, we have

$$\nu^{-1} \|A^{\frac{1}{2}} u^{(1)}\|_{L_p(0,1;E)} \leq C[\|\nu^{-2} u^{(2)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)}].$$

Moreover, by virtue of Theorem 1.1 again, there exists a $C > 0$ such that for $0 < h \leq h_0$,

$$\|\nu^{-1} u^1\|_{L_p(0,1;E)} \leq C(h^{\frac{1}{2}} \|\nu^{-2} u^{(2)}\|_{L_p(0,1;E)} + h^{-\frac{1}{2}} \|u\|_{L_p(0,1;E)}).$$

Therefore, by using (5.4) we can conclude that

$$\begin{aligned} \|\nu^{-1} B_1 u^{(1)}\|_{L_p(0,1;E)} &\leq \varepsilon \|\nu^{-1} A^{\frac{1}{2}} u^{(1)}\|_{L_p(0,1;E)} + C(\varepsilon) \|\nu^{-1} u^{(1)}\|_{L_p(0,1;E)} \\ &\leq \varepsilon[\|\nu^{-2} u^{(2)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)}] + C(\varepsilon) \|u\|_{L_p(0,1;E)}. \end{aligned}$$

Moreover, from condition (1.2), it is clear that

$$\begin{aligned} \|B_2 u\|_{L_p(0,1;E)} &\leq \varepsilon \|Au\|_{L_p(0,1;E)} + C(\varepsilon) \|u\|_{L_p(0,1;E)} \\ &\leq \varepsilon[\|\nu^{-2} u^{(2)}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)}] + C(\varepsilon) \|u\|_{L_p(0,1;E)}. \end{aligned} \tag{5.5}$$

By choosing a suitable ε from (5.3)–(5.5), and by the substitution of variable, we obtain (5.1).

(b) Let O_0, O denote operators in $L_p(\sigma; E)$ generated by problems (4.1)–(4.2) and (2.3)–(2.4), respectively. Let

$$O_1 u = B_1 u^{(1)} + B_2 u, \quad u \in W_p^2(\sigma; E(A), E).$$

We can conclude from Theorem 4.1 that operator $O_0(s) + d$, for sufficiently large $d > 0$ has an inverse from $X = L_p(\sigma; E) \times E_1 \times E_2$ onto $W_p^2(\sigma; E(A), E)$. By estimates (5.3)–(5.4), for every $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$, such that for $u \in W_p^2(\sigma; E(A), E)$,

$$\|O_1 u\|_{L_p(\sigma;E)} \leq \varepsilon \|u\|_{W_p^2(\sigma;E(A),E)} + C(\varepsilon) \|u\|_{L_p(\sigma;E)}.$$

Then from Theorem 1.2 and [39] it follows that the operator $O_1(t)$ from $W_p^2(\sigma; E(A), E)$ into X is compact. Then in view of Theorem 4.1 and by the perturbation theory of linear operator [17, Section 14], we obtain that the operator $O_1(s)$ from $W_p^2(\sigma; E(A), E)$ into X is a Fredholm operator.

6 Free BVPs for Partial DOE

Let us now consider the nonlocal BVP (2.1)–(2.2).

Theorem 6.1 *Assume that the following conditions are satisfied:*

(1) *E is a UMD space; $A(s)=A_s(x)$ is a uniformly R-positive operator in E, $A_s(x)A_s^{-1}(x_0)$ is bounded and uniformly continuous with respect to a collection of s and $x\in[0,T]$;*

(2) *for any $\varepsilon>0$, there is $C(\varepsilon)>0$ such that for a.e. $x\in G$ and for $u\in(E(A),E)_{\frac{1}{2},\infty}$,*

$$\|A_k(s,x,y)u\|\le\varepsilon\|u\|_{(E(A),E)_{\frac{1}{2},\infty}}+C(\varepsilon)\|u\|;$$

(3) *$\eta_k=(-1)^{m_1}\alpha_{k1}\beta_{k2}-(-1)^{m_2}\alpha_{k2}\beta_{k1}\ne0$, $p\in(1,\infty)$, $k=1,2$;*

(4) *$s_i\in C^\gamma[0,T]$, $0<\gamma<1$ and there is a $b>0$ so that $\nu(s)\ge b>0$ and $\nu(s)$, $\nu^{-1}(s)$ are uniformly continuous with respect to s and $x\in[0,T]$.*

Then, for all $f\in L_p(G_s;E)$, $|\arg\lambda|\le\varphi$ and sufficiently large $|\lambda|$, problem (2.1)–(2.2) has a unique solution u belonging to $W_p^2(G_s;E(A),E)$ and the following uniform coercive estimate holds:

$$\sum_{j=0}^{2}|\lambda|^{1-\frac{j}{2}}\Big[\sum_{j=0}^{2}\|D_x^ju\|_{L_p(G_s;E)}+\|D_y^ju\|_{L_p(G_s;E)}\Big]+\|Au\|_{L_p(G_s;E)}\le C\|f\|_{L_p(G_s;E)}. \tag{6.1}$$

Proof Consider the principal part of the BVP (2.1)–(2.2), i.e., consider the following problem:

$$\begin{aligned}&L_0u=-D_x^2u(x,y)-D_y^2u(x,y)+(A(s)+\lambda)u(x,y)=f(x,y),\\&L_{jk}u=0,\quad j,k=1,2,\end{aligned} \tag{6.2}$$

where L_{jk} are defined by equalities (2.2). Let $Q_0(s)$ and $Q(s)$ be differential operators in $L_p(G_s;E)$ generated by BVP (6.2) and (2.1)–(2.1), respectively. Since $L_p(0,T;L_p(\sigma(s);E))=L_p(G_s;E)=X$, the BVP (6.2) can be expressed as the following ordinary DOE with variable coefficient:

$$\begin{aligned}&-D^2u(x)+[B(s(x))+\lambda]u(x)=f(x),\\&L_{1k}u=\sum_{i=0}^{m_k}\nu^{\delta_i}\Big[\alpha_{1ji}u^{(i)}(0)+\beta_{1ji}u^{(i)}(T)+\sum_{j=1}^{N_k}\delta_{1ji}u^{(i)}(x_{1kj})\Big]=0,\quad k=1,2,\end{aligned} \tag{6.3}$$

where $x_{1kj}\in(0,T)$ and $B(s)=B_s(x)$ is the differential operator in $L_p(\sigma;E)=E_s$ generated by BVP problem (4.15). Consider the operator B_s generated by problem (4.16). By estimate (4.3), we have the following uniform estimate:

$$\|A_s(x_0)B_s^{-1}(x_0)\|_{B(F)}\le C. \tag{6.4}$$

For $u\in F$, we have

$$\begin{aligned}&\|B_s(x)B_s^{-1}(x_0)u-B_s(\tau)B_s^{-1}(x_0)u\|_F\\&=\|[B_s(x)-B_s(\tau)]B_s^{-1}(x_0)u\|_F\\&\le\Big\|[\nu^{-2}(x)-\nu^{-2}(\tau)]\frac{\mathrm{d}^2}{\mathrm{d}x^2}B_s^{-1}(x_0)u\Big\|_F+\|[A_s(x)-A_s(\tau)]B_s^{-1}(x_0)u\|_F\\&\le\|[A_s(x)-A_s(\tau)]A_s^{-1}(x_0)\|_{B(F)}\|A_s(x_0)B_s^{-1}(x_0)u\|_F.\end{aligned} \tag{6.5}$$

By virtue of Result 4.1, we obtain that $D(B_s)$ is independent of s and x and B_s is uniformly positive in F. By conditions (1.1)–(2.2), in view of (6.4) and (6.5) and by estimate (4.3), we obtain that the function $B_s(x)B_s^{-1}(x_0)$ is bounded and uniformly continuous with respect to the collection of s and $x \in [0,T]$. It implies that $B_s(x)B_s^{-1}(x_0)$ is bounded and uniformly continuous with respect to the collection of s and $x \in [0,T]$. By virtue of [3, Theorem 4.5.2], $E_s \in \mathrm{UMD}$ provided $E \in \mathrm{UMD}$, $p \in (1,\infty)$. Theorem 4.2 implies that the operator B_s is uniformly R-positive in E_s. Therefore, by virtue of [26, Theorem 3] for all $f \in L_p(0,T;E_s)$, $|\arg\lambda| \le \varphi$ and sufficiently large $|\lambda|$, problem (6.2) has a unique solution u belonging to $W_p^2(0,T;D(B_s),F)$ and the following uniform coercive estimate holds:

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \|D_x^j u\|_{L_p(0,T;F)} + \|B_s u\|_{L_p(0,T;F)} \le C\|f\|_{L_p(0,T;F)}. \tag{6.6}$$

Then, by estimate (6.6) and by the substitution of variable $y = s_0 + (s_1 - s_0)t,\ t \in (0,1)$, we obtain that for $f \in L_p(G_s;E)$, $|\arg\lambda| \le \varphi$ and sufficiently large $|\lambda|$, problem (6.2) has a unique solution u belonging to $W_p^2(G_s;E(A),E)$ and the uniform in s coercive estimate (6.1) holds for the solution to problem (6.3). This implies the estimate

$$\sum_{j=0}^{2}(|\lambda|^{1-\frac{j}{2}}\|D_x^j(Q_0+\lambda)^{-1}f\|_X + |\lambda|^{1-\frac{j}{2}}\|D_y^j(Q_0+\lambda)^{-1}f\|_X) + \|A(Q_0+\lambda)^{-1}f\|_X \le C\|f\|_X. \tag{6.7}$$

By condition (1.1) and by virtue of Theorem 1.1 for all $u \in W_p^2(G_s;E(A),E)$, for any $\varepsilon > 0$ there is a $C(\varepsilon) > 0$, such that

$$\|A_1 D_x u\|_X + \|A_2 D_y u\|_X \le \varepsilon\|u\|_{W_p^2(G_s;E(A),E)} + C(\varepsilon)\|u\|_X. \tag{6.8}$$

By using estimates (6.7)–(6.8) for sufficiently large $|\lambda|$, we obtain

$$\|A_1 D_x u\|_X + \|A_2 D_y u\|_X \le \varepsilon\|(Q_0+\lambda)u\|_X. \tag{6.9}$$

Then, in view of estimates (6.8)–(6.9) and in virtue of the perturbation theory of linear operators [17, Theorem 14.1], we obtain the estimate (6.1).

Theorem 6.2 *Let all conditions of Theorem* 6.1 *be satisfied and* $A^{-1} \in \sigma_\infty(E)$. *Then, problem* (2.1)–(2.2) *is Fredholm in* $L_p(G_s;E)$ *for* $\lambda = 0$.

Proof Theorem 6.1 implies that the differential operator $Q(s)$ has a bounded inverse from $L_p(G_s;E)$ to $W_p^2(G_s;D(A),E)$ for sufficiently large $|\lambda|$. Fredholmness of the operator Q is obtained then from Theorem 5.1 by virtue of compactness of embedding of $W_p^2(G_s;E(A),E)$ into $L_p(G_s;E)$ (see [28, Theorem 2]) and by the perturbation theory of linear operators [16, Theorem 14.1].

Result 6.1 Theorem 6.1 implies that the differential operator $Q = Q(s)$ has a resolvent operator $(Q+\lambda)^{-1}$ for $\lambda \in S(\varphi)$, $\varphi \in [0,\pi)$ and the following uniform estimate holds:

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}}(\|D_x^j(Q+\lambda)^{-1}\|_X + \|D_y^j(Q+\lambda)^{-1}\|_X) + \|A(Q+\lambda)^{-1}\|_X \le C.$$

Remark 6.1 Assume that all conditions of Theorem 6.1 are satisfied. Then in virtue of R-positivity of A, by using the representation of the solution to problem (2.1)–(2.2) (see [11, Lemma 7.1]) and a similar technique as in [11, Theorem 7.4], we conclude that the operator Q is R-positive in $L_p(G_s;E)$.

7 Free Boundary Value Problems for Anisotropic Elliptic Equations

The Fredholm property of BVPs for elliptic equations with parameters in smooth domains were studied e.g. in [1, 9], and for nonsmooth domains these questions were investigated e.g. in [12, 13].

Let $\Omega \subset \mathbb{R}^n$ be an open connected set with compact $\mathbb{C}^{2m}$-boundary $\partial\Omega$. Let us consider the nonlocal free BVPs on cylindrical domain $\widetilde{\Omega} = G_s \times \Omega$ for the following anisotropic elliptic equation:

$$\begin{aligned} Lu &= -\sum_{k=1}^{2} \frac{\partial^2 u(x,y)}{\partial x_k^2} + \sum_{k=1}^{2} d_k \frac{\partial u(x,y)}{\partial x_k} \\ &\quad + \sum_{|\alpha| \le 2m} a_\alpha(y) D_y^\alpha u(x,y) = f(x,y), \quad x \in G_s,\ y \in \Omega, \end{aligned} \tag{7.1}$$

$$\begin{aligned} L_{1k}u &= \sum_{i=0}^{m_{1k}} \nu^{i-\frac{1}{p}} [\alpha_{1ki} u_{x_1}^{(i)}(0, x_2, y) + \beta_{1ki} u_{x_1}^{(i)}(T, x_2, y)] \\ &\quad + \sum_{j=1}^{N_k} \delta_{1kj} u_{x_1}^{(i)}(x_{kj}, x_2, y) = 0, \\ L_{2k}u &= \sum_{i=0}^{m_{2k}} \nu^{i-\frac{1}{p}} [\alpha_{2ki} u_{x_2}^{(i)}(x_1, s_0(x), y) + \beta_{2ki} u_{x_2}^{(i)}(x_1, s_1(x), y)] \\ &\quad + \sum_{j=1}^{N_k} \delta_{2kj} u_{x_2}^{(i)}(x_1, y_{kj}, y) = 0, \quad k = 1, 2, \end{aligned} \tag{7.2}$$

$$B_j u = \sum_{|\beta| \le m_j} b_{j\beta}(y) D_y^\beta u(x,y) = 0, \quad x \in G_s,\ y \in \partial\Omega,\ j = 1, 2, \cdots, m, \tag{7.3}$$

where $G_s = \{x = (x_1, x_2),\ 0 < x_1 < T,\ x_2 \in \sigma(s)\}$,

$$\begin{aligned} &s = (s_0, s_1), \quad s_0 = s_0(x), \quad s_1 = s_1(x), \quad \sigma(s) = (s_0(x), s_1(x)), \\ &\nu = \nu(s) = s_1(x) - s_0(x), \quad x \in [0, T], \quad x_{kj} \in (0, T), \quad y_{kj} \in (s_1(x), s_2(x)), \end{aligned}$$

Γ is a boundary of the region $\Omega \in \mathbb{R}^n$, a_k, α_{ikj} and β_{ikj} are complex-valued function on G_s, $D_j = -i\frac{\partial}{\partial y_j}$, $m_k \in \{0, 1\}$, $y = (y_1, \cdots, y_n)$.

Let $\Omega \subset \mathbb{R}^n$ be an open connected set with compact $\mathbb{C}^{2m}$-boundary $\partial\Omega$. Recall that for all $y_0 \in \partial\Omega$ local coordinates corresponding to y_0 are defined as coordinates obtained from the original ones by a rotation and a shift which transfers y_0 to the origin and after which the positive y_l-axis has the direction of the interior normal to $\partial\Omega$ at y_0.

If $\widetilde{\Omega}_s = G_s \times \Omega$, $\mathbf{p} = (p_1, p)$, $L_{\mathbf{p}}(\widetilde{\Omega}_s)$ will denote the space of all $\mathbf{p}$-summable scalar-valued functions with mixed norm (see, e.g., [7, Section 1]), i.e., the space of all measurable functions f defined on $\widetilde{\Omega}_s$, for which

$$\|f\|_{L_{\mathbf{p}}(\widetilde{\Omega}_s)} = \Big(\int_{G_s} \Big(\int_\Omega |f(x,y)|^{p_1} \mathrm{d}y \Big)^{\frac{p}{p_1}} \mathrm{d}x \Big)^{\frac{1}{p}} < \infty.$$

Analogously, $W_{\mathbf{p}}^{2,2m}(\widetilde{\Omega}_s)$ denotes the anisotropic Sobolev space with corresponding mixed norm (see [7, Section 10]).

Theorem 7.1 *Let the following conditions be satisfied:*

(1) *$s_i \in C^\gamma[0,T]$, $0<\gamma<1$ and there is a $b>0$ so that $\nu(s)\geq b>0$ and $\nu(s)$, $\nu^{-1}(s)$ are uniformly continuous with respect to s and $x\in[0,T]$ and $\eta_k=(-1)^{m_1}\alpha_{k1}\beta_{k2}-(-1)^{m_2}\alpha_{k2}\beta_{k1}\neq 0$, $p,p_1\in(1,\infty)$, where $\alpha_{ki}=\alpha_{kim_{ik}},\beta_{ki}=\beta_{kim_{ik}}$;*

(2) *$a_\alpha\in C(\overline{\Omega})$ for each $|\alpha|=2m$ and $a_\alpha\in[L_\infty+L_{r_k}](\Omega)$ for each $|\alpha|=k<2m$ with $r_k\geq q$ and $2m-k>\frac{l}{r_k}$;*

(3) *$b_{j\beta}\in C^{2m-m_j}(\partial\Omega)$ for each j, β and $m_j<2m$, $B_{j0}(y',\xi)=\sum\limits_{|\beta|=m_j} b_{j\beta}(y',\xi)\neq 0$ for $y'\in\partial\Omega$, where $\xi=(\xi_1,\xi_2,\cdots,\xi_n)\in\mathbb{R}^n$ is normal to $\partial\Omega$;*

(4) *for $y\in\overline{\Omega}$, $\xi\in\mathbb{R}^n$, $\lambda\in S(\varphi)$, $\varphi\in(0,\pi)$, $|\xi|+|\lambda|\neq 0$, let $\lambda+A_0(y,\xi)\neq 0$, where $A_0(y,\xi)=\sum\limits_{|\alpha|=2m} a_\alpha(y)\xi^\alpha$;*

(5) *for each $y_0\in\partial\Omega$, local BVP in local coordinates corresponding to y_0*

$$\lambda+A_0(y_0,\xi',D_{n+1})\vartheta(y)=0,\quad y>0,$$
$$B_{j0}(y_0,\xi',D_{n+1})\vartheta(0)=h_j,\quad j=1,2,\cdots,m$$

has a unique solution $\vartheta\in C_0(R_+)$ for all $h=(h_1,h_2,\cdots,h_n)\in\mathbb{R}^n$ and $\lambda\in S(\varphi)$, $\xi'\in\mathbb{R}^n$ with $|\xi'|+|\lambda|\neq 0$.

Then, we have that

(a) *for all $f\in L_{\mathbf{p}}(\widetilde{\Omega}_s;E)$, $|\arg\lambda|\leq\varphi$ and sufficiently large $|\lambda|$, problem (7.1)–(7.3) has a unique solution u that belongs to $W_{\mathbf{p}}^{2,2m}(\widetilde{\Omega}_s)$ and the following coercive uniform estimate holds:*

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\Big\|\frac{\partial^i u}{\partial x_k}\Big\|_{L_{\mathbf{p}}(\widetilde{\Omega}_s)}+\sum_{|\beta|=2m}\|D_y^\beta u\|_{L_{\mathbf{p}}(\widetilde{\Omega}_s)}\leq C\|f\|_{L_{\mathbf{p}}(\widetilde{\Omega}_s)};$$

(b) *problem (7.1)–(7.3) is Fredholm in $L_{\mathbf{p}}(\widetilde{\Omega}_s)$.*

Proof Let $E=L_{p_1}(\Omega)$. By [7], the space $L_{p_1}(\Omega)$, $p_1\in(1,\infty)$ is UMD. Consider the operator A defined by

$$D(A)=W_{p_1}^{2m}(\Omega;B_ju=0),\quad Au=\sum_{|\alpha|\leq 2m}a_\alpha(y)D^\alpha u(y).$$

For $x\in\Omega$ also consider the following operators:

$$A_k(x)u=d_k(x,y)u(y),\quad k=1,2,\cdots,n.$$

Problem (7.1)–(7.3) can be rewritten in the form (2.1)–(2.2), where $u(x)=u(x,\cdot)$, $f(x)=f(x,\cdot)$ are functions with values in $E=L_{p_1}(\Omega)$. By virtue of [1], problem

$$\lambda u(y)+\sum_{|\alpha|\leq 2m}a_\alpha(y)D_y^\alpha u(y)=f(y),$$
$$B_ju=\sum_{|\beta|\leq m_j}b_{j\beta}(y)D_y^\beta u(y)=0,\quad j=1,2,\cdots,m$$

has a unique solution for $f\in L_{p_1}(\Omega)$ and arg $\lambda\in S(\varphi)$, $|\lambda|\to\infty$. Moreover, in view of [9, Theorem 8.2], the differential operator A is R-positive in L_{p_1}. It is known that the embedding $W_{p_1}^{2m}(\Omega)\subset L_{p_1}(\Omega)$ is compact (see e.g. [30, Theorem 3.2.5]). Then by using interpolation properties of Sobolev spaces (see e.g. [31, Section 4]) it is clear to see that condition (2) of Theorem 6.1 holds. Conditions (1)–(5) imply that the other conditions of Theorem 6.1 are fulfilled too. Then from Theorems 6.1–6.2, the assertions are obtained.

8 Nonlocal Free Boundary Value Problems for Infinite Systems of Elliptic Equations

The Fredholm property of boundary value problems for elliptic equations with parameters in smooth domains was studied in [2, 9] and for non-smooth domains it was treated e.g. in [13]. In this section, the maximal regularity of nonlocal BVPs for finite and infinite systems of elliptic equations are established. Consider the following infinite system of nonlocal boundary value problems:

$$
\begin{aligned}
&- D_x^2 u_m(x,y) - D_y^2 u_m(x,y) + \sum_{j=1}^{\infty} d_{1mj}(x,y) D_x u_j(x,y) \\
&+ \sum_{j=1}^{\infty} d_{2mj}(x,y) D_y u_j(x,y) + \sum_{j=1}^{\infty} [d_j + \lambda] u_j(x,y) = f_m(x,y), \quad x, y \in G_s, \qquad (8.1)
\end{aligned}
$$

$$
\begin{aligned}
L_{1k} u_m &= \sum_{i=0}^{m_{1k}} \nu^{\delta_i} \Big[\alpha_{1ki} u_m^{(i)}(0,y) + \beta_{1k} u^{(i)}(T,y) + \sum_{j=1}^{N_k} \delta_{1kj} u_m^{(i)}(x_{kj}, y) \Big] = 0, \\
L_{2k} u_m &= \sum_{i=0}^{m_{2k}} \nu^{\delta_i} \Big[\alpha_{2ki} u_m^{(i)}(x, s_0(x)) + \beta_{2k} u_m^{(i)}(x, s_1(x)) + \sum_{j=1}^{N_k} \delta_{2kj} u_m^{(i)}(x, y_{kj}) \Big] \qquad (8.2) \\
&= 0, \quad k = 1, 2,
\end{aligned}
$$

where

$$
\Omega_s = \{ (x,y) \in \mathbb{R}^2,\ x \in [0,T],\ x_{kj} \in (0,T),\ y \in \sigma(s) \}
$$

is moving domain and

$$
\begin{aligned}
&y_{kj} \in \sigma(s), \quad \sigma(s) = (s_0(x), s_1(x)), \quad \delta_i = i - \frac{1}{p}, \\
&D = \{d_m\}, \quad d_m > 0, \quad u = \{u_m\}, \quad Du = \{d_m u_m\}, \quad m = 1, 2, \cdots, \\
&l_q(D) = \Big\{ u \colon u \in l_q, \|u\|_{l_q(D)} = \|Du\|_{l_q} = \Big(\sum_{m=1}^{\infty} |d_m u_m|^q \Big)^{\frac{1}{q}} < \infty \Big\}, \quad 1 < q < \infty.
\end{aligned}
$$

Let $O(s)$ denote the differential operator in $L_p(\Omega_s; l_q)$ generated by BVP (8.1)–(8.2).

Theorem 8.1 *Suppose that $s_i \in C^\gamma[0,T]$, $0 < \gamma < 1$ and there is a $b > 0$ so that $\nu(s) \geq b > 0$ and $\nu(s)$, $\nu^{-1}(s)$ are uniformly continuous with respect to s and $x \in [0,T]$ and*

$$
\begin{aligned}
&(-1)^{m_1} \alpha_{i1} \beta_{i2} - (-1)^{m_2} \alpha_{i2} \beta_{i1} \neq 0, \quad i = 1, 2, \\
&\max_k \sup_m \sum_{j=1}^{\infty} d_{kmj}(x) d_j^{-(\frac{1}{2} - \mu)} < M, \quad 0 < \mu < \frac{1}{2},\ x \in [0,T].
\end{aligned}
$$

Then, for all $f(x) = \{f_m(x)\}_1^\infty \in L_p(\Omega_s; l_q)$ and for sufficiently large $|\lambda|$, problem (8.1)–(8.2) has a unique solution $u = \{u_m(x)\}_1^\infty$ that belongs to space $W_p^2(\Omega_s, l_q(D), l_q)$, and the following uniform coercive estimate holds:

$$
\Big[\Big(\int_{\Omega_s} \sum_{m=1}^{\infty} |D_x^2 u_m(x)|^q \Big)^{\frac{p}{q}} \mathrm{d}x \Big]^{\frac{1}{p}} + \Big[\Big(\int_{\Omega_s} \sum_{m=1}^{\infty} |D_y^2 u_m(x)|^q \Big)^{\frac{p}{q}} \mathrm{d}x \Big]^{\frac{1}{p}}
$$

$$+\Big[\Big(\int_{\Omega_s}\sum_{m=1}^{\infty}|d_m u_m(x)|^q\Big)^{\frac{p}{q}}\mathrm{d}x\Big]^{\frac{1}{p}}$$

$$\le C\Big[\Big(\int_{\Omega_s}\sum_{m=1}^{\infty}|f_m(x)|^q\Big)^{\frac{p}{q}}\mathrm{d}x\Big]^{\frac{1}{p}}. \tag{8.3}$$

Moreover, problem (8.1)–(8.2) *is Fredholm in* $L_p(\Omega_s; l_q)$ *for* $\lambda = 0$.

Proof Really, let $E = l_q$, A and A_k be infinite matrices, such that

$$A = d_m\delta_{jm}, \quad A_k = [d_{kmj}(x)], \quad m, j = 1, 2, \cdots, \infty.$$

It is easy to see that this operator A is R-positive in l_q. Therefore, by virtue of Theorem 6.1, we obtain that the problem (8.1)–(8.2) for $f \in L_p(\Omega_s; l_q)$, $|\arg\lambda| \le \varphi$ and sufficiently large $|\lambda|$, has a unique solution u that belongs to $W_p^l(\Omega_s; l_q(D), l_q)$ and the following coercive estimate holds:

$$\|D_x^2 u\|_{L_p(\Omega_s;l_q)} + \|D_y^2 u\|_{L_p(\Omega_s;l_q)} + \|Du\|_{L_p(\Omega_s;l_q)} \le C\|f\|_{L_p(\Omega_s;l_q)}. \tag{8.4}$$

Namely, we obtain the estimate (8.3). Moreover, by Theorem 6.2, problem (8.1)–(8.2) for $\lambda = 0$ is Fredholm in $L_p(\Omega_s; l_q)$.

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