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Density Results in Sobolev Spaces Whose Elements Vanish on a Part of the Boundary

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Abstract This paper is devoted to the study of the subspace of $W^{m,r}$ of functions that vanish on a part γ_0 of the boundary. The author gives a crucial estimate of the Poincaré constant in balls centered on the boundary of γ_0. Then, the convolution-translation method, a variant of the standard mollifier technique, can be used to prove the density of smooth functions that vanish in a neighborhood of γ_0, in this subspace. The result is first proved for $m = 1$, then generalized to the case where $m \geq 1$, in any dimension, in the framework of Lipschitz-continuous domain. However, as may be expected, it is needed to make additional assumptions on the boundary of γ_0, namely that it is locally the graph of some Lipschitz-continuous function.

Keywords Density results, Boundary value problems, Sobolev spaces
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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, whose boundary is Lipschitz-continuous. This article mainly deals with functions of $W^{m,r}(\Omega, \gamma_0)$, where $r > 1$ and $m \geq 1$, which are functions of $W^{m,r}(\Omega)$ that vanish on an open part γ_0 of the boundary $\partial\Omega$. More precisely, we study the density of smooth functions that vanish on a neighborhood of γ_0 in the space $W^{m,r}(\Omega, \gamma_0)$. This density is well-known in particular cases and is used in [1]. It is proven in two dimensions for $m = 1$ in [1] by introducing a convolution-translation operator. The aim of the present paper is to prove the density result in the general case, dimension $d \geq 3$ and $m \geq 1$, in the same way as in [1]. Indeed, this method of convolution-translation is very interesting because it allows us to really construct the approximation by smooth functions and it is understandable also for nonspecialists. It thus seems useful to give a detailed proof, by a constructive method, within easy reach, of these significant results.

Let γ_1 denote the complementary set of $\overline{\gamma}_0$ in the boundary $\partial\Omega$. In two dimensions, it is generally assumed, as in [1], that $\overline{\gamma}_0 \cap \overline{\gamma}_1$ is composed of a finite number of points. In this article, we assume that the intersection $\overline{\gamma}_0 \cap \overline{\gamma}_1$ has a finite number of connected components and that the boundary of γ_0 is locally the graph of some Lipschitz-continuous function, which allows us to derive a basic estimate of the Poincaré constant in balls centered on $\overline{\gamma}_0 \cap \overline{\gamma}_1$. We use a modified mollification technique, initiated by [6] and rediscovered simultaneously in [1, 4], which consists

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in combining a convolution and a translation. First, we localize and establish a partition of unity, which allows us to distinguish three parts in the boundary. On a neighborhood of a point of γ_0, we make a translation outside the domain, in a neighborhood of a point of γ_1, make a translation inside the domain, and next apply, in both cases, the mollification technique. On the third part of the boundary, which is composed of neighborhoods of the connected components of $\overline{\gamma}_0 \cap \overline{\gamma}_1$, because of Poincaré's inequality, we approximate the function by 0.

In dimension $d \geq 3$, the neighborhoods of the connected components of $\overline{\gamma}_0 \cap \overline{\gamma}_1$ are no longer balls, which complicate the previous approximation by 0: we consider an optimal covering by balls and a special technique of permutation and partition to deal with the intersections of balls in the estimates.

In this paper, the main result is Theorem 3.1, which establishes the density in $W^{1,r}(\Omega, \gamma_0)$, that is to say the density result for $m = 1$. The generalization to the case $m \geq 1$, which is Theorem 4.1, is straightforward.

This article is organized as follows. In Section 2, we define the adequate covering of $\overline{\Omega}$ and the partition of unity subordinated to this covering. In Section 3, we prove our main density result in $W^{1,r}(\Omega, \gamma_0)$. Finally, Section 4 is devoted to the generalization of this result to the space $W^{m,r}(\Omega, \gamma_0)$, with $m \geq 1$.

We end this introduction with some notation that we shall use further on. We recall that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$, whose boundary is Lipschitz-continuous. Let γ_0 and γ_1 be two non-empty open parts of $\partial\Omega$ that have a finite number of connected components and verify

$$\partial\Omega = \overline{\gamma}_0 \cup \overline{\gamma}_1, \quad \gamma_0 \cap \gamma_1 = \emptyset, \quad \overline{\gamma}_0 \cap \overline{\gamma}_1 = \bigcup_{k=1}^{q} K_k, \tag{1.1}$$

where K_k, $1 \leq k \leq q$, denote the connected components of $\overline{\gamma}_0 \cap \overline{\gamma}_1$ and, for $1 \leq k \leq q$, let us set

$$\forall\, \alpha > 0, \quad G_{k,\alpha} = \{\mathbf{x} \in \mathbb{R}^d,\ d(\mathbf{x}, K_k) < \alpha\}, \tag{1.2}$$

where $d(\,\cdot\,,\,\cdot\,)$ is the Euclidian distance in $\mathbb{R}^d$. Afterwards, we choose α such that

$$0 < \alpha < \alpha_0' = \frac{1}{2} \min_{\substack{1 \leq i,j \leq q \\ i \neq j}} d(K_i, K_j) \quad \text{and} \quad \alpha \leq 1. \tag{1.3}$$

We define for each real $r > 1$ and each integer $m \geq 1$,

$$W^{m,r}(\Omega, \gamma_0) = \Big\{ v \in W^{m,r}(\Omega),\ \Big(\frac{\partial^j v}{\partial n^j}\Big)\Big|_{\gamma_0} = 0,\ j = 0, \cdots, m-1 \Big\}, \tag{1.4}$$

$$\mathcal{D}(\overline{\Omega}, \gamma_0) = \{v \in \mathcal{D}(\overline{\Omega}),\ v \text{ is equal to 0 in a neighborhood of } \gamma_0\}. \tag{1.5}$$

2 Partition of Unity

2.1 First covering of $\overline{\Omega}$

Since the boundary of Ω is Lipschitz-continuous, for every $\mathbf{x} \in \partial\Omega$, there exist an open hypercube $C_{\mathbf{x}}$, which is a neighborhood of $\mathbf{x}$ in $\mathbb{R}^d$, and new orthogonal coordinates $\mathbf{y} = (\mathbf{y}', y_d)$, where $\mathbf{y}' = (y_1, \cdots, y_{d-1})$, such that

(i) $C_{\mathbf{x}} = \prod_{j=1}^{d}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[.$

(ii) There exists a Lipschitz-continuous function $\Phi^{\mathbf{x}}$ defined in $\prod_{j=1}^{d-1}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[$ of constant $L_{\mathbf{x}}$, such that $\forall \mathbf{y}' \in \prod_{j=1}^{d-1}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[,\ |\Phi^{\mathbf{x}}(\mathbf{y}')| \leq \frac{a_{\mathbf{x},d}}{2}$ and

$$\Omega \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d < \Phi^{\mathbf{x}}(\mathbf{y}')\}, \quad \partial\Omega \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d = \Phi^{\mathbf{x}}(\mathbf{y}')\}. \tag{2.1}$$

Moreover, $\forall\, \mathbf{x} \in \gamma_0 \cup \gamma_1$, $\forall j = 1, \cdots, d$, we choose the real numbers $a_{\mathbf{x},j}$ such that $C_{\mathbf{x}} \cap \overline{\gamma}_0 \cap \overline{\gamma}_1 = \emptyset$. Since $\forall x \in \gamma_0$, $C_{\mathbf{x}} \cap \overline{\gamma}_1 = \emptyset$ and $\forall\, \mathbf{x} \in \gamma_1$, $C_{\mathbf{x}} \cap \overline{\gamma}_0 = \emptyset$, we have

$$\begin{aligned} \forall \mathbf{x} \in \gamma_0, &\quad \gamma_0 \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d = \Phi^{\mathbf{x}}(\mathbf{y}')\}, \\ \forall \mathbf{x} \in \gamma_1, &\quad \gamma_1 \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d = \Phi^{\mathbf{x}}(\mathbf{y}')\}. \end{aligned}$$

In addition, for $d > 2$, denoting $\mathbf{y} = (\mathbf{y}'', y_{d-1}, y_d)$, we assume that, for every $\mathbf{x} \in \overline{\gamma}_0 \cap \overline{\gamma}_1$, the previous open hypercube $C_{\mathbf{x}}$ is such that there exists a second Lipschitz-continuous function $\Psi^{\mathbf{x}}$ defined in the set $\prod_{j=1}^{d-2}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[$ of constant $M_{\mathbf{x}}$, such that $\forall \mathbf{y}'' \in \prod_{j=1}^{d-2}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[,\ |\Psi^{\mathbf{x}}(\mathbf{y}'')| \leq \frac{a_{\mathbf{x},d-1}}{2}$ and

$$\gamma_0 \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d = \Phi^{\mathbf{x}}(\mathbf{y}'),\ y_{d-1} > \Psi^{\mathbf{x}}(\mathbf{y}'')\}, \tag{2.2}$$

$$\gamma_1 \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d = \Phi^{\mathbf{x}}(\mathbf{y}'),\ y_{d-1} < \Psi^{\mathbf{x}}(\mathbf{y}'')\}. \tag{2.3}$$

For $d = 2$, we set 0 in the place of $\Psi^{\mathbf{x}}(\mathbf{y}'')$ in (2.2) and (2.3).

For every strictly positive real number α verifying (1.3), let us define a finite open covering of $\overline{\Omega}$ as follows.

First, we have

$$\partial\Omega \subset \Big(\bigcup_{\mathbf{x} \in \gamma_0 \cup \gamma_1} C_{\mathbf{x}} \Big) \bigcup \Big(\bigcup_{k=1}^{q} G_{k,\frac{\alpha}{2}} \Big).$$

Note that, owing to (1.3), $G_{i,\alpha} \cap G_{j,\alpha} = \emptyset$, $1 \leq i, j \leq q,\ i \neq j$. Second, the compactness implies that there exists a finite open covering of $\partial\Omega$:

$$\partial\Omega \subset \Big(\bigcup_{k=1}^{q} G_{k,\frac{\alpha}{2}} \Big) \bigcup \Big(\bigcup_{k=q+1}^{r_\alpha} C_{\mathbf{m}_{k,\alpha}} \Big), \tag{2.4}$$

where the open sets $C_{\mathbf{x}}$ are defined by (2.1) and $G_{k,\alpha}$ is defined by (1.2). Moreover, there exists an open set $C_{0,\alpha}$, such that

$$\overline{C}_{0,\alpha} \subset \Omega \quad \text{and} \quad \overline{\Omega} \subset C_{0,\alpha} \bigcup \Big(\bigcup_{k=1}^{q} G_{k,\frac{\alpha}{2}} \Big) \bigcup \Big(\bigcup_{k=q+1}^{r_\alpha} C_{\mathbf{m}_{k,\alpha}} \Big), \tag{2.5}$$

which is an open covering of $\overline{\Omega}$ denoted by $\mathcal{R}_\alpha$.

2.2 Second covering of $\overline{\Omega}$ and associated partition of unity

Let ρ be a standard mollifier, which means that ρ is a positive C^∞ function in $\mathbb{R}^d$ supported in the unit ball, such that $\int_{\mathbb{R}^d} \rho(\mathbf{x})\mathrm{d}\mathbf{x} = 1$. For every $p \in \mathbb{N}^*$, we define

$$\forall \mathbf{x} \in \mathbb{R}^d, \quad \rho_p(\mathbf{x}) = p^d \rho(p\mathbf{x}). \tag{2.6}$$

Let φ belong to $C^1(\mathbb{R}_+)$, such that

$$\forall t \in \Big[0, \frac{9}{16}\Big], \quad \varphi(t) = 1, \ \forall t \geq \frac{11}{16}, \quad \varphi(t) = 0 \quad \text{and} \quad \forall t \in \mathbb{R}_+, \quad |\varphi'(t)| \leq A.$$

For example, we can choose φ defined on $[\frac{9}{16}, \frac{11}{16}]$ by $\varphi(t) = \frac{1+\cos(8\pi t - \frac{9\pi}{2})}{2}$, with $A = 4\pi$. Let us recall that, for $k = 1, \cdots, q$ and $i = 1, \cdots, d$, $\mathbf{x} \mapsto \partial_i d(\mathbf{x}, K_k)$ belongs to $L^\infty(\mathbb{R}^d)$ and verifies

$$\forall i = 1, \cdots, d, \ \forall \mathbf{x} \in \mathbb{R}^d, \quad |\partial_i d(\mathbf{x}, K_k)| \leq 1 \tag{2.7}$$

(see [4]). Then, we set

$$\forall k, \ 1 \leq k \leq q, \quad \theta_{\alpha,k} = \varphi\Big(\frac{1}{\alpha} d(\,\cdot\,, K_k)\Big) * \rho_{p_\alpha} \tag{2.8}$$

with $p_\alpha = [\frac{16}{\alpha}] + 1$, where $[x]$ denotes the integral part of the real number x, and ρ_p is defined by (2.6). This function belongs to $\mathcal{D}(G_{k,\alpha})$ and verifies, for $i = 1, \cdots, d$,

$$\begin{aligned} &\forall \mathbf{x} \in G_{k,\frac{\alpha}{2}}, \quad \theta_{\alpha,k}(\mathbf{x}) = 1, \quad \forall \mathbf{x} \notin G_{k,\frac{3\alpha}{4}}, \quad \theta_{\alpha,k}(\mathbf{x}) = 0, \\ &\forall \mathbf{x} \in \mathbb{R}^d, \quad |\partial_i \theta_{\alpha,k}(\mathbf{x})| \leq \frac{A}{\alpha}. \end{aligned} \tag{2.9}$$

Considering successively that $\theta_{\alpha,j} + (1 - \theta_{\alpha,j}) = 1$, for $j = 1, \cdots, q$, we obtain

$$\theta_{\alpha,1} + (1 - \theta_{\alpha,1})\theta_{\alpha,2} + \cdots + \Big(\prod_{j=1}^{q-1}(1 - \theta_{\alpha,j})\Big)\theta_{\alpha,q} + \prod_{j=1}^{q}(1 - \theta_{\alpha,j}) = 1.$$

But, since the sets $G_{j,\alpha}$ are disconnected and since $\theta_{\alpha,j}$ belongs to $\mathcal{D}(G_{j,\alpha})$, for $1 \leq j \leq q$, we have $\Big(\prod_{j=1}^{k-1}(1 - \theta_{\alpha,j})\Big)\theta_{\alpha,k} = \theta_{\alpha,k}$. Thus, we obtain

$$\theta_{\alpha,1} + \theta_{\alpha,2} + \cdots + \theta_{\alpha,q} + \prod_{j=1}^{q}(1 - \theta_{\alpha,j}) = 1.$$

Hence, we derive, for every $u \in W^{1,r}(\Omega, \gamma_0)$,

$$u = \theta_{\alpha,1}u + \theta_{\alpha,2}u + \cdots + \theta_{\alpha,q}u + \Big(\prod_{j=1}^{q}(1 - \theta_{\alpha,j})\Big)u. \tag{2.10}$$

Let $\{\beta_{\alpha,j}\}_{j=0}^{r_\alpha}$ be a partition of unity on $\overline{\Omega}$ (see [2] or [3]), subordinated to the covering $\mathcal{R}_\alpha$ defined by (2.5). Substituting the functions $\beta_{\alpha,j}$ in (2.10) yields

$$u = \theta_{\alpha,1}u + \theta_{\alpha,2}u + \cdots + \theta_{\alpha,q}u + \sum_{k=0}^{r_\alpha}\Big(\prod_{j=1}^{q}(1 - \theta_{\alpha,j})\Big)\beta_{\alpha,k}u.$$

Considering that, for every $1 \le k \le q$, $\prod_{j=1}^{q}(1-\theta_{\alpha,j})\beta_{\alpha,k} = 0$, since, if $\mathbf{x} \in G_{k,\frac{\alpha}{2}}$, $\theta_{\alpha,k}(x) = 1$, we obtain

$$u = \sum_{k=0}^{r_\alpha} \varphi_{\alpha,k} u, \tag{2.11}$$

where $\varphi_{\alpha,k} = \Big(\prod_{j=1}^{q}(1-\theta_{\alpha,j})\Big)\beta_{\alpha,k}$, $k=0$ or $q+1 \le k \le r_\alpha$ and $\varphi_{\alpha,k} = \theta_{\alpha,k},\ 1 \le k \le q$. Thus, for α verifying (1.3), $\mathcal{P}_\alpha = \{\varphi_{\alpha,k}\}_{k=0}^{r_\alpha}$ is a partition of unity on $\overline{\Omega}$, subordinated to the covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$, with

$$\begin{aligned} &\mathcal{O}_{0,\alpha} = C_{0,\alpha}, \quad \mathcal{O}_{k,\alpha} = G_{k,\alpha} \quad \text{for } 1 \le k \le q, \\ &\mathcal{O}_{k,\alpha} = C_{\mathbf{m}_{k,\alpha}} \quad \text{for } q+1 \le k \le r_\alpha, \end{aligned} \tag{2.12}$$

where the sets $C_{0,\alpha}$, $G_{k,\alpha}$ and $C_{\mathbf{x}}$ are respectively defined by (2.5), (1.2) and (2.1).

3 Density Result in $W^{1,r}(\Omega, \gamma_0)$

Theorem 3.1 *Let $r > 1$ be a real number. Let Ω be a bounded domain in $\mathbb{R}^d$ whose boundary is Lipschitz-continuous and let γ_0 be an open part of $\partial\Omega$ verifying* (1.1). *Let the spaces $W^{1,r}(\Omega, \gamma_0)$ and $\mathcal{D}(\overline{\Omega}, \gamma_0)$ be defined respectively by* (1.4) *and* (1.5). *Then the space $\mathcal{D}(\overline{\Omega}, \gamma_0)$ is dense in $W^{1,r}(\Omega, \gamma_0)$.*

Proof From now on, we suppose that α verifies (1.3), so we can consider the partition $\mathcal{P}_\alpha$ defined by (2.12). For every real number $\varepsilon > 0$, let us define a real $\alpha_\varepsilon > 0$, such that for $0 < \alpha \le \alpha_\varepsilon$, the partition of unity $\mathcal{P}_\alpha$ subordinated to the covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$ allows us to construct an approximation $u_\varepsilon \in \mathcal{D}(\overline{\Omega},\ \gamma_0)$ of $u \in W^{1,r}(\Omega,\ \gamma_0)$ in $W^{1,r}$ norm.

Let us prove a first lemma which allows us to define, for every $k > 1$, an extension $v_\alpha \in W^{1,r}(B(\mathbf{0}, k\alpha))$ of $v \in W^{1,r}(B(\mathbf{0}, \alpha))$, such that the norm of v_α in $W^{1,r}(B(\mathbf{0}, k\alpha))$ is bounded by the norm of v in $W^{1,r}(B(\mathbf{0}, \alpha))$ multiplied by a constant independant of α.

Lemma 3.1 *For every $\mathbf{y} \in \mathbb{R}^d$, $\alpha > 0$ and $k > 1$, there exists a constant $C(k, d, r)$ independant of α such that, $\forall v \in W^{1,r}(B(\mathbf{y}, \alpha))$, there exists an extension $v_\alpha \in W^{1,r}(B(\mathbf{y}, k\alpha))$ of v verifying*

$$\begin{aligned} \|v_\alpha\|_{L^r(B(\mathbf{y},k\alpha))} &\le C(k,d,r)\|v\|_{L^r(B(\mathbf{y},\alpha))}, \\ \|\nabla v_\alpha\|_{L^r(B(\mathbf{y},k\alpha))} &\le C(k,d,r)\|\nabla v\|_{L^r(B(\mathbf{y},\alpha))}. \end{aligned} \tag{3.1}$$

Proof First, considering the map $\mathbf{x} \mapsto v(\mathbf{y}+\mathbf{x})$, we can assume that $\mathbf{y} = \mathbf{0}$. Let us define, for $\alpha \le \beta$, the set $Cr(\alpha, \beta)$ by

$$Cr(\alpha,\beta) = \{\mathbf{x} \in \mathbb{R}^d,\ \alpha \le \|\mathbf{x}\| \le \beta\} \tag{3.2}$$

and the function v_α, which extends the function v on $B(\mathbf{0}, k\alpha)$ by

$$\forall \mathbf{x} \in Cr(\alpha, k\alpha), \quad v_\alpha(\mathbf{x}) = v\Big(\Big(-\frac{1}{2(k-1)} + \frac{2k-1}{2(k-1)}\frac{\alpha}{\|\mathbf{x}\|}\Big)\mathbf{x}\Big). \tag{3.3}$$

This definition is justified because if $\|\mathbf{x}\| = \alpha$, $(-\frac{1}{2(k-1)} + \frac{2k-1}{2(k-1)}\frac{\alpha}{\|\mathbf{x}\|})\mathbf{x} = \mathbf{x}$ and we can verify

$$\mathbf{y}(\mathbf{x}) = \Big(-\frac{1}{2(k-1)} + \frac{2k-1}{2(k-1)}\frac{\alpha}{\|\mathbf{x}\|}\Big)\mathbf{x} \in Cr(\alpha, k\alpha) \Longleftrightarrow \mathbf{x} \in Cr\Big(\frac{a}{2}, \alpha\Big). \tag{3.4}$$

By taking derivatives in the sense of distributions and applying Green's formula in the sets $B(\mathbf{0},\alpha)$ and $Cr(\alpha, k\alpha)$, we prove that v_α belongs to $W^{1,r}(B(\mathbf{0}, k\alpha))$:

$$\begin{aligned} \forall \mathbf{x} \in B(\mathbf{0},\alpha), &\quad \frac{\partial v_\alpha}{\partial x_i}(\mathbf{x}) = \frac{\partial v}{\partial x_i}(\mathbf{x}), \\ \forall \mathbf{x} \in Cr(\alpha, k\alpha), &\quad \frac{\partial v_\alpha}{\partial x_i}(\mathbf{x}) = \frac{\partial v}{\partial x_i}(\mathbf{y}(\mathbf{x})), \end{aligned}$$

where $\mathbf{y}(\mathbf{x})$ is defined in (3.4). In order to compute the norm in $L^r(Cr(\alpha, k\alpha))$ of v_α and ∇v_α, we consider the mapping

$$\Phi: \ \mathbf{x} \mapsto \mathbf{y} = \Big(-\frac{1}{2(k-1)} + \frac{2k-1}{2(k-1)}\frac{\alpha}{\|\mathbf{x}\|}\Big)\mathbf{x}$$

defined on $Cr(\alpha, k\alpha)$. Since, for $1 \le i \le d$,

$$x_i = -2(k-1)y_i + (2k-1)\frac{\alpha y_i}{\|\mathbf{y}\|},$$

we derive

$$\frac{\partial x_i}{\partial y_i}(\mathbf{y}) = -2(k-1) + (2k-1)\alpha\frac{\sum_{j\neq i} y_j^2}{\|\mathbf{y}\|^3} \le 2(k-1) + (2k-1)\alpha\frac{1}{\|\mathbf{y}\|}$$

and, for $j \neq i$,

$$\frac{\partial x_i}{\partial y_j}(\mathbf{y}) = -(2k-1)\alpha\frac{y_i y_j}{\|\mathbf{y}\|^3} \le \frac{(2k-1)\alpha}{2\|\mathbf{y}\|}.$$

Hence, in view of $\|\mathbf{y}\| \ge \frac{\alpha}{2}$, we obtain, $\forall \mathbf{y} \in Cr(\frac{\alpha}{2}, \alpha)$,

$$\forall \mathbf{y} \in Cr\Big(\frac{\alpha}{2}, \alpha\Big), \quad \Big|\frac{\partial x_i}{\partial y_i}(\mathbf{y})\Big| \le 2(3k-2), \quad \Big|\frac{\partial x_i}{\partial y_j}(\mathbf{y})\Big| \le 2k-1. \tag{3.5}$$

In the same way, we derive

$$\forall \mathbf{x} \in Cr(\alpha, k\alpha), \quad \Big|\frac{\partial y_i}{\partial x_i}(\mathbf{x})\Big| \le \frac{k}{k-1}, \quad \Big|\frac{\partial y_j}{\partial x_i}(\mathbf{x})\Big| \le \frac{2k-1}{4(k-1)} \le \frac{k}{k-1}. \tag{3.6}$$

Therefore, the one-to-one mapping Φ from $Cr(\alpha, k\alpha)$ to $Cr(\frac{\alpha}{2}, \alpha)$ is of class C^1 and its inverse Φ^{-1} is also of class C^1 on $Cr(\frac{\alpha}{2}, \alpha)$. Moreover, considering the Jacobian determinant $J(\mathbf{y}) = \det((\Phi^{-1})'(\mathbf{y}))$, there exists a constant $C(k,d)$ such that

$$\forall \mathbf{y} \in Cr\Big(\frac{\alpha}{2}, \alpha\Big), \quad |J(\mathbf{y})| \le C(k,d). \tag{3.7}$$

Then, we have

$$\int_{Cr(\alpha,k\alpha)} |v_\alpha(\mathbf{x})|^r \mathrm{d}\mathbf{x} = \int_{Cr(\frac{\alpha}{2},\alpha)} |v(\mathbf{y})|^r |J(\mathbf{y})| \mathrm{d}\mathbf{y} \le C(k,d) \int_{Cr(\frac{\alpha}{2},\alpha)} |v(\mathbf{y})|^r \mathrm{d}\mathbf{y},$$

which gives

$$\|v_\alpha\|^r_{L^r(B(\mathbf{0},k\alpha))} \leq (1 + C(k,d))\|v\|^r_{L^r(B(\mathbf{0},\alpha))}. \tag{3.8}$$

Next, we can write, for $1 \leq i \leq d$, $\forall \mathbf{x} \in Cr(\alpha, k\alpha)$,

$$\frac{\partial v_\alpha}{\partial x_i}(\mathbf{x}) = \sum_{j=1}^{d} \frac{\partial v}{\partial y_j}(\mathbf{y}(\mathbf{x}))\frac{\partial y_j}{\partial x_i}(\mathbf{x}).$$

Hölder's inequality and the estimations (3.6) yield

$$\begin{aligned}\Big|\frac{\partial v_\alpha}{\partial x_i}(\mathbf{x})\Big|^r &\leq \Big(\sum_{j=1}^{d}\Big|\frac{\partial y_j}{\partial x_i}(\mathbf{x})\Big|^{\frac{r}{r-1}}\Big)^{r-1}\Big(\sum_{j=1}^{d}\Big|\frac{\partial v}{\partial y_j}(\mathbf{y}(\mathbf{x}))\Big|^r\Big)\\ &\leq d^{r-1}\Big(\frac{k}{k-1}\Big)^r\Big(\sum_{j=1}^{d}\Big|\frac{\partial v}{\partial y_j}(\mathbf{y}(\mathbf{x}))\Big|^r\Big).\end{aligned}$$

Then, owing to (3.7), we obtain

$$\int_{Cr(\alpha,k\alpha)}\Big|\frac{\partial v_\alpha}{\partial x_i}(\mathbf{x})\Big|^r \mathrm{d}\mathbf{x} \leq d^{r-1}\Big(\frac{k}{k-1}\Big)^r C(k,d)\sum_{j=1}^{d}\int_{Cr(\frac{\alpha}{2},\alpha)}\Big|\frac{\partial v}{\partial y_j}(\mathbf{y})\Big|^r \mathrm{d}\mathbf{y},$$

which implies

$$\sum_{i=1}^{d}\int_{Cr(\alpha,k\alpha)}\Big|\frac{\partial v_\alpha}{\partial x_i}(\mathbf{x})\Big|^r \mathrm{d}\mathbf{x} \leq \Big(\frac{d\,k}{k-1}\Big)^r C(k,d)\sum_{j=1}^{d}\int_{Cr(\frac{\alpha}{2},\alpha)}\Big|\frac{\partial v}{\partial y_j}(\mathbf{y})\Big|^r \mathrm{d}\mathbf{y}.$$

Finally, we have

$$\|\nabla v_\alpha\|^r_{L^r(B(\mathbf{0},k\alpha))} \leq \Big(\Big(\frac{d\,k}{k-1}\Big)^r C(k,d) + 1\Big)\|\nabla v\|^r_{L^r(B(\mathbf{0},\alpha))},$$

and with (3.8) in addition, the lemma follows with $C(k,d,r) = ((\frac{d\,k}{k-1})^r\, C(k,d) + 1)^{\frac{1}{r}}$.

Let $v \in W^{1,r}(\mathbb{R}^d)$ such that $v|_\Omega$ belongs to $W^{1,r}(\Omega,\gamma_0)$. Let $\mathbf{y}$ belong to $\overline{\gamma}_0 \cap \overline{\gamma}_1$. The next lemma proves that the norm of v in $L^r(B(\mathbf{y},\alpha))$ is bounded by the norm of ∇v in $L^r(B(\mathbf{y},\alpha))$ with a constant linear with respect to α.

Lemma 3.2 *Let* $\mathbf{y}$ *belong to* $\overline{\gamma}_0\cap\overline{\gamma}_1$*, where* γ_0 *and* γ_1 *are defined by* (1.1)*, and* $v \in W^{1,r}(\mathbb{R}^d)$ *such that* $v|_\Omega$ *belongs to* $W^{1,r}(\Omega,\gamma_0)$*. For* $0 < \alpha \leq \alpha_0$*, where* α_0 *depends on* Ω*, there exists a constant* C_1 *depending on* r*,* d *and* Ω*, such that*

$$\|v\|^r_{L^r(B(\mathbf{y},\alpha))} \leq C_1\,\alpha^r\,\|\nabla v\|^r_{L^r(B(\mathbf{y},\alpha))}. \tag{3.9}$$

Proof First, let us assume $d > 2$. For all $\mathbf{x} \in \overline{\gamma}_0 \cap \overline{\gamma}_1$, we consider the hypercube $C'_{\mathbf{x}} = \prod_{j=1}^{d}]-\frac{a_{\mathbf{x},j}}{2}, \frac{a_{\mathbf{x},j}}{2}[$, where the real $a_{\mathbf{x},j}$, $j = 1, \cdots, d$, are defined in (2.1)–(2.3). The compactness of $\overline{\gamma}_0 \cap \overline{\gamma}_1$ implies that there exists a finite open covering of $\overline{\gamma}_0 \cap \overline{\gamma}_1$

$$\overline{\gamma}_0 \cap \overline{\gamma}_1 \subset \bigcup_{i=1}^{s} C'_{\mathbf{x}_i}. \tag{3.10}$$

Therefore, $\forall \mathbf{y} \in \overline{\gamma}_0 \cap \overline{\gamma}_1$, there exists an integer $i_{\mathbf{y}}$, denoted i for simplifying the notation, such that $\mathbf{y}$ belongs to $C'_{\mathbf{x}_i}$. Considering α'_0 defined in (1.3), we choose α such that

$$0 < \alpha \leq \alpha_0 = \min(\alpha'_0, \alpha''_0, 1), \quad \text{where } \alpha''_0 = \frac{1}{2} \min_{\substack{1 \leq j \leq d \\ 1 \leq i \leq s}} a_{x_i, j}. \tag{3.11}$$

This choice of α, since $\mathbf{y}$ belongs to $C'_{\mathbf{x}_i}$, yields that

$$C(\mathbf{y}, \alpha) = \prod_{j=1}^{d}]y_j - \alpha, y_j + \alpha[\subset C_{\mathbf{x}_i}. \tag{3.12}$$

Let us set

$$M = \max\Big(1, \max_{1 \leq j \leq s} M_{\mathbf{x}_j}\Big), \quad L = \max\Big(1, \max_{1 \leq j \leq s} L_{\mathbf{x}_j}\Big). \tag{3.13}$$

For every $\mathbf{x}$ in $C(\mathbf{y}, \alpha)$, let us define the point $\mathbf{z} = (z_1, \cdots, z_d) = \mathbf{z}(\mathbf{x})$ by

$$\forall 1 \leq j \leq d-2, \quad z_j = \frac{1}{4\, M\, L\sqrt{d-2}} x_j + \Big(1 - \frac{1}{4\, M\, L\sqrt{d-2}}\Big) y_j, \tag{3.14}$$

$$z_{d-1} = \frac{1}{8L} x_{d-1} + \Big(1 - \frac{1}{8L}\Big) y_{d-1} + \frac{3\alpha}{8L}, \quad z_d = \Phi^{\mathbf{x}_i}(\mathbf{z}'). \tag{3.15}$$

Since $\mathbf{x} \in C(\mathbf{y}, \alpha)$, in view of (2.2), (2.3) and (3.13), we derive

$$\begin{aligned} &\forall 1 \leq j \leq d-1, \quad |z_j - y_j| < \alpha, \\ &y_{d-1} + \frac{\alpha}{4L} < z_{d-1} < y_{d-1} + \frac{\alpha}{2L}, \quad d(\mathbf{z}'', \mathbf{y}'') < \frac{\alpha}{4ML}. \end{aligned} \tag{3.16}$$

Then, we have

$$|\Psi^{\mathbf{x}_i}(\mathbf{z}'') - y_{d-1}| = |\Psi^{\mathbf{x}_i}(\mathbf{z}'') - \Psi^{\mathbf{x}_i}(\mathbf{y}'')| \leq M_{\mathbf{x}_i}\, d(\mathbf{z}'', \mathbf{y}'') \leq \frac{\alpha}{4L},$$

which implies $\Psi^{\mathbf{x}_i}(\mathbf{z}'') \leq y_{d-1} + \frac{\alpha}{4L}$, and, therefore,

$$z_{d-1} > \Psi^{\mathbf{x}_i}(\mathbf{z}''). \tag{3.17}$$

From (3.16), we derive $d(\mathbf{z}', \mathbf{y}') < \frac{\alpha}{L}$. Since $|z_d - y_d| = |\Phi^{\mathbf{x}_i}(\mathbf{z}') - \Phi^{\mathbf{x}_i}(\mathbf{y}')| \leq L_{\mathbf{x}_i}\, d(\mathbf{z}', \mathbf{y}')$, we obtain $|z_d - y_d| < \alpha$. Hence, with (3.11), (3.16) and (3.17) , we derive the implication

$$\mathbf{x} \in C(\mathbf{y}, \alpha) \Longrightarrow \mathbf{z} \in C(\mathbf{y}, \alpha) \cap \gamma_0. \tag{3.18}$$

Next, let us set

$$\forall \mathbf{x} \in C(\mathbf{y}, \alpha), \quad \mathbf{f}_1(t) = (t, x_2, \cdots, x_d), \quad \mathbf{f}_d(t) = (z_1, \cdots, z_{d-1}, t), \tag{3.19}$$

$$\forall 1 < i < d, \quad \mathbf{f}_i(t) = (z_1, \cdots, z_{i-1}, t, x_{i+1}, \cdots, x_d), \tag{3.20}$$

where $\mathbf{z}$ is defined by (3.14) and (3.15). Let v belong to $W^{1,r}(\mathbb{R}^d)$. Since $\forall \mathbf{x} \in C(\mathbf{y}, \alpha)$, $\mathbf{f}_d(z_d) = \mathbf{z}$ belongs to γ_0, that is to say, $v(\mathbf{z}) = 0$ and $\mathbf{f}_1(x_1) = \mathbf{x}$, we can write

$$v(\mathbf{x}) = \sum_{i=1}^{d} (v(\mathbf{f}_i(x_i)) - v(\mathbf{f}_i(z_i))) = \sum_{i=1}^{d} \int_{z_i}^{x_i} \frac{\mathrm{d}v}{\mathrm{d}t}(\mathbf{f}_i(t))\, \mathrm{d}t = \sum_{i=1}^{d} \int_{z_i}^{x_i} \frac{\partial v}{\partial x_i}(\mathbf{f}_i(t))\, \mathrm{d}t.$$

Hence, we derive

$$|v(\mathbf{x})|^r \leq d^{r-1} \sum_{i=1}^{d} \Big| \int_{z_i}^{x_i} \frac{\partial v}{\partial x_i}(\mathbf{f}_i(t))\,\mathrm{d}t \Big|^r. \tag{3.21}$$

Next, we have, for $1 \leq i \leq d$,

$$\Big| \int_{z_i}^{x_i} \frac{\partial v}{\partial x_i}(\mathbf{f}_i(t))\,\mathrm{d}t \Big|^r \leq (x_i - y_i + \alpha)^{r-1} \int_{y_i-\alpha}^{y_i+\alpha} \Big| \frac{\partial v}{\partial x_i}(\mathbf{f}_i(t)) \Big|^r \mathrm{d}t.$$

Integrating with respect to x_i yields

$$\int_{y_i-\alpha}^{y_i+\alpha} \Big| \int_{z_i}^{x_i} \frac{\partial v}{\partial x_i}(\mathbf{f}_i(t))\,\mathrm{d}t \Big|^r \mathrm{d}x_i \leq \frac{2^r \alpha^r}{r} \int_{y_i-\alpha}^{y_i+\alpha} \Big| \frac{\partial v}{\partial x_i}(\mathbf{m}_i(\mathbf{x})) \Big|^r \mathrm{d}x_i,$$

where $\mathbf{m}_i(\mathbf{x}) = \mathbf{f}_i(x_i)$. Then, we obtain

$$\int_{C(\mathbf{y},\alpha)} \Big| \int_{z_i}^{x_i} \frac{\partial v}{\partial x_i}(\mathbf{f}_i(t))\,\mathrm{d}t \Big|^r \mathrm{d}\mathbf{x} \leq \frac{2^r \alpha^r}{r} \int_{C(\mathbf{y},\alpha)} \Big| \frac{\partial v}{\partial x_i}(\mathbf{m}_i(\mathbf{x})) \Big|^r \mathrm{d}\mathbf{x}.$$

On the one hand, $\forall \mathbf{x} \in C(\mathbf{y}, \alpha)$, $\mathbf{m}_i(\mathbf{x})$ belongs to $C(\mathbf{y}, \alpha)$. On the other hand, the Jacobian determinant J_i of the transformation $\mathbf{m}_i^{-1}$ is such that

$$\forall 1 \leq i \leq d-1, \quad J_i = \det((\mathbf{m}_i^{-1})') = (4ML\sqrt{d-2}\,)^{i-1}, \quad J_d = 8L(4ML\sqrt{d-2}\,)^{d-2}.$$

Then, we derive, for $1 \leq i \leq d$,

$$\int_{C(\mathbf{y},\alpha)} \Big| \int_{z_i}^{x_i} \frac{\partial v}{\partial x_i}(\mathbf{f}_i(t))\,\mathrm{d}t \Big|^r \mathrm{d}\mathbf{x} \leq \Big(\frac{1}{r}\Big) 2^{r+3} L(4ML\sqrt{d-2}\,)^{d-2} \alpha^r \int_{C(\mathbf{y},\alpha)} \Big| \frac{\partial v}{\partial x_i}(\mathbf{m}_i) \Big|^r \mathrm{d}\mathbf{m}_i.$$

Hence, owing to (3.21), we obtain

$$\int_{C(\mathbf{y},\alpha} |v(\mathbf{x})|^r \mathrm{d}\mathbf{x} \leq \Big(\frac{1}{r}\Big) d^{r-1} 2^{r+3} L(4ML\sqrt{d-2}\,)^{d-2} \alpha^r \sum_{i=1}^{d} \int_{C(\mathbf{y},\alpha)} \Big| \frac{\partial v}{\partial x_i}(\mathbf{m}_i) \Big|^r \mathrm{d}\mathbf{m}_i,$$

that is to say

$$\|v\|^r_{L^r(C(\mathbf{y},\alpha))} \leq K(r, d, \Omega) \alpha^r \|\nabla v\|_{L^r(C(\mathbf{y},\alpha))}, \tag{3.22}$$

where $K(r, d, \Omega) = (\frac{1}{r}) d^{r-1} 2^{r+3} L(4ML\sqrt{d-2}\,)^{d-2}$.

Next, in view of Lemma 3.1, we extend $v|_{B(\mathbf{y},\alpha)} \in W^{1,r}(B(\mathbf{y}, \alpha))$ by $v_\alpha \in W^{1,r}(B(\mathbf{y}, (\sqrt{d}\,)\alpha))$. Owing to (3.22) and considering that

$$B(\mathbf{y}, \alpha) \subset C(\mathbf{y}, \alpha) \subset B(\mathbf{y}, (\sqrt{d}\,)\alpha),$$

we derive

$$\begin{aligned}
\|v\|^r_{L^r(B(\mathbf{y},\alpha))} &\leq \|v_\alpha\|^r_{L^r(C(\mathbf{y},\alpha))} \leq K(r, d, \Omega)\alpha^r \|\nabla v_\alpha\|^r_{L^r(C(\mathbf{y},\alpha))} \\
&\leq K(r, d, \Omega)\alpha^r \|\nabla v_\alpha\|^r_{L^r(B(\mathbf{y},(\sqrt{d})\alpha))} \\
&\leq K(r, d, \Omega) C(\sqrt{d}, d, r) \alpha^r \|\nabla v\|^r_{L^r(B(\mathbf{y},\alpha))}
\end{aligned}$$

and the result of the lemma follows for $d > 2$, with $C_1 = K(r,d,\Omega)\, C(\sqrt{d}, d, r)$.

Finally, for $d = 2$, in view of $y_1 = 0$, we set $z_1 = \frac{1}{4L}x_1 + \frac{3}{4L}\alpha$ and $z_2 = \Phi^{\mathbf{x}_i}(z_1)$, where L is defined as in (3.13). Then, we obtain $0 < \frac{\alpha}{2L} < z_1 < \alpha$ and we still have the implication (3.18). In the same way as the previous, we can write, since $v(\mathbf{z}) = 0$,

$$\begin{aligned}\forall \mathbf{x} \in C(\mathbf{y},\alpha), \quad |v(\mathbf{x})|^r &\le 2^{r-1}\Big(\Big|\int_{z_1}^{x_1} \frac{\partial v}{\partial x_1}(t, x_2)\,\mathrm{d}t\Big|^r + \Big|\int_{z_2}^{x_2} \frac{\partial v}{\partial x_2}(z_1, t)\,\mathrm{d}t\Big|^r\Big)\\ &\le 2^{r-1}\Big((x_1 - y_1 + \alpha)^{r-1}\int_{y-1-\alpha}^{y_1+\alpha}\Big|\frac{\partial v}{\partial x_1}(t, x_2)\Big|^r \mathrm{d}t\\ &\quad + (x_2 - y_2 + \alpha)^{r-1}\int_{y_2-\alpha}^{y_2+\alpha}\Big|\frac{\partial v}{\partial x_2}(z_1, t)\Big|^r \mathrm{d}t\Big).\end{aligned}$$

Then, integrating on $C(\mathbf{y},\alpha)$ (note that on the right-hand side, we integrate the first term of the sum, first with respect to x_1, and the second term, first with respect to x_2) yields

$$\int_{C(\mathbf{y},\alpha} |v(\mathbf{x})|^r \mathrm{d}\mathbf{x} \le \frac{2^{2r-1}\alpha^r}{r}\Big(\int_{C(\mathbf{y},\alpha)}\Big|\frac{\partial v}{\partial x_2}(x_1,x_2)\Big|^r \mathrm{d}\mathbf{x} + \int_{C(\mathbf{y},\alpha)}\Big|\frac{\partial v}{\partial x_2}(z_1,x_2)\Big|^r \mathrm{d}x_1\mathrm{d}x_2\Big).$$

In view of $\mathrm{d}x_1 = 4L\mathrm{d}z_1$, we derive

$$\|v\|^r_{L^r(C(\mathbf{y},\alpha))} \le K(r,2,\Omega)\alpha^r \|\nabla v\|_{L^r(C(\mathbf{y},\alpha))}$$

with $K(r,2,\Omega) = \frac{2^{2r+1}L}{r}$, and we end the proof in the same way as the previous.

Let $\widetilde{u} \in W^{1,r}(\mathbb{R}^d)$ be an extension of $u \in W^{1,r}(\Omega,\gamma_0)$ outside Ω. The two previous lemmas allow us to establish the next lemma which gives an approximation of u by zero in $G_{k,\alpha}$.

Lemma 3.3 *For every real number $\varepsilon > 0$, there exists a real number α_ε verifying* (1.3) *such that, for every $0 < \alpha \le \alpha_\varepsilon$,*

$$\forall k = 1, \cdots, q, \quad \|\theta_{\alpha,k} u\|_{W^{1,r}(G_{k,\alpha}\cap\Omega)} \le \frac{\varepsilon}{4q}. \tag{3.23}$$

Proof For $k = 1, \cdots, q$, let $\{B(\mathbf{x}_i,\alpha)\}_{i=1}^p$ be an open optimal covering of $\overline{G}_{k,\alpha}$, where $B(\mathbf{x}_i,\alpha)$ denotes the open ball with center $\mathbf{x}_i$ and radius α. This means that there is no covering of $\overline{G}_{k,\alpha}$ with less than p balls of radius α. Let $i \in \mathbb{N}^*$ such that $1 \le i \le p$. Note that $B(\mathbf{x}_i,\alpha) \cap \overline{G}_{k,\alpha} \ne \emptyset$ and let $\mathbf{z}_i$ belong to $B(\mathbf{x}_i,\alpha) \cap \overline{G}_{k,\alpha}$. Then, there exists $\mathbf{y}_i \in K_k$, such that $d(\mathbf{z}_i,\mathbf{y}_i) \le \alpha$, which implies $d(\mathbf{x}_i,\mathbf{y}_i) < 2\alpha$. Hence, we derive

$$G_{k,\alpha} \subset \bigcup_{i=1}^p B(\mathbf{x}_i,\alpha) \subset \bigcup_{i=1}^p B(\mathbf{y}_i, 3\alpha), \tag{3.24}$$

such that the covering $\{B(\mathbf{x}_i,\alpha)\}_{i=1}^p$ is maximal and the covering $\{B(\mathbf{y}_i,3\alpha)\}_{i=1}^p$ verifies, $\forall i = 1, \cdots, p$,

$$\mathbf{y}_i \text{ belongs to } K_k. \tag{3.25}$$

Note that, $\forall \mathbf{x} \in \mathbb{R}^d$, $\forall n \in \mathbb{N}^*$ and $\forall \alpha > 0$, there exists a covering $\{B(\mathbf{x}'_i,\alpha)\}_{i=1}^{p_{n,d}}$ of the ball $B(\mathbf{x}, n\alpha)$ with $p_{n,d} = ([n\sqrt{d}\,] + 1)^d$, where $[x]$ denotes the integral part of the real number x.

Indeed, the ball of radius $n\alpha$ is inscribed in a hypercube of edge $2n\alpha$ and the hypercube of edge $\frac{2\alpha}{\sqrt{d}}$ is inscribed in a ball of radius α. Let $i \in \mathbb{N}^*$ such that $1 \leq i \leq p$ and let us set

$$N_i = \{j \in \mathbb{N}^*, 1 \leq j \leq p, B(\mathbf{y}_j, 3\alpha) \cap B(\mathbf{y}_i, 3\alpha) \neq \emptyset\}. \tag{3.26}$$

On the one hand, we have

$$\bigcup_{j \in N_i} B(\mathbf{x}_j, \alpha) \subset \bigcup_{j \in N_i} B(\mathbf{y}_j, 3\alpha) \subset B(\mathbf{y}_i, 9\alpha) \subset B(\mathbf{x}_i, 11\alpha).$$

On the other hand, the previous note implies

$$B(\mathbf{x}_i, 11\alpha) \subset \bigcup_{j=1}^{p_{11,d}} B(\mathbf{x}'_j, \alpha).$$

Since the covering $\{B(\mathbf{x}_i, \alpha)\}_{i=1}^{p}$ of $\overline{G}_{k,\alpha}$ is maximal, we derive

$$\forall i \in \mathbb{N}^*,\ 1 \leq i \leq p, \quad \operatorname{card} N_i \leq p_{11,d} = ([11\sqrt{d}\,] + 1)^d = M_d, \tag{3.27}$$

where N_i is defined by (3.26). Applying the crucial Lemma 3.2 yields

$$\|\widetilde{u}\|^r_{L^(B(\mathbf{y}_i,3\alpha)} \leq C'_1\, \alpha^r \|\nabla \widetilde{u}\|^r_{L^(B(\mathbf{y}_i,3\alpha)}, \tag{3.28}$$

where $C'_1 = 3^r C_1$. Then, from (3.24), we derive

$$\|u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq \|\widetilde{u}\|^r_{L^r(G_{k,\alpha})} \leq \sum_{i=1}^{p} \|\widetilde{u}\|^r_{L^r(B(\mathbf{y}_i,3\alpha))},$$

and in view of (3.28), we obtain

$$\|u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq C'_1 \alpha^r \sum_{i=1}^{p} \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_i,3\alpha))}. \tag{3.29}$$

Now, we can assume that the integrals $\|\nabla \widetilde{u}\|_{L^r(B(\mathbf{y}_{\psi(i)},3\alpha))}$ are in decreasing order with respect to i where ψ is a permutation of the set $\{1, \cdots, p\}$. To simplify the notation, we still denote the index i instead of $\psi(i)$. Thus, we assume that, for $i = 1, \cdots, p-1$,

$$\|\nabla \widetilde{u}\|_{L^r(B(\mathbf{y}_i,3\alpha))} \geq \|\nabla \widetilde{u}\|_{L^r(B(\mathbf{y}_{i+1},3\alpha))}. \tag{3.30}$$

Next, we construct by finite induction a partition of $I = \{i \in \mathbb{N}^*,\ 1 \leq i \leq p\}$ in the following way. We define $I_0 = I$, $i_1 = 1$ and for $k \geq 1$

$$J_k = \{j \in I_{k-1},\ B(\mathbf{y}_j, 3\alpha) \cap B(\mathbf{y}_{i_k}, 3\alpha) \neq \emptyset\}, \quad I_k = \{j \in I_{k-1},\ B(\mathbf{y}_j, 3\alpha) \cap B(\mathbf{y}_{i_k}, 3\alpha) = \emptyset\}$$

and $i_{k+1} = \min I_k$ if $I_k \neq \emptyset$. Note that $i_{k+1} > i_k$, because, by construction, $i_{k+1} \geq i_k$ and $i_k \notin I_k$. Let $l \geq 1$ such that $I_l = \emptyset$ and $I_{l-1} \neq \emptyset$. Considering that $I_{k-1} = J_k \cup I_k$ for $k = 1, \cdots, l$, we obtain the following partition of I:

$$I = \bigcup_{k=1}^{l} J_k. \tag{3.31}$$

Moreover, by construction, the balls $B(y_{i_k}, 3\alpha)$, $k = 1, \cdots, l$ are disconnected two by two. Hence, on the one hand, we derive

$$\|\nabla \widetilde{u}\|^r_{L^r\left(\bigcup_{i=1}^{p} B(\mathbf{y}_i, 3\alpha)\right)} \geq \sum_{k=1}^{l} \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_{i_k}, 3\alpha))}. \tag{3.32}$$

On the other hand, we have

$$\sum_{i=1}^{p} \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_i, 3\alpha))} = \sum_{k=1}^{l} \Big(\sum_{j \in J_k} \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_j, 3\alpha))} \Big).$$

But, in view of (3.27) and (3.30), we can write

$$\sum_{j \in J_k} \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_j, 3\alpha))} \leq M_d \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_{i_k}, 3\alpha))}.$$

Thus, we derive

$$\sum_{i=1}^{p} \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_i, 3\alpha))} \leq M_d \sum_{k=1}^{l} \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_{i_k}, 3\alpha))}.$$

Then, owing to (3.32), we obtain the crucial estimate

$$\sum_{i=1}^{p} \|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{y}_i, 3\alpha))} \leq M_d \|\nabla \widetilde{u}\|^r_{L^r\left(\bigcup_{i=1}^{p} B(\mathbf{y}_i, 3\alpha)\right)} \leq M_d \|\nabla \widetilde{u}\|^r_{L^r(G_{k,3\alpha})},$$

which gives, in view of (3.29),

$$\|u\|^r_{L^r(G_{k,\alpha} \cap \Omega)} \leq C'_1 M_d \alpha^r \|\nabla \widetilde{u}\|^r_{L^r(G_{k,3\alpha})}. \tag{3.33}$$

Finally, for $i = 1, \cdots, d$, $\partial_i(\theta_{\alpha,k} u) = \partial_i(\theta_{\alpha,k}) u + \theta_{\alpha,k} \partial_i u$, where $\theta_{\alpha,k}$ is defined by (2.8). From (2.9) and (3.33), we derive

$$\|\partial_i(\theta_{\alpha,k}) u\|^r_{L^r(G_{k,\alpha} \cap \Omega)} \leq \frac{A^r}{\alpha^r} \|u\|^r_{L^r(G_{k,\alpha} \cap \Omega)} \leq C'_1 M_d A^r \|\nabla \widetilde{u}\|^r_{L^r(G_{k,3\alpha})}.$$

Considering (2.9) again and

$$\|\partial_i(\theta_{\alpha,k} u)\|^r_{L^r(G_{k,\alpha} \cap \Omega)} \leq 2^{r-1} (\|\partial_i(\theta_{\alpha,k}) u\|^r_{L^r(G_{k,\alpha} \cap \Omega)} + \|\theta_{\alpha,k} \partial_i u\|^r_{L^r(G_{k,\alpha} \cap \Omega)}),$$

we obtain, for $k = 1, \cdots, q$,

$$\|\theta_{\alpha,k} u\|^r_{W^{1,r}(G_{k,\alpha} \cap \Omega)} \leq \|\widetilde{u}\|^r_{L^r(G_{k,3\alpha})} + 2^{r-1} \, (C'_1 \, M_d \, A^r + 1) d \|\nabla \widetilde{u}\|^r_{L^r(G_{k,3\alpha})}.$$

Note that

$$\bigcap_{\alpha > 0} G_{k,3\alpha} = K_k$$

and the measure of K_k is 0 in $\mathbb{R}^d$. Since $\widetilde{u}$ belongs to $W^{1,r}(\mathbb{R}^d)$, for $k = 1, \cdots, q$, we have

$$\lim_{\alpha \to 0} \|\theta_{\alpha,k} u\|_{W^{1,r}(G_{k,\alpha} \cap \Omega)} = 0.$$

Thus, there exists a real $\alpha_\varepsilon > 0$, such that the inequalities (3.23) and (1.3) are verified.

Let us note that, considering the partition of unity $\mathcal{P}_\alpha$ defined by (2.12), such that $0 < \alpha \leq \alpha_\varepsilon$, and in view of $\theta_{\alpha,k} \in \mathcal{D}(G_{k,\alpha})$, (3.23) can be written as

$$\forall k = 1, \cdots, q, \quad \|\varphi_{\alpha,k} u\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4q}, \tag{3.34}$$

so that, for every $k = 1, \cdots, q$, we can approximate $\boldsymbol{\varphi}_{\alpha,k} u$ by 0 in $\mathcal{O}_{k,\alpha} = G_{k,\alpha}$.

We now deal with the case $k = 0$, that is to say, we want approximate $\varphi_{\alpha,0} u$ in $\mathcal{O}_{0,\alpha}$. Let us recall that $\varphi_{\alpha,0} u$ has a compact support in $\mathcal{O}_{0,\alpha}$ with $\overline{\mathcal{O}}_{0,\alpha} \subset \Omega$. Therefore, we have

$$d(\operatorname{supp}(\varphi_{\alpha,0} u), \partial \mathcal{O}_{0,\alpha}) = \mu_0 > 0, \tag{3.35}$$

and we can note that $\widetilde{\varphi_{\alpha,0u}}$ belongs to $W^{1,r}(\mathbb{R}^d)$, where the latter denotes the extension by zero. Then, for every $p \in \mathbb{N}^*$, we define u_p by

$$\forall \mathbf{x} \in \mathbb{R}\mathbb{R}^d, \quad u_p(\mathbf{x}) = ((\widetilde{\varphi_{\alpha,0} u}) * \rho_p)(\mathbf{x}) = \int_{B(\mathbf{0}, \frac{1}{p})} \widetilde{\varphi_{\alpha,0} u}(\mathbf{x} - \mathbf{y}) \rho_p(\mathbf{y}) \mathrm{d}\mathbf{y},$$

where ρ_p is defined by (2.6). In a standard way, we obtain that

$$\lim_{p \to +\infty} u_p = \widetilde{\varphi_{\alpha,0} u}, \quad \text{in } W^{1,r}(\mathbb{R}\mathbb{R}^d),$$

which implies that there exists a $P_\varepsilon \in N^*$, such that $\forall p \geq P_\varepsilon$,

$$\|\varphi_{\alpha,0} u - u_p\|_{W^{1,r}(\mathcal{O}_{0,\alpha})} \leq \frac{\varepsilon}{4}. \tag{3.36}$$

Next, concerning the support of u_p, we choose $p \geq \frac{3}{\mu_0}$ and define the set $E = \{\mathbf{x} \in \overline{\mathcal{O}}_{0,\alpha},\ d(\mathbf{x}, \partial \mathcal{O}_{0,\alpha}) \leq \frac{\mu_0}{3}\}$. This implies that $\forall \mathbf{y} \in B(\mathbf{0}, \frac{1}{p})$ and $\forall \mathbf{x} \in E$,

$$d(\mathbf{x} - \mathbf{y}, \operatorname{supp}(\varphi_{\alpha,0} u)) \geq d(\partial \mathcal{O}_{0,\alpha}, \operatorname{supp}(\varphi_{\alpha,0} u)) - d(\mathbf{x} - \mathbf{y}, \mathbf{x}) - d(\mathbf{x}, \partial \mathcal{O}_{0,\alpha}) \geq \frac{\mu_0}{3} > 0.$$

In the same way, we have $\forall \mathbf{y} \in B(\mathbf{0}, \frac{1}{p})$ and $\forall \mathbf{x} \in \overline{\Omega} \setminus \mathcal{O}_{0,\alpha}$,

$$d(\mathbf{x} - \mathbf{y}, \operatorname{supp}(\varphi_{\alpha,0} u)) \geq \frac{2\mu_0}{3} > 0.$$

Hence, we derive that u_p vanishes on $E \cup (\overline{\Omega} \setminus \mathcal{O}_{0,\alpha})$. Setting $u_{\varepsilon,0} = u_{m_\varepsilon}$, where m_ε is defined by $m_\varepsilon = \max([\frac{3}{\mu_0}], P_\varepsilon)$ ($[r]$ is the integral part of r), and considering the supports of $\varphi_{\alpha,0} u$ and $u_{\varepsilon,0}$, yield

$$\|\varphi_{\alpha,0} u - u_{\varepsilon,0}\|_{W^{1,r}(\mathcal{O}_{0,\alpha} \cap \Omega)} = \|\varphi_{\alpha,0} u - u_{\varepsilon,0}\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4} \quad \text{with } u_{\varepsilon,0} \in \mathcal{D}(\mathcal{O}_{0,\alpha}), \tag{3.37}$$

where $\overline{\mathcal{O}}_{0,\alpha} \subset \Omega$.

The next lemma gives an approximation of $\varphi_{\alpha,k} u$ in $\mathcal{O}_{k,\alpha}$ for $k = q+1, \cdots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_1$, that is, an approximation of u localized around γ_1.

Lemma 3.4 *Let α be a real number verifying* (1.3). *For every real number $\varepsilon > 0$ and for every $k = q+1, \cdots, r_\alpha$, such that $\mathbf{m}_{k,\alpha} \in \gamma_1$, there exists a function $u_{\varepsilon,k} \in \mathcal{D}(\overline{\Omega})$ with compact support in $\mathcal{O}_{k,\alpha} \cap \overline{\Omega}$, such that*

$$\|\varphi_{\alpha,k} u - u_{\varepsilon,k}\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}, \tag{3.38}$$

where r_α is defined by (2.4).

Proof For $k = q+1, \cdots, r_\alpha$ with $\mathbf{m}_{k,\alpha} \in \gamma_1$, we want to approximate $\varphi_{\alpha,k} u$. To simplify the notations, we drop the indexes, replacing $\varphi_{\alpha,k}\, u$ by u and $\mathcal{O}_{k,\alpha}$ by $\mathcal{O}$, so that we may assume that u has compact support in $\mathcal{O} \cap \overline{\Omega}$ and set

$$d(\partial\mathcal{O} \cap \overline{\Omega}, \text{supp } u) = \mu > 0. \tag{3.39}$$

Considering (2.1) and (2.12), we may assume that $\mathcal{O}$ is an open hypercube, neighborhood of a point of γ_1, such that, in new orthogonal coordinates $\mathbf{y} = (\mathbf{y}', y_d)$, we have

$$\mathcal{O} \cap \Omega = \{\mathbf{y} \in \mathcal{O},\ y_d < \Phi(\mathbf{y}')\} \quad \text{and} \quad \gamma_1 \cap \mathcal{O} = \{\mathbf{y} \in \mathcal{O},\ y_d = \Phi(\mathbf{y}')\}, \tag{3.40}$$

where Φ is a Lipschitz-continuous function, defined in $\prod_{j=1}^{d-1}]-a_j, a_j[$, of constant L.

Let $n \in \mathbb{N}^*$. We set

$$u_n(\mathbf{y}) = u\Big(\mathbf{y}', y_d - \frac{1}{n}\Big), \tag{3.41}$$

which is a function defined on

$$\Omega_n = \Big\{\mathbf{y} \in \mathbb{R}^d,\ \Big(\mathbf{y}', y_d - \frac{1}{n}\Big) \in \mathcal{O} \cap \Omega\Big\}.$$

The set Ω_n is obtained by translating $\mathcal{O} \cap \Omega$ to the direction of positive y_d. We denote by $\widetilde{u}_n$ the extension of u_n by zero. Considering the support of u, we can see that the restriction of $\widetilde{u}_n$ to $\mathcal{O} \cap \Omega$ belongs to $W^{1,r}(\mathcal{O} \cap \Omega)$.

Next, since the translation is continuous on $L^r(\mathbb{R}^d)$, we derive

$$\lim_{n \to +\infty} \widetilde{u}_{n|\mathcal{O}\cap\Omega} = u, \quad \text{in } L^r(\mathcal{O} \cap \Omega).$$

Moreover, as $\partial_i(\widetilde{u}_{n|\mathcal{O}\cap\Omega}) = (\widetilde{\partial_i u})_{n|\mathcal{O}\cap\Omega}$, where the wide latter denotes the extension by zero of $(\partial_i u)_n$ in $\mathcal{O} \cap \Omega \setminus \Omega_n$, as we can verify by deriving in the sense of distribution, we have the same convergence for the partial derivatives. Thus, we obtain

$$\lim_{n \to +\infty} \widetilde{u}_{n|\mathcal{O}\cap\Omega} = u, \quad \text{in } W^{1,r}(\mathcal{O} \cap \Omega). \tag{3.42}$$

For every $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$, we define

$$u_{n,p} = \widetilde{u}_n * \rho_p. \tag{3.43}$$

The standard properties of the convolution imply

$$\lim_{p \to +\infty} u_{n,p} = \widetilde{u}_n, \quad \text{in } L^r(\mathbb{R}^d). \tag{3.44}$$

Next

$$\partial_i u_{n,p} = \partial_i \widetilde{u}_n * \rho_p.$$

We cannot pass to the limit in $L^r(\mathbb{R}^d)$, because, usually, $\partial_i \widetilde{u}_n$ is not in $L^r(\mathbb{R}^d)$. First, let us show that, for p large enough, $\widetilde{u}_{n|\mathcal{O}_p}$ belongs to $W^{1,r}(\mathcal{O}_p)$, where $\mathcal{O}_p$ is defined by

$$\mathcal{O}_p = \Big\{\mathbf{y} \in \mathbb{R}^d,\ d(\mathbf{y}, \mathcal{O} \cap \Omega) < \frac{1}{p}\Big\}. \tag{3.45}$$

We set

$$\Gamma_n = \Big\{\mathbf{y} \in \mathbb{R}^d, \Big(\mathbf{y}', y_d - \frac{1}{n}\Big) \in \partial\Omega \cap \mathcal{O}\Big\}, \tag{3.46}$$

and thus, we can write

$$\partial\Omega_n = \overline{\Gamma_n} \cup \overline{\Gamma'_n} \quad \text{with } \Gamma_n \cap \Gamma'_n = \emptyset.$$

We can note that, since $\forall \mathbf{y} \in \Gamma'_n$, $(\mathbf{y}', y_d - \frac{1}{n}) \in \Omega \cap (\partial\mathcal{O})$,

$$\forall \mathbf{y} \in \Gamma'_n, \quad u_n(\mathbf{y}) = 0. \tag{3.47}$$

Let us estimate, for every $\mathbf{z} \in \Gamma_n$, the distance $d(\mathbf{z}, \overline{\mathcal{O} \cap \Omega}) = d(\mathbf{z}, \overline{\mathcal{O}} \cap \partial\Omega)$. Indeed, $\forall \mathbf{y} \in \overline{\mathcal{O} \cap \Omega}$, $[\mathbf{z}, \mathbf{y}] \cap (\overline{\mathcal{O}} \cap \partial\Omega) \neq \emptyset$.

$$\forall \mathbf{z} \in \Gamma_n,\ \forall \mathbf{y} \in (\overline{\mathcal{O}} \cap \partial\Omega), \quad \|\mathbf{z} - \mathbf{y}\|^2 = \|\mathbf{z}' - \mathbf{y}'\|^2 + \Big(\frac{1}{n} + \Phi(\mathbf{z}') - \Phi(\mathbf{y}')\Big)^2.$$

The properties of Φ yield

$$\frac{1}{n} + \Phi(\mathbf{z}') - \Phi(\mathbf{y}') \geq \frac{1}{n} - L\|\mathbf{z}' - \mathbf{y}'\|.$$

Then, if $\|\mathbf{z}' - \mathbf{y}'\| \leq \frac{1}{2nL}$, we have $\|\mathbf{z} - \mathbf{y}\| \geq \frac{1}{2n}$, and if $\|\mathbf{z}' - \mathbf{y}'\| \geq \frac{1}{2nL}$, we have $\|\mathbf{z} - \mathbf{y}\| \geq \frac{1}{2nL}$. Therefore, we obtain

$$d(\Gamma_n, \overline{\mathcal{O} \cap \Omega}) \geq \min\Big(\frac{1}{2n}, \frac{1}{2nL}\Big). \tag{3.48}$$

Next, we have by definition

$$\forall \psi \in \mathcal{D}(\mathcal{O}_p), \quad \langle \partial_i \widetilde{u}_n, \psi \rangle_{\mathcal{D}(\mathcal{O}_p)} = -\int_{\mathcal{O}_p} \widetilde{u}_n(\mathbf{x}) \partial_i \psi(\mathbf{x}) \mathrm{d}\mathbf{x} = -\int_{\mathcal{O}_p \cap \Omega_n} u_n(\mathbf{x}) \partial_i \psi(\mathbf{x}) \mathrm{d}\mathbf{x}.$$

Since u_n belongs to $W^{1,r}(\Omega_n)$, Green's formula yields

$$\langle \partial_i \widetilde{u}_n, \psi \rangle_{\mathcal{D}(\mathcal{O}_p)} = \int_{\mathcal{O}_p \cap \Omega_n} \partial_i u_n(\mathbf{x}) \psi(\mathbf{x}) \mathrm{d}\mathbf{x} - \int_{\partial(\mathcal{O}_p \cap \Omega_n)} u_n(\mathbf{s}) \psi(\mathbf{s})\, n_i \mathrm{d}\mathbf{s}.$$

Let us choose

$$\frac{1}{p} < \min\Big(\frac{1}{2n}, \frac{1}{2nL}\Big). \tag{3.49}$$

Then, owing to (3.48), we have for every $\mathbf{y} \in \overline{\mathcal{O}}_p$,

$$d(\mathbf{y}, \overline{\mathcal{O} \cap \Omega}) \leq \frac{1}{p} < \min\Big(\frac{1}{2n}, \frac{1}{2nL}\Big) \leq d(\Gamma_n, \overline{\mathcal{O} \cap \Omega}),$$

which implies

$$\Gamma_n \cap \overline{\mathcal{O}}_p = \emptyset.$$

Hence, we obtain

$$\partial(\mathcal{O}_p \cap \Omega_n) \subset (\partial(\mathcal{O}_p) \cup \partial(\Omega_n)) \cap \overline{\mathcal{O}}_p \subset (\partial(\mathcal{O}_p) \cup \overline{\Gamma'_n}).$$

Therefore, with (3.47) in addition, $u_n\psi$ vanishes on $\partial(\mathcal{O}_p \cap \Omega_n)$. We derive

$$\langle \partial_i \widetilde{u}_n, \psi \rangle_{\mathcal{D}(\mathcal{O}_p)} = \int_{\mathcal{O}_p \cap \Omega_n} \partial_i u_n(\mathbf{x}) \psi(\mathbf{x}) \mathrm{d}\mathbf{x},$$

that is to say,

$$\widetilde{u}_{n|\mathcal{O}_p} \text{ belongs to } W^{1,r}(\mathcal{O}_p) \quad \text{and} \quad \partial_i \widetilde{u}_{n|\mathcal{O}_p} = \widetilde{\partial_i u_n}, \tag{3.50}$$

where the wide latter is the extension by zero of $\partial_i u_n \in L^r(\Omega_n \cap \mathcal{O}_p)$ in $\mathcal{O}_p$.

Second, let us show that, for p large enough, $\partial_i u_{n,p} = \widetilde{\partial_i u_n} * \rho_p$. In view of Fubini's Theorem,

$$\begin{aligned} \forall \psi \in \mathcal{D}(\mathcal{O} \cap \Omega), \quad \langle \partial_i u_{n,p}, \psi \rangle_{\mathcal{D}(\mathcal{O}\cap\Omega)} &= - \int_{\mathcal{O}\cap\Omega} \Big(\int_{B(\mathbf{0},\frac{1}{p})} \widetilde{u}_n(\mathbf{x}-\mathbf{y}) \rho_p(\mathbf{y}) \mathrm{d}\mathbf{y} \Big) \partial_i \psi(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= - \int_{B(\mathbf{0},\frac{1}{p})} \rho_p(\mathbf{y}) \Big(\int_{\mathcal{O}\cap\Omega} \widetilde{u}_n(\mathbf{x}-\mathbf{y}) \partial_i \psi(\mathbf{x}) \mathrm{d}\mathbf{x} \Big) \mathrm{d}\mathbf{y}. \end{aligned}$$

Considering (3.50), for every $\mathbf{y} \in B(\mathbf{0}, \frac{1}{p})$, $\mathbf{x} \mapsto \widetilde{u}_n(\mathbf{x}-\mathbf{y})$ belongs to $W^{1,r}(\mathcal{O} \cap \Omega)$ and

$$\forall \mathbf{x} \in \mathcal{O} \cap \Omega, \ \forall \mathbf{y} \in B\Big(\mathbf{0}, \frac{1}{p}\Big), \quad \partial_i \widetilde{u}_n(\mathbf{x}-\mathbf{y}) = \widetilde{\partial_i u_n}(\mathbf{x}-\mathbf{y}).$$

Then, Green's formula and Fubini's Theorem yield

$$\langle \partial_i u_{n,p}, \psi \rangle_{\mathcal{D}(\mathcal{O}\cap\Omega)} = \int_{\mathcal{O}\cap\Omega} \Big(\int_{B(\mathbf{0},\frac{1}{p})} \widetilde{\partial_i u_n}(\mathbf{x}-\mathbf{y}) \rho_p(\mathbf{y}) \mathrm{d}\mathbf{y} \Big) \psi(\mathbf{x}) \mathrm{d}\mathbf{x},$$

which implies, for every p verifying (3.49),

$$\partial_i u_{n,p} = \widetilde{\partial_i u_n} * \rho_p.$$

From the standard properties of the convolution, we derive

$$\lim_{p\to+\infty} (\partial_i u_{n,p})|_{\mathcal{O}\cap\Omega} = \widetilde{\partial_i u_n} \quad \text{and} \quad L^r(\mathcal{O} \cap \Omega)$$

and in view of (3.44),

$$\lim_{p\to+\infty} (u_{n,p})|_{\mathcal{O}\cap\Omega} = \widetilde{u}_n, \quad \text{in } W^{1,r}(\mathcal{O} \cap \Omega). \tag{3.51}$$

Then, (3.42) and (3.51) yield that there exists an $N_\varepsilon \in \mathbb{N}^*$ such that, for $\min(n,p) \geq N_\varepsilon$,

$$\|u - u_{n,p}\|_{W^{1,r}(\mathcal{O}\cap\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}. \tag{3.52}$$

Finally, we set $\mathbf{z}_n = (\mathbf{0}, \frac{1}{n})$ and for $\min(n,p) \geq \frac{6}{\mu}$ where μ is defined by (3.39), we consider the set

$$E = \Big\{ \mathbf{x} \in \overline{\mathcal{O}} \cap \overline{\Omega},\ d(\mathbf{x}, \partial\mathcal{O} \cap \overline{\Omega}) \leq \frac{\mu}{3} \Big\}.$$

$\forall \mathbf{x} \in E,\ \forall \mathbf{y} \in B(\mathbf{0}, \frac{1}{p})$, and we have

$$d(\mathbf{x} - \mathbf{y} - \mathbf{z}_n, \mathrm{supp}\ u) \geq d(\partial\mathcal{O} \cap \overline{\Omega}, \mathrm{supp}\ u) - d(\mathbf{x}, \partial\mathcal{O} \cap \overline{\Omega}) - d(\mathbf{x}, \mathbf{x} - \mathbf{y} - \mathbf{z}_n) \geq \frac{\mu}{3} > 0.$$

In the same way, $\forall \mathbf{x} \in \overline{\Omega} \setminus \mathcal{O},\ \forall \mathbf{y} \in B(\mathbf{0}, \frac{1}{p})$, and we have

$$d(\mathbf{x} - \mathbf{y} - \mathbf{z}_n, \mathrm{supp}\ u) \geq d(\mathbf{x}, \mathrm{supp}\ u) - d(\mathbf{x}, \mathbf{x} - \mathbf{y} - \mathbf{z}_n) \geq \frac{2\mu}{3} > 0.$$

Hence, we derive that, for every $\mathbf{x} \in E \cup (\overline{\Omega} \setminus \mathcal{O})$ and $\mathbf{y} \in B(\mathbf{0}, \frac{1}{p})$, $\mathbf{x} - \mathbf{y} - \mathbf{z}_n$ does not belong to $\mathrm{supp}\, u$, which implies $u_{n,p}(\mathbf{x}) = 0$. Thus, the function $u_\varepsilon = u_{m_\varepsilon, m_\varepsilon}$, where $u_{n,p}$ is defined by (3.43) and m_ε by $m_\varepsilon = \max([\frac{6}{\mu}] + 1, N_\varepsilon)$, belongs to $\mathcal{D}(\overline{\Omega})$ with a compact support in $\mathcal{O} \cap \overline{\Omega}$ and verifies

$$\|u - u_\varepsilon\|_{W^{1,r}(\mathcal{O}\cap\Omega)} = \|u - u_\varepsilon\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha},$$

which ends the proof of the lemma.

The next lemma deals with an approximation of $\varphi_{\alpha,k} u$ in $\mathcal{O}_{k,\alpha}$ for $k = q+1, \cdots, r_\alpha$, such that $\mathbf{m}_{k,\alpha} \in \gamma_0$, that is, an approximation of u localized around γ_0, which is the part of the boundary where u vanishes.

Lemma 3.5 *Let α be a real number verifying* (1.3). *For every real number $\varepsilon > 0$ and for every $k = q+1, \cdots, r_\alpha$, such that $\mathbf{m}_{k,\alpha} \in \gamma_0$, there exists a function $u_{\varepsilon,k} \in \mathcal{D}(\Omega \cap \mathcal{O}_{k,\alpha})$, such that*

$$\|\varphi_{\alpha,k} u - u_{\varepsilon,k}\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}, \tag{3.53}$$

where r_α is defined by (2.4).

Proof As in the previous lemma, to simplify the notations, we drop the indexes, replacing for $k = q+1, \cdots, r_\alpha$, $\varphi_{\alpha,k}\, u$ by u and $\mathcal{O}_{k,\alpha}$ by $\mathcal{O}$, so that we may assume that u has compact support in $\mathcal{O} \cap \overline{\Omega}$, and set

$$d(\partial\mathcal{O} \cap \overline{\Omega}, \mathrm{supp}\ u) = \nu > 0. \tag{3.54}$$

Considering (2.1) and (2.12), we may assume that $\mathcal{O}$ is an open hypercube, such that, in new orthogonal coordinates $\mathbf{y} = (\mathbf{y}', y_d)$, we have

$$\mathcal{O} \cap \Omega = \{\mathbf{y} \in \mathcal{O},\ y_d < \Phi(\mathbf{y}')\} \quad \text{and} \quad \gamma_0 \cap \mathcal{O} = \{\mathbf{y} \in \mathcal{O},\ y_d = \Phi(\mathbf{y}')\}, \tag{3.55}$$

where Φ is a Lipschitz-continuous function, defined in $\prod_{j=1}^{d-1}]-a_j, a_j[$ of constant L.

Let $n \in \mathbb{N}^*$. We set

$$u_n(\mathbf{y}) = u\Big(\mathbf{y}', y_d + \frac{1}{n}\Big), \tag{3.56}$$

which is a function defined on

$$\Omega_n = \Big\{\mathbf{y} \in \mathbb{R}^d,\ \Big(\mathbf{y}', y_d + \frac{1}{n}\Big) \in \mathcal{O} \cap \Omega\Big\}.$$

The set Ω_n is obtained by translating $\mathcal{O} \cap \Omega$ in the direction of negative y_d, that is to say, contrary to the previous, inside the domain Ω. We denote by $\widetilde{u}_n$ the extension of u_n by zero outside Ω_n. Considering the support of u and since u vanishes on γ_0, we can see that the restriction of $\widetilde{u}_n$ to $\mathcal{O} \cap \Omega$ belongs to $W^{1,r}(\mathcal{O} \cap \Omega)$, and as in the previous lemma, we have

$$\lim_{n\to+\infty} \widetilde{u}_{n|\mathcal{O}\cap\Omega} = u, \quad \text{in } W^{1,r}(\mathcal{O} \cap \Omega). \tag{3.57}$$

Note that, if $\frac{1}{n} \le \nu$, where ν is defined by (3.54), then u_n has a compact support in $\mathcal{O} \cap \Omega$, and therefore, $\widetilde{u}_n$ belongs to $W^{1,r}(\mathbb{R}^d)$. Hence, setting

$$u_{n,p} = \widetilde{u}_n * \rho_p,$$

we derive

$$\lim_{p\to+\infty} (u_{n,p})|_{\mathcal{O}\cap\Omega} = (\widetilde{u}_n)|_{\mathcal{O}\cap\Omega}, \quad \text{in } W^{1,r}(\mathcal{O} \cap \Omega),$$

which implies that, in view of (3.57), there exists an $N'_\varepsilon \in \mathbb{N}^*$, such that, for $\min(n,p) \ge N'_\varepsilon$,

$$\|u - u_{n,p}\|_{W^{1,r}(\mathcal{O}\cap\Omega)} \le \frac{\varepsilon}{4\, r_\alpha}. \tag{3.58}$$

We set

$$\Gamma_n^* = \Big\{\mathbf{y} \in \mathbb{R}^d, \Big(\mathbf{y}', y_d + \frac{1}{n}\Big) \in \partial\Omega \cap \mathcal{O}\Big\}. \tag{3.59}$$

Note that $d(\partial\Omega \cap \overline{\mathcal{O}}, \partial\Omega_n) = d(\partial\Omega \cap \overline{\mathcal{O}}, \Gamma_n^*)$ because $\forall \mathbf{z} \in \partial\Omega \cap \overline{\mathcal{O}}$ and $\forall \mathbf{y} \in \Omega_n$, $[\mathbf{z}, \mathbf{y}] \cap \Gamma_n^* \ne \emptyset$. Moreover, in the same way as for Γ_n, we obtain the analogue of (3.48)

$$d(\partial\Omega \cap \overline{\mathcal{O}}, \partial\Omega_n) = d(\partial\Omega \cap \overline{\mathcal{O}}, \Gamma_n^*) \ge \min\Big(\frac{1}{2n}, \frac{1}{2nL}\Big) = \delta_n. \tag{3.60}$$

We recall that

$$u_{n,p}(\mathbf{x}) = \int_{B(\mathbf{0},\frac{1}{p})} \widetilde{u}_n(\mathbf{x} - \mathbf{y}) \rho_p(\mathbf{y}) \mathrm{d}\mathbf{y}.$$

Let us define the following two sets:

$$E = \Big\{\mathbf{x} \in \overline{\Omega \cap \mathcal{O}},\ d(\mathbf{x}, \partial\Omega \cap \overline{\mathcal{O}}) \le \frac{\delta_n}{3}\Big\} \quad \text{and} \quad F = \Big\{\mathbf{x} \in \overline{\Omega \cap \mathcal{O}},\ d(\mathbf{x}, \partial\mathcal{O} \cap \overline{\Omega}) \le \frac{\nu}{3}\Big\}.$$

On the one hand, choosing $p \ge \frac{3}{\delta_n}$, $\forall \mathbf{y} \in B(\mathbf{0}, \frac{1}{p})$ and $\forall \mathbf{x} \in E$, we have

$$d(\mathbf{x} - \mathbf{y}, \partial\Omega_n) \ge d(\partial\Omega \cap \overline{\mathcal{O}}, \partial\Omega_n) - d(\mathbf{x}, \partial\Omega \cap \overline{\mathcal{O}}) - d(\mathbf{x}, \mathbf{x} - \mathbf{y}) \ge \frac{\delta_n}{3} > 0,$$

which implies $\widetilde{u}_n(\mathbf{x} - \mathbf{y}) = 0$. Thus, we obtain

$$\forall \mathbf{x} \in E, \quad u_{n,p}(\mathbf{x}) = 0. \tag{3.61}$$

On the other hand, setting $\mathbf{z}_n = (\mathbf{0}, \frac{1}{n})$ and choosing n and p large enough, such that $\frac{1}{n}+\frac{1}{p} \le \frac{\nu}{3}$, $\forall \mathbf{y} \in B(\mathbf{0}, \frac{1}{p})$ and $\forall \mathbf{x} \in E$, we have

$$d(\mathbf{x}-\mathbf{y}+\mathbf{z}_n, \operatorname{supp} u) \ge d(\partial\Omega \cap \overline{\Omega}, \operatorname{supp} u) - d(\mathbf{x}, \partial\Omega \cap \overline{\Omega}) - d(\mathbf{x}, \mathbf{x}-\mathbf{y}+\mathbf{z}_n) \ge \frac{\nu}{3} > 0,$$

which implies $\widetilde{u}_n(\mathbf{x}-\mathbf{y}) = 0$ and therefore

$$\forall \mathbf{x} \in F,\ u_{n,p}(\mathbf{x}) = 0. \tag{3.62}$$

Thus, since $\partial(\Omega \cap \mathcal{O}) = (\partial\mathcal{O} \cap \overline{\Omega}) \cup (\partial\Omega \cap \overline{\mathcal{O}})$, owing to (3.61) and (3.62), for $n \ge \frac{6}{\nu}$ and $p \ge \max(\frac{6}{\nu}, \frac{3}{\delta_n})$ with δ_n defined in (3.60), $u_{n,p}$ belongs to $\mathcal{D}(\Omega \cap \mathcal{O})$. Finally, in view of (3.58), the function $u_\varepsilon = u_{n_\varepsilon, p_\varepsilon}$, where

$$n_\varepsilon = \max\Big(\Big[\frac{6}{\nu}\Big]+1, N'_\varepsilon\Big) \quad \text{and} \quad p_\varepsilon = \max\Big(\Big[\frac{6}{\nu}\Big]+1, \Big[\frac{3}{\min\big(\frac{1}{2n_\varepsilon}, \frac{1}{2n_\varepsilon L}\big)}\Big]+1, N'_\varepsilon\Big),$$

belongs to $\mathcal{D}(\Omega \cap \mathcal{O})$ and verifies

$$\|u-u_\varepsilon\|_{W^{1,r}(\mathcal{O}\cap\Omega)} = \|u-u_\varepsilon\|_{W^{1,r}(\Omega)} \le \frac{\varepsilon}{4\,r_\alpha}.$$

Hence, the lemma follows.

We can now complete the proof of Theorem 3.1. Let $\varepsilon > 0$ be a given real number. Lemma 3.3 leads us to define a partition of unity $\mathcal{P}_\alpha$, with $\alpha \le \alpha_\varepsilon$, where $\mathcal{P}_\alpha$ is defined by (2.12). Next, (3.37), Lemmas 3.4 and 3.5 allow us to construct a function u_ε of $\mathcal{D}(\overline{\Omega})$ defined by

$$u_\varepsilon = u_{\varepsilon,0} + \sum_{q+1 \le k \le r_\alpha} u_{\varepsilon,k}. \tag{3.63}$$

Then, we have

$$\|u-u_\varepsilon\|_{W^{1,r}(\Omega)} \le \|\varphi_{\alpha,0} u - u_{\varepsilon,0}\|_{W^{1,r}(\Omega)} + \sum_{q+1\le k\le r_\alpha} \|\varphi_{\alpha,k}\, u - u_{\varepsilon,k}\|_{W^{1,r}(\Omega)} + \sum_{k=1}^{q} \|\boldsymbol{\varphi}_{\alpha,k}\, u\|_{W^{1,r}(\Omega)},$$

which implies, in view of (3.34), (3.37)–(3.38) and (3.53),

$$\|u-u_\varepsilon\|_{W^{1,r}(\Omega)} \le \varepsilon. \tag{3.64}$$

Moreover, owing to Lemma 3.4, we obtain that, for every $k = q+1, \cdots, r_\alpha$ with $\mathbf{m}_{k,\alpha} \in \gamma_1$ (note that by construction $\mathcal{O}_{k,\alpha} \cap \overline{\gamma}_0 = C_{\mathbf{m}_{k,\alpha}} \cap \overline{\gamma}_0 = \emptyset$), $u_{\varepsilon,k}$ belongs to $\mathcal{D}(\overline{\Omega}, \gamma_0)$ and, consequently, u_ε belongs to $\mathcal{D}(\overline{\Omega}, \gamma_0)$, where $\mathcal{D}(\overline{\Omega}, \gamma_0)$ is defined by (1.5). Thus, Theorem 3.1 is proved.

4 Density Result in $W^{m,r}(\Omega, \gamma_0)$

Let $k \ge 1$ be an integer and let us suppose that the boundary $\partial\Omega$ is of class $C^{k,1}$, which means that, for every $\mathbf{x} \in \partial\Omega$, the functions $\Phi^{\mathbf{x}}$, defined by (2.1), are of class $C^{k,1}$. The following theorem generalizes Theorem 3.1.

Theorem 4.1 *Let $r > 1$ be a real number, and $m \geq 1$ be an integer. Let Ω be a bounded domain in $\mathbb{R}^d$ whose boundary is of class $C^{k,1}$, where k is an integer such that $k+1 \geq m$, and let γ_0 be an open part of $\partial\Omega$ verifying* (1.1). *Let the spaces $W^{m,r}(\Omega,\gamma_0)$ and $\mathcal{D}(\overline{\Omega},\gamma_0)$ be defined respectively by* (1.4) *and* (1.5). *Then the space $\mathcal{D}(\overline{\Omega},\gamma_0)$ is dense in $W^{m,r}(\Omega,\gamma_0)$.*

Proof Let us prove the result for $m = 2$, the extension to the general case is straightforward. We suppose that u belongs to $W^{2,r}(\Omega,\gamma_0)$. The proof of this theorem is analogous to that of Theorem 3.1. Indeed, we use the same covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$ defined by (2.12), and an associated partition of unity $\widetilde{\mathcal{P}}_\alpha$ analogous to $\mathcal{P}_\alpha$, defined as follows.

First, we define the functions $\widetilde{\theta}_{\alpha,k}$, for $k = 1,\cdots,q$, by

$$\forall k,\ 1 \leq k \leq q, \quad \widetilde{\theta}_{\alpha,k} = \widetilde{\varphi}\Big(\frac{1}{\alpha}d(\,.\,,K_k)\Big) * \rho_{p_\alpha} \tag{4.1}$$

with $p_\alpha = [\frac{16}{\alpha}]+1$ and ρ_p defined by (2.6), where the function $\widetilde{\varphi}$ belongs to $C^2(\mathbb{R}^+)$ and verifies

$$\forall t \in \Big[0,\frac{9}{16}\Big],\ \ \widetilde{\varphi}(t) = 1, \quad \forall t \geq \frac{11}{16},\ \ \widetilde{\varphi}(t) = 0 \quad \text{and} \quad \forall t \in \mathbb{R}_+,\ \ |\widetilde{\varphi}'(t)| \leq A,\ \ |\widetilde{\varphi}''(t)| \leq B.$$

For example, we can choose $\widetilde{\varphi}$ defined on $[\frac{9}{16},\frac{11}{16}]$ by

$$\widetilde{\varphi}(t) = 15(16^4)\int_t^{\frac{11}{16}}\Big(x-\frac{9}{16}\Big)^2\Big(x-\frac{11}{16}\Big)^2\mathrm{d}x.$$

Since the boundary is at least of class $C^{1,1}$, the first and second order partial derivatives of the function $\mathbf{x} \mapsto d(\mathbf{x},K_k)$ belong to $L^\infty(\mathbb{R}^d)$ (see [5]). Setting $M = \|\partial^2 d(\,\cdot\,,K_k)\|_{L^\infty(\mathbb{R}^d)}$, we derive the following estimations for the functions $\widetilde{\theta}_{\alpha,k} \in \mathcal{D}(G_{k,\alpha})$ and its derivatives, for $k = 1,\cdots,q$ and for $i,j = 1,\cdots,d$,

$$\begin{aligned}&\forall \mathbf{x} \in G_{k,\frac{\alpha}{2}}, \quad \widetilde{\theta}_{\alpha,k}(\mathbf{x}) = 1, \quad \forall \mathbf{x} \notin G_{k,\frac{3\alpha}{4}}, \quad \widetilde{\theta}_{\alpha,k}(\mathbf{x}) = 0,\\ &\forall \mathbf{x} \in \mathbb{R}^d, \quad |\partial_i\widetilde{\theta}_{\alpha,k}(\mathbf{x})| \leq \frac{A}{\alpha}, \quad |\partial_i\partial_j\widetilde{\theta}_{\alpha,k}(\mathbf{x})| \leq \frac{C}{\alpha^2},\end{aligned} \tag{4.2}$$

where $G_{k,\alpha}$ is defined by (1.2) and $C = B + AM$.

Second, we set $\widetilde{\mathcal{P}}_\alpha = \{\widetilde{\varphi}_{\alpha,k}\}_{k=0}^{r_\alpha}$ with

$$\begin{aligned}&\widetilde{\varphi}_{\alpha,k} = \Big(\prod_{j=1}^{q}(1-\widetilde{\theta}_{\alpha,j})\Big)\beta_{\alpha,k}, \quad k = 0 \text{ or } q+1 \leq k \leq r_\alpha,\\ &\widetilde{\varphi}_{\alpha,k} = \widetilde{\theta}_{\alpha,k}, \quad 1 \leq k \leq q.\end{aligned} \tag{4.3}$$

As previously discussed, for every real ε, we must compute a parameter α'_ε, allowing us to construct an adequate partition of unity $\widetilde{\mathcal{P}}_\alpha$ with $\alpha \leq \alpha'_\varepsilon$. Thus, we prove an analogous lemma to Lemma 3.3.

Lemma 4.1 *For every real number $\varepsilon > 0$, there exists a real number α'_ε verifying* (1.3) *such that, for every $0 < \alpha \leq \alpha'_\varepsilon$,*

$$\forall k = 1,\cdots,q, \quad \|\widetilde{\theta}_{\alpha,k}u\|_{W^{2,r}(G_{k,\alpha}\cap\Omega)} \leq \frac{\varepsilon}{4q}. \tag{4.4}$$

Proof In the same way as in Lemma 3.3, using an extension $\widetilde{u} \in W^{2,r}(\mathbb{R}^d)$ of $u \in W^{2,r}(\Omega, \gamma_0)$, we prove

$$\lim_{\alpha \to 0} \|\widetilde{\theta}_{\alpha,k} u\|_{W^{1,r}(G_{k,\alpha} \cap \Omega)} = 0. \tag{4.5}$$

On the one hand, for $j = 1, \cdots, d$ and $i = 1, \cdots, p$, $\partial_j \widetilde{u}$ vanishes on $B(\mathbf{x}_i, 2\alpha) \cap \gamma_0$, which has a strictly positive measure, and we can use Poincaré's inequality to deduce

$$\|\nabla \widetilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))} \le C_1 \alpha^r \|\partial^2 \widetilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))},$$

where C_1 is the constant defined in (3.28). As in the proof of Lemma 3.3, setting the integrals $\|\partial^2 \widetilde{u}\|_{L^r(B(\mathbf{x}_i,2\alpha))}$ in decreasing order, by an analogous method, we obtain

$$\|\nabla \widetilde{u}\|^r_{L^r(G_{k,\alpha})} \le C_1 \alpha^r M_d \|\partial^2 \widetilde{u}\|^r_{L^r(G_{k,4\alpha})}, \tag{4.6}$$

where M_d is defined by (3.27). Moreover, owing to (3.33), we derive

$$\|u\|^r_{L^r(G_{k,\alpha} \cap \Omega)} \le 4^r \, C_1^2 \alpha^{2r} M_d^2 \|\partial^2 \widetilde{u}\|^r_{L^r(G_{k,16\alpha})}. \tag{4.7}$$

On the other hand, we can write

$$\partial_i \partial_j (\widetilde{\theta}_{\alpha,k}\, u) = \partial_i \partial_j (\widetilde{\theta}_{\alpha,k})\, u + \partial_i (\widetilde{\theta}_{\alpha,k}) \partial_j u + \partial_j (\widetilde{\theta}_{\alpha,k})\, \partial_i u + (\widetilde{\theta}_{\alpha,k}) \partial_i \partial_j u.$$

Then, in view of (4.2), (4.6) and (4.7), we obtain

$$\|\partial^2 (\widetilde{\theta}_{\alpha,k}\, u)\|^r_{L^r(G_{k,\alpha} \cap \Omega)} \le 4^{r-1} ((4\,C)^r \,(d\,C_1\,M_d)^2 + 2d\,C_1\,M_d\,A^r + 1) \|\partial^2 \widetilde{u}\|^r_{L^r(G_{k,16\alpha})}.$$

Hence, since

$$\lim_{\alpha \to 0} \|\partial^2 \widetilde{u}\|_{L^r(G_{k,16\alpha})} = 0,$$

we derive

$$\lim_{\alpha \to 0} \|\partial^2 (\widetilde{\theta}_{\alpha,k}\, u)\|_{L^r(G_{k,\alpha} \cap \Omega)} = 0,$$

which implies, owing to (4.5),

$$\lim_{\alpha \to 0} \|\widetilde{\theta}_{\alpha,k} u\|_{W^{2,r}(G_{k,\alpha} \cap \Omega)} = 0,$$

and the result of the lemma follows.

We consider a partition of unity $\widetilde{\mathcal{P}}_\alpha$ defined by (4.1) with $0 < \alpha \le \alpha'_\varepsilon$, subordinated to the covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$, defined by (2.12), where α'_ε is defined in Lemma 4.1. Since $\widetilde{\theta}_{k,\alpha}$ belongs to $\mathcal{D}(G_{k,\alpha})$, (4.4) can be written as, with the notation of the partition $\widetilde{\mathcal{P}}_\alpha$,

$$\forall k = 1, \cdots, q, \quad \|\widetilde{\varphi}_{\alpha,k} u\|_{W^{2,r}(\Omega)} \le \frac{\varepsilon}{4q}, \tag{4.8}$$

so that, for every $k = 1, \cdots, q$, we can approximate $\widetilde{\varphi}_{\alpha,k} u$ by 0 in $\mathcal{O}_{k,\alpha} = G_{k,\alpha}$.

We now deal with the case $k = 0$, that is to say, we want approximate $\widetilde{\varphi}_{\alpha,0} u$ in $\mathcal{O}_{0,\alpha}$. In the same way as in the proof of Theorem 3.1, we set $u_p = (\widetilde{\widetilde{\varphi}_{\alpha,0} u}) * \rho_p$, where the wide latter denotes the extension by zero. In a standard way, considering that $\widetilde{\widetilde{\varphi}_{\alpha,0} u} \in W^{2,r}(\mathbb{R}^d)$, we obtain that

$$\lim_{p \to +\infty} u_p = \widetilde{\widetilde{\varphi}_{\alpha,0} u}, \quad \text{in } W^{2,r}(\mathbb{R}^d),$$

which implies that there exists a $P'_\varepsilon \in N^*$, such that $\forall p \geq P'_\varepsilon$,

$$\|\widetilde{\varphi}_{\alpha,0}u - u_p\|_{W^{2,r}(\mathcal{O}_{0,\alpha})} \leq \frac{\varepsilon}{4}. \tag{4.9}$$

Then considering $\mu'_0 = d(\text{supp}(\widetilde{\varphi}_{\alpha,0}\, u), \partial\mathcal{O}_{0,\alpha}) > 0$ and setting $u_{\varepsilon,0} = u_{m'_\varepsilon}$, where m'_ε is defined by $m'_\varepsilon = \max([\frac{3}{\mu'_0}], P'_\varepsilon)$ ($[r]$ is the integral part of r), yield

$$\|\widetilde{\varphi}_{\alpha,0}u - u_{\varepsilon,0}\|_{W^{2,r}(\mathcal{O}_{0,\alpha}\cap\Omega)} = \|\widetilde{\varphi}_{\alpha,0}u - u_{\varepsilon,0}\|_{W^{2,r}(\Omega)} \leq \frac{\varepsilon}{4} \quad \text{with } u_{\varepsilon,0} \in \mathcal{D}(\mathcal{O}_{0,\alpha}), \tag{4.10}$$

where $\overline{\mathcal{O}}_{0,\alpha} \subset \Omega$.

Next, we are taking an approximation of $\widetilde{\varphi}_{\alpha,k}\, u$ in $\mathcal{O}_{k,\alpha}$ for $k = q+1, \cdots, r_\alpha$, such that $\mathbf{m}_{k,\alpha} \in \gamma_1$, that is, an approximation of u localized around γ_1. As in Lemma 3.4, to simplify the notations, we drop the indexes, replacing $\widetilde{\varphi}_{\alpha,k}\, u$ by u and $\mathcal{O}_{k,\alpha}$ by $\mathcal{O}$, so that we may assume that u has compact support in $\mathcal{O} \cap \overline{\Omega}$ and set

$$d(\partial\mathcal{O} \cap \overline{\Omega}, \text{supp } u) = \mu' > 0. \tag{4.11}$$

We define u_n, Ω_n by (3.41) and denote by $\widetilde{u}_n$ the extension of u_n by zero. We can verify, by deriving in the sense of distribution, that

$$\partial_i(\widetilde{u}_{n|\mathcal{O}\cap\Omega}) = \widetilde{(\partial_i u)}_{n|\mathcal{O}\cap\Omega}, \quad \partial_j\partial_i(\widetilde{u}_{n|\mathcal{O}\cap\Omega}) = \widetilde{(\partial_j\partial_i u)}_{n|\mathcal{O}\cap\Omega},$$

where the wide latter denotes the extension by zero in $\mathcal{O} \cap \Omega \setminus \Omega_n$, which implies that the restriction of $\widetilde{u}_n$ to $\mathcal{O}\cap\Omega$ belongs to $W^{2,r}(\mathcal{O}\cap\Omega)$ and the following convergence:

$$\lim_{n\to+\infty} \widetilde{u}_{n|\mathcal{O}\cap\Omega} = u, \quad \text{in } W^{2,r}(\mathcal{O}\cap\Omega). \tag{4.12}$$

Next, we define $u_{n,p}$ by (3.43) and in the same way as in Lemma 3.4, we prove that $\widetilde{u}_{n|\mathcal{O}_p}$ belongs to $W^{2,r}(\mathcal{O}_p)$, where $\mathcal{O}_p$ is defined by (3.45), and $\partial_j\partial_i\widetilde{u}_{n|\mathcal{O}_p} = \widetilde{\partial_j\partial_i u_n}$, where the wide latter is the extension by zero of $\partial_j\partial_i u_n \in L^r(\Omega_n \cap \mathcal{O}_p)$ in $\mathcal{O}_p$. Moreover, as in the proof of (3.51), we can show that for p verifying (3.49),

$$\partial_j\partial_i u_{n,p} = \widetilde{\partial_j\partial_i u_n} * \rho_p, \quad \text{almost everywhere in } \mathcal{O}\cap\Omega,$$

and we obtain

$$\lim_{p\to+\infty} (u_{n,p})|_{\mathcal{O}\cap\Omega} = \widetilde{u}_n, \quad \text{in } W^{2,r}(\mathcal{O}\cap\Omega).$$

Hence, with (4.12), we derive that there exists an $N'_\varepsilon \in \mathbb{N}^*$, such that for $\min(n,p) \geq N'_\varepsilon$,

$$\|u - u_{n,p}\|_{W^{2,r}(\mathcal{O}\cap\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}. \tag{4.13}$$

Thus, the function $u_\varepsilon = u_{m'_\varepsilon, m'_\varepsilon}$, where m'_ε is defined by

$$m'_\varepsilon = \max\left(\left[\frac{6}{\mu'}\right] + 1, N'_\varepsilon\right)$$

with μ' defined by (4.11), belongs to $\mathcal{D}(\overline{\Omega})$ with a compact support in $\mathcal{O}\cap\overline{\Omega}$ and verifies

$$\|u - u_\varepsilon\|_{W^{2,r}(\mathcal{O}\cap\Omega)} = \|u - u_\varepsilon\|_{W^{2,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}.$$

Then, with the initial notation, we obtain, for every $k = q+1, \cdots, r_\alpha$, such that $\mathbf{m}_{k,\alpha} \in \gamma_1$,

$$\|\widetilde{\varphi}_{\alpha,k}\, u - u_{\varepsilon,k}\|_{W^{2,r}(\mathcal{O}\cap\Omega)} \le \frac{\varepsilon}{4\, r_\alpha}, \tag{4.14}$$

where the function $u_{\varepsilon,k}$ belongs to $\mathcal{D}(\overline{\Omega})$ with compact support in $\mathcal{O}_{k,\alpha} \cap \overline{\Omega}$, which ends the problem of the approximation of u localized around γ_1.

Finally, we still have an approximation of $\widetilde{\varphi}_{\alpha,k}\, u$ in $\mathcal{O}_{k,\alpha}$ to do, for $k = q+1, \cdots, r_\alpha$, such that $\mathbf{m}_{k,\alpha} \in \gamma_0$, that is, an approximation of u localized around γ_0, which is the part of the boundary where u vanishes.

As previously done, to simplify the notations, we replace, for $k = q+1, \cdots, r_\alpha$, $\varphi_{\alpha,k}\, u$ by u and $\mathcal{O}_{k,\alpha}$ by $\mathcal{O}$, so that we may assume that u has compact support in $\mathcal{O} \cap \overline{\Omega}$ and set

$$d(\partial\mathcal{O} \cap \overline{\Omega}, \text{supp } u) = \nu' > 0. \tag{4.15}$$

We again define u_n by (3.56) and $\widetilde{u}_n$ again denotes the extension of u_n by zero. In the same way as in the proof of Lemma 3.5, we have

$$\lim_{n\to+\infty} \widetilde{u}_n|_{\mathcal{O}\cap\Omega} = u, \quad \text{in } W^{2,r}(\mathcal{O}\cap\Omega) \tag{4.16}$$

and for $\frac{1}{n} \le \nu'$, u_n has a compact support in $\mathcal{O} \cap \Omega$. Moreover, setting again $u_{n,p} = \widetilde{u_n} * \rho_p$ yields that there exists an $N''_\varepsilon \in \mathbb{N}^*$, such that for $\min(n,p) \ge N''_\varepsilon$,

$$\|u - u_{n,p}\|_{W^{2,r}(\mathcal{O}\cap\Omega)} \le \frac{\varepsilon}{4\, r_\alpha}. \tag{4.17}$$

Then, the function $u_\varepsilon = u_{n'_\varepsilon, p'_\varepsilon}$, where

$$n'_\varepsilon = \max\Big\{\Big[\frac{6}{\nu'}\Big] + 1, N''_\varepsilon\Big\} \quad \text{and} \quad p'_\varepsilon = \max\Big\{\Big[\frac{6}{\nu'}\Big] + 1, \Big[\frac{3}{\min\Big(\frac{1}{2n'_\varepsilon}, \frac{1}{2n'_\varepsilon L}\Big)}\Big] + 1, N''_\varepsilon\Big\},$$

belongs to $\mathcal{D}(\Omega \cap \mathcal{O})$ and verifies

$$\|u - u_\varepsilon\|_{W^{2,r}(\mathcal{O}\cap\Omega)} = \|u - u_\varepsilon\|_{W^{2,r}(\Omega)} \le \frac{\varepsilon}{4\, r_\alpha}.$$

With the initial notation, we obtain, for every $k = q+1, \cdots, r_\alpha$, such that $\mathbf{m}_{k,\alpha} \in \gamma_0$,

$$\|\widetilde{\varphi}_{\alpha,k}\, u - u_{\varepsilon,k}\|_{W^{2,r}(\Omega)} \le \frac{\varepsilon}{4\, r_\alpha}, \tag{4.18}$$

where the function $u_{\varepsilon,k}$ belongs to $\mathcal{D}(\Omega \cap \mathcal{O}_{k,\alpha})$, which ends the problem of the approximation of u localized around γ_0.

We can complete the proof of Theorem 4.1. Let $\varepsilon > 0$ be a given real number. Lemma 4.1 leads us to define an adequate partition of unity $\widetilde{\mathcal{P}}_\alpha$, with $0 < \alpha \le \alpha_\varepsilon$. Next (4.9), (4.14) and (4.18) allow us to construct a function u_ε of $\mathcal{D}(\overline{\Omega})$ defined by

$$u_\varepsilon = u_{\varepsilon,0} + \sum_{q+1 \le k \le r_\alpha} u_{\varepsilon,k}$$

that verifies

$$\|u - u_\varepsilon\|_{W^{2,r}(\Omega)} \le \varepsilon.$$

With the same argument as at the end of the proof of Theorem 3.1, we prove that u_ε belongs to $\mathcal{D}(\overline{\Omega}, \gamma_0)$, where $\mathcal{D}(\overline{\Omega}, \gamma_0)$ is defined by (1.5). Thus, Theorem 4.1 is proved.

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