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Estimates for the Tail Probability of the Supremum of a Random Walk with Independent Increments*

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Abstract The authors investigate the tail probability of the supremum of a random walk with independent increments and obtain some equivalent assertions in the case that the increments are independent and identically distributed random variables with O-subexponential integrated distributions. A uniform upper bound is derived for the distribution of the supremum of a random walk with independent but non-identically distributed increments, whose tail distributions are dominated by a common tail distribution with an O-subexponential integrated distribution.

Keywords Random walk, O-Subexponential distribution, Integrated distribution, Supremum
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1 Introduction and Main Results

In various applied fields, such as queueing theory and risk theory among others, the model of a random walk occurs in a natural way. For a nice review, one can refer to [1, 4] among others. For a random walk, it is important to estimate the tail probability of its supremum. As applications, in queueing theory, this tail probability is the so-called overflow probability; and in risk theory, it may be interpreted as the ruin probability. In the case that a random walk has independent and identically distributed (i.i.d.) increments, many results were derived (see [3, 5, 8, 9, 11, 14, 15] among others). Recently, Foss et al. [6] studied a random walk with independent but non-identically distributed increments, and obtained some important and skillful results. All of the above results required that the integrated distributions of the increments of a random walk be subexponential (belonging to the class $\mathcal{S}$, see the definition below), or their tail distributions are dominated by a common tail distribution, whose integrated distribution belongs to the class $\mathcal{S}$. In this paper, we consider a wider distribution class and generalize some existing results.

Klüppelberg [7] introduced a weak idempotent distribution class, which extends the class $\mathcal{S}$. From the view of regular variation, as Shimura and Watanabe [10] stated, it may be more

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appropriate to be called the O-subexponential class (denoted by $\mathcal{OS}$). Shimura and Watanabe [10] systematically investigated the class $\mathcal{OS}$ and obtained some fundamental and important results, especially on random sums and infinite divisible distributions. However, they did not deal with the supremum of random walks, which attracts us greatly. In this paper, we establish a result corresponding to [6, Proposition 2.1] under the condition that the integrated distribution of the dominant distribution belongs to the class $\mathcal{OS}$. To this end, we need to extend the classical Pakes-Veraverbeke-Embrechts Theorem (P-V-E Theorem) from the class $\mathcal{S}$ to the class $\mathcal{OS}$. In order to better illuminate some existing results and our main results, we firstly introduce some notions and notation.

Throughout this paper, all the limits and asymptotic relations hold for x tending to ∞ unless stated otherwise. For two nonnegative functions $a(x)$ and $b(x)$ on $(-\infty,\infty)$, we write $a(x) = O(b(x))$, if $\limsup \frac{a(x)}{b(x)} < \infty$; $a(x) \approx b(x)$, if $a(x) = O(b(x))$ and $b(x) = O(a(x))$; $a(x) \sim b(x)$, if $\lim \frac{a(x)}{b(x)} = 1$; $a(x) = o(b(x))$, if $\lim \frac{a(x)}{b(x)} = 0$. The indicator function of an event A is denoted by $\mathbf{1}_A$.

Set $D = (-\infty,\infty)$ or $[0,\infty)$. A distribution V on D is said to be proper, if $V(\infty) = 1$. Denote its tail distribution and integrated distribution by $\overline{V}(x) = 1 - V(x)$ and $V^{\mathrm{I}}(x) = 1 - \min(1, \int_x^\infty \overline{V}(y)\mathrm{d}y)$, $x \in (-\infty,\infty)$, respectively. For some $\gamma \geq 0$, a distribution V of a random variable (r.v.) X is said to belong to the class $\mathcal{L}(\gamma)$, if $\overline{V}(x-y) \sim \mathrm{e}^{\gamma y}\overline{V}(x)$ for any $y \in (-\infty,\infty)$. Especially, the class $\mathcal{L}(0)$ is the so-called long-tailed distribution class, denoted by $\mathcal{L}$. For some $\gamma \geq 0$, V is said to belong to the class $\mathcal{S}(\gamma)$ on $[0,\infty)$, if $V \in \mathcal{L}(\gamma)$, $\int_0^\infty \mathrm{e}^{\gamma y} V \mathrm{d}y < \infty$ and $\overline{V^{*2}}(x) \sim 2\overline{V}(x)\int_0^\infty \mathrm{e}^{\gamma y} V \mathrm{d}y$, where V^{*n} denotes the n-fold convolution of V, $n \geq 2$, $V^{*1} = V$ and V^{*0} is the distribution degenerate at zero. V is said to belong to the class $\mathcal{S}(\gamma)$ on $(-\infty,\infty)$, if the distribution V_+ of $X\mathbf{1}_{\{X\geq 0\}}$ belongs to the class $\mathcal{S}(\gamma)$. Especially, $\mathcal{S}(0)$ is called the subexponential distribution class, denoted by $\mathcal{S}$. We point out that in the case $\gamma = 0$, $V \in \mathcal{L}$ on $[0,\infty)$ is not necessary, since it can be proved that $\mathcal{S} \subset \mathcal{L}$.

The classes $\mathcal{L}(\gamma)$ and $\mathcal{S}(\gamma)$ can be naturally extended to the following. A distribution V is said to belong to the class $\mathcal{OL}$, if $\overline{V}(x-y) = O(\overline{V}(x))$ for any $y \in (-\infty,\infty)$. V is said to belong to the class $\mathcal{OS}$, if $\overline{V_+^{*2}}(x) = O(\overline{V}_+(x))$. Clearly, we have the relations that $\mathcal{OS} \subset \mathcal{OL}$, $\mathcal{L}(\gamma) \subset \mathcal{OL}$ and $\mathcal{S}(\gamma) \subset \mathcal{OS}$, but the reverse is not true. For example, the class $\mathcal{OS}$ includes another important distribution class $\mathcal{D}$ consisting of all distributions with dominated variation ($V \in \mathcal{D}$, if $\overline{V}(xy) = O(\overline{V}(x))$ for any $y \in (0,1)$), but $\mathcal{D}\backslash\mathcal{S} \neq \emptyset$.

Let $\{\xi_i : i \geq 1\}$ be a sequence of random variables. Denote

$$S_0 = 0, \quad S_n = \sum_{i=1}^n \xi_i, \quad n \geq 1.$$

Then the sequence $\{S_n : n \geq 0\}$ constitutes a random walk. Define the supremum of the random walk $M = \sup_{n\geq 0} S_n$ with distribution W on $[0,\infty)$. Denote the upwards first passage time by $\tau_+ = \inf\{n \geq 0 : S_n > 0\}$, if $\{n \geq 0 : S_n > 0\} \neq \emptyset$; otherwise, $\tau_+ = \infty$. The random variable S_{τ_+} is called the first upwards ladder height of the random walk. If $\{\xi_i : i \geq 1\}$ are indepentlent and identically distributed random variables with finite mean $\mathrm{E}\xi_1 = -a < 0$, it is well-known that S_{τ_+} is defective random variable, i.e., $0 < \mathrm{P}(S_{\tau_+} < \infty) = \mathrm{P}(\tau_+ < \infty) = p < 1$. Define $\overline{G}(x) = \mathrm{P}(S_{\tau_+} > x \mid \tau_+ < \infty)$, $x \geq 0$. Then G is a proper distribution.

We firstly introduce some existing results. In the case that $\{\xi_i : i \geq 1\}$ are indepentlent and identically distributed random variables, some classical results were derived. For simplicity, we only state P-V-E Theorem for the heavy-tailed case.

Theorem 1.1 *Let $\{\xi_i : i \geq 1\}$ be independent and identically distributed random variables with common distribution F on $(-\infty, \infty)$ and finite negative mean $\mathrm{E}\xi_1 = -a$. Then, the following assertions are equivalent:*
(i) $F^{\mathrm{I}} \in \mathcal{S}$; (ii) $G \in \mathcal{S}$; (iii) $\overline{W}(x) \sim a^{-1}\overline{F^{\mathrm{I}}}(x)$; (iv) $W \in \mathcal{S}$.

Theorem 1.1 provides a perfect equivalence result. For the wider class $\mathcal{OS}$, although the precise equivalence in Theorem 1.1(iii) may not be maintained, a similar weak equivalence result also holds. As in [10], for a distribution V on $[0, \infty)$, denote

$$l^*(V) = \limsup \frac{\overline{V^{*2}}(x)}{\overline{V}(x)}.$$

Theorem 1.2 *Let $\{\xi_i : i \geq 1\}$ be the same as those in Theorem 1.1. If $F^{\mathrm{I}} \in \mathcal{OL}$, then*
(a) $\overline{F^{\mathrm{I}}}(x) = O(\overline{W}(x))$;
(b) (i) $F^{\mathrm{I}} \in \mathcal{OS}$ *and* (ii) $G \in \mathcal{OS}$ *are equivalent*;
(c) (iii) $\overline{F^{\mathrm{I}}}(x) \approx \overline{G}(x) \approx \overline{W}(x)$ *yields* (i) *or* (ii).
If $F^{\mathrm{I}} \in \mathcal{L}$ and $a^{-1}\big(l^*(F_+^{\mathrm{I}}) - 1\big) < 1$, then (i) *or* (ii) *yields* (iii). *In this case,* (i) ((ii) *or* (iii)) *implies $W \in \mathcal{OS}$.*

Remark 1.1 It is known that the class $\mathcal{S}$ is closed under convolution roots, but as remarked by Shimura and Watanabe [10] that the class $\mathcal{OS}$ does not own this property, which essentially leads to the difference between Theorems 1.1 and 1.2.

Foss et al. [6] considered a random walk with independent but non-identically distributed increments, and provided a uniform upper bound for the supremum.

Theorem 1.3 *Let F be a distribution on $(-\infty, \infty)$ such that $\int_0^\infty \overline{F}(y)\mathrm{d}y < \infty$ and its integrated distribution $F^{\mathrm{I}} \in \mathcal{S}$. Let α and β be two fixed positive constants. Consider any sequence $\{\xi_i : i \geq 1\}$ of independent random variables such that, for each i, the distribution F_i of ξ_i satisfies the conditions*

$$\overline{F_i}(x) \leq \overline{F}(x) \quad \textit{for all } x \in (-\infty, \infty), \tag{1.1}$$

$$\int_{-\infty}^{\infty} \max(y, -\beta) F_i \mathrm{d}y \leq -\alpha. \tag{1.2}$$

Then, there exists a positive constant r_0, depending only on F, α and β, such that for all sequences $\{\xi_i : i \geq 1\}$ above,

$$\overline{W}(x) \leq r_0 \overline{F^{\mathrm{I}}}(x) \quad \textit{for all } x \in (-\infty, \infty). \tag{1.3}$$

Along the line of Theorem 1.3, from Theorem 1.2, we can obtain a more general result. To this end, we require some more notions and notation. Denote by $\mathcal{F}$ the class of all distributions concentrated on $(-\infty, \infty)$. Denote the light-tailed distribution class by

$$\mathcal{K}^c = \Big\{V \in \mathcal{F} : \int_0^\infty \mathrm{e}^{\alpha y} V \mathrm{d}y < \infty \text{ for some } \alpha > 0\Big\},$$

and the heavy-tailed distribution class by $\mathcal{K} = \mathcal{F}\backslash\mathcal{K}^c$. Set

$$\mathcal{DK}^c = \{V \in \mathcal{F} : \overline{V}_1(x) = O(\overline{V}(x)) \text{ for all } V_1 \in \mathcal{K}^c\}.$$

Wang et al. [12] gave another representation of $\mathcal{DK}^c$, i.e.,

$$\mathcal{DK}^c = \{V \in \mathcal{F} : \lim \mathrm{e}^{\alpha x}\overline{V}(x) = \infty \text{ for all } \alpha > 0\}. \tag{1.4}$$

Theorem 1.4 *Under the conditions of Theorem* 1.3, *replacing* $F^{\mathrm{I}} \in \mathcal{S}$ *by* $F^{\mathrm{I}} \in \mathcal{OS} \cap \mathcal{DK}^c$, (1.3) *still holds.*

Remark 1.2 We point out that since $\mathcal{L} \cup \mathcal{D} \subset \mathcal{OS} \cap \mathcal{DK}^c$, Theorem 1.4 substantially generalizes Theorem 1.3. Additionally, it is easy to see that $\mathcal{OS} \cap \mathcal{DK}^c \subset \mathcal{K}$, hence F^{I} in Theorem 1.4 is heavy-tailed. However, we do not know under what conditions, (1.3) still holds for some other distributions in $\mathcal{OS}$.

The proofs of Theorems 1.2 and 1.4 will be given in Sections 2 and 3, respectively.

2 Proof of Theorem 1.2

According to the famous Wiener-Hopf factorization (see [11]), we rewrite $\overline{W}(x)$ as

$$\overline{W}(x) = (1-p)\sum_{n=0}^{\infty} p^n \overline{G^{*n}}(x), \quad x \geq 0. \tag{2.1}$$

Before the proof of Theorem 1.2, it is necessary to make clear the relation between G and F^{I}. To this end, we give two lemmas below. By the standard argument, we can easily derive the first lemma.

Lemma 2.1 *If* $F^{\mathrm{I}} \in \mathcal{OL}$, *then for any constant* d_1, *any positive constant* d_2 *and nonnegative integer* m, *it holds that*

$$\overline{F^{\mathrm{I}}}(x) \approx \sum_{n=m}^{\infty} \overline{F}(x + d_1 + d_2 n),$$

and so does $\overline{F}(x) = O(\overline{F^{\mathrm{I}}}(x))$.

Lemma 2.2 *Let* $\{\xi_i : i \geq 1\}$ *be independent and identically distributed random variables with common distribution* F *on* $(-\infty, \infty)$ *and finite mean* $\mathrm{E}\xi_1 = -a < 0$. *If* $F^{\mathrm{I}} \in \mathcal{OL}$, *then* $\overline{G}(x) \approx \overline{F^{\mathrm{I}}}(x)$.

Proof We firstly prove $\overline{F^{\mathrm{I}}}(x) = O(\overline{G}(x))$ along the line of Zachary [15]. By $\mathrm{E}\xi_1 = -a < 0$ and the strong law of large numbers (SLLN), we have that for any $\varepsilon > 0$ and $0 < \delta < 1 - p$, there exists a positive constant K such that for all $n \geq 1$,

$$\mathrm{P}(S_n > -K - n(a + \varepsilon)) \geq 1 - \delta. \tag{2.2}$$

Denote $M_n = \max_{0 \leq k \leq n} S_k$, $n \geq 0$. From (2.2), $F^{\mathrm{I}} \in \mathcal{OL}$ and Lemma 2.1, we obtain that for sufficiently large x,

$$\begin{aligned}
\mathrm{P}(S_{\tau_+} > x) &= \sum_{n=1}^{\infty} \mathrm{P}(S_n > x, M_{n-1} = 0) \\
&\geq \sum_{n=1}^{\infty} \overline{F}(x + K + (n-1)(a+\varepsilon))\mathrm{P}(M_{n-1} = 0, S_{n-1} > -K - (n-1)(a+\varepsilon)) \\
&\geq \sum_{n=1}^{\infty} \overline{F}(x + K + (n-1)(a+\varepsilon))(\mathrm{P}(S_{n-1} > -K - (n-1)(a+\varepsilon)) - \mathrm{P}(M_{n-1} > 0)) \\
&\geq (1 - \delta - \mathrm{P}(M > 0)) \sum_{n=1}^{\infty} \overline{F}(x + K + (n-1)(a+\varepsilon)) \\
&= (1 - \delta - p) \sum_{n=0}^{\infty} \overline{F}(x + K + n(a+\varepsilon)) \approx \overline{F^{\mathrm{I}}}(x).
\end{aligned} \tag{2.3}$$

Now we estimate the upper bound of $\mathrm{P}(S_{\tau_+} > x)$ by using the ideas in [2] or [13]. Denote $A_n = \{S_j \leq 0 : j = 0, 1, \cdots, n\}$. Consider the measures $H_0(B) = \mathbf{1}_{\{0 \in B\}}$ and $H_n(B) = \mathrm{P}(A_n, S_n \in B)$ with any set $B \subset (-\infty, 0]$, $n \geq 1$. Define the taboo renewal function $H(B) = \sum\limits_{n=0}^{\infty} H_n(B)$. By Blackwell's renewal theorem and $a > 0$, we have

$$H(-j + (0,1]) \to a^{-1}(1-p), \quad \text{as } j \to \infty.$$

Hence, there exists a positive constant C_1, such that for all $j \geq 0$,

$$H(-j + (0,1]) \leq C_1. \tag{2.4}$$

From Fubini Theorem, (2.4), $F^{\mathrm{I}} \in \mathcal{OL}$ and Lemma 2.1, we derive that

$$\begin{aligned}
\mathrm{P}(S_{\tau_+} > x) &= \sum_{n=0}^{\infty} \int_{-\infty}^{0} \overline{F}(x-y)\mathrm{P}(S_n \in \mathrm{d}y, A_n) \\
&= \sum_{j=0}^{\infty} \int_{-j-1}^{-j} \overline{F}(x-y)H\mathrm{d}y \\
&\leq \sum_{j=0}^{\infty} \overline{F}(x+j)H(-j + (0,1]) \\
&\leq C_1 \sum_{j=0}^{\infty} \overline{F}(x+j) \approx \overline{F^{\mathrm{I}}}(x).
\end{aligned} \tag{2.5}$$

Therefore, $\overline{G}(x) \approx \overline{F^{\mathrm{I}}}(x)$ follows from (2.3) and (2.5).

Now we continue to prove Theorem 1.2. Clearly, (2.1) implies $\overline{G}(x) = O(\overline{W}(x))$, which, combining with Lemma 2.2, yields that

$$\overline{F^{\mathrm{I}}}(x) \approx \overline{G}(x) = O(\overline{W}(x)).$$

This completes the proof of (a).

As for (b), by Lemma 2.2 of this paper and [10, Proposition 2.3(ii)], which is due to K-lüppelberg [7], we have that $F^{\mathrm{I}} \in \mathcal{OS}$ holds if and only if $G \in \mathcal{OS}$, that is, (i) and (ii) are equivalent.

Now we prove (c). If (iii) holds, then by (2.1) we have that $\overline{W}(x) \geq (1-p)p^2\overline{G^{*2}}(x)$. This and $\overline{W}(x) \approx \overline{G}(x)$ give

$$\limsup \frac{\overline{G^{*2}}(x)}{\overline{G}(x)} \leq \limsup \frac{\overline{W}(x)}{(1-p)p^2\overline{G}(x)} < \infty,$$

which means (ii) $G \in \mathcal{OS}$.

We mainly prove that (i) and (ii) yield (iii) under the additional condition $F^{\mathrm{I}} \in \mathcal{L}$ and $a^{-1}(l^*(F_+^{\mathrm{I}})-1) < 1$. Using similar arguments in [15], we aim to prove $\overline{W}(x) = O(\overline{F^{\mathrm{I}}}(x))$.

For any $0 < \varepsilon < a$ and fixed $R > 0$, define the renewal times $0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots$ for the process $\{S_n : n \geq 0\}$ by

$$\tau_1 = \min\{n \geq 1 : \ S_n > R - n(a-\varepsilon)\} \leq \infty,$$

and for $m \geq 2$, $\tau_m = \infty$ if $\tau_{m-1} = \infty$; otherwise, define

$$\tau_m = \tau_{m-1} + \min\{n \geq 1 : \ S_{\tau_{m-1}+n} - S_{\tau_{m-1}} > R - n(a-\varepsilon)\}.$$

By $\mathrm{E}\xi_1 = -a < 0$ and the SLLN, we have

$$\begin{aligned} r = \mathrm{P}(\tau_1 < \infty) &\to 0, \quad \text{as } R \to \infty, \\ S_n \xrightarrow{\text{a.s.}} -\infty &\equiv S_\infty, \qquad \text{as } n \to \infty. \end{aligned}$$

Since $\overline{F}(x)$ is non-increasing in x, we have that for sufficiently large x,

$$\begin{aligned} \mathrm{P}(S_{\tau_1} > x) &= \sum_{n=1}^{\infty} \mathrm{P}(S_n > x, \tau_1 = n) \\ &\leq \sum_{n=1}^{\infty} \mathrm{P}(S_n > x, S_{n-1} \leq R - (n-1)(a-\varepsilon)) \\ &\leq \sum_{n=1}^{\infty} \overline{F}(x - R + (n-1)(a-\varepsilon)) \\ &\leq (a-\varepsilon)^{-1}\overline{F^{\mathrm{I}}}(x - R - a + \varepsilon). \end{aligned} \tag{2.6}$$

Let $\{Y_i : i \geq 1\}$ be indepentlent and identically distributed random variables with common tail distribution $\mathrm{P}(Y_1 > x) = \mathrm{P}(S_{\tau_1} > x \mid \tau_1 < \infty)$. By (2.6) and $F^{\mathrm{I}} \in \mathcal{L}$, for the above $\varepsilon > 0$, there exists a positive constant $x_0 = x_0(R)$, such that for all $x \geq x_0$,

$$\overline{F^{\mathrm{I}}}(x - R - a + \varepsilon) \leq (1+\varepsilon)\overline{F^{\mathrm{I}}}(x).$$

Hence, if we choose

$$\overline{L}(x) = \begin{cases} (1+\varepsilon)r^{-1}(a-\varepsilon)^{-1}\overline{F^{\mathrm{I}}}(x), & x \geq x_0, \\ 1, & x < x_0, \end{cases} \tag{2.7}$$

then $L(x)$ is a distribution on $[x_0,\infty)$, which may depend on R, satisfying $\mathrm{P}(Y_1>x)\le \overline{L}(x)$ for any $x\in(-\infty,\infty)$, and

$$\overline{L}(x)\sim(1+\varepsilon)r^{-1}(a-\varepsilon)^{-1}\overline{F^{\mathrm{I}}}(x). \tag{2.8}$$

Denote the distribution of Y_1 by F_Y. Let $F_Y^{-1}(y)=\sup\{x: F_Y(x)\le y\}$ be the generalized inverse of F_Y, and define L^{-1} similarly. For each n, Y_n can be rewritten as $F_Y^{-1}(F_Y(Y_n))$ a.s., and $F_Y(Y_n)$ is a random variable with uniform distribution on the unit interval $[0,1]$ almost surely. Let

$$Z_n=L^{-1}(F_Y(Y_n)),\quad n\ge 1.$$

Then, the independent and identically distributed random variables Z_n have a common distribution L and $Y_n\le Z_n$ a.s., because of $\overline{F_Y}(x)\le\overline{L}(x)$ for all $x\in(-\infty,\infty)$. Furthermore, define two random variables

$$n_0=\min\{n: S_n=M\}\quad\text{and}\quad m_0=\max\{n:\tau_n\le n_0\}.$$

If $\tau_{m_0}<n_0$, then from the definition of m_0, it holds that $M-S_{\tau_{m_0}}=S_{n_0}-S_{\tau_{m_0}}\le R-(n_0-\tau_{m_0})(a-\varepsilon)<R$; if $\tau_{m_0}=n_0$, then $M-S_{\tau_{m_0}}=0<R$ is trivial. Thus,

$$M=S_{\tau_{m_0}}+(M-S_{\tau_{m_0}})<S_{\tau_{m_0}}+R.$$

Hence, we have that for all $x>R$,

$$\overline{W}(x)\le\mathrm{P}(S_{\tau_{m_0}}>x-R)\le\sum_{n=1}^{\infty}r^n\mathrm{P}(Y_1+\cdots+Y_n>x-R)\le\sum_{n=1}^{\infty}r^n\overline{L^{*n}}(x-R). \tag{2.9}$$

It follows from $L\in\mathcal{OS}$ and [10, Proposition 2.4] that there exist two positive constants $C_2=C_2(R)$ and $\lambda=\lambda(R)$, such that

$$\frac{\overline{L^{*n}}(x)}{\overline{L}(x)}\le C_2\lambda^n \tag{2.10}$$

holds uniformly for all $n\ge1$ and all $x\ge0$, where $\lambda=l^*(L)+\varepsilon-1$ and $l^*(L)=\limsup\frac{\overline{L^{*2}}(x)}{\overline{L}(x)}$. According to (2.7) and $F^{\mathrm{I}}\in\mathcal{L}$, we have that for any fixed R and sufficiently large x,

$$\begin{aligned}
l^*(L)&=1+\limsup\Big(\int_{x_0}^{x-x_0}+\int_{x-x_0}^{x}\Big)\frac{\overline{L}(x-t)}{\overline{L}(x)}\mathrm{d}L(t)\\
&=1+\limsup\Big((1+\varepsilon)r^{-1}(a-\varepsilon)^{-1}\int_{x_0}^{x-x_0}\frac{\overline{F^{\mathrm{I}}}(x-t)}{\overline{F^{\mathrm{I}}}(x)}\mathrm{d}F^{\mathrm{I}}(t)+\int_{x-x_0}^{x}\frac{1}{\overline{F^{\mathrm{I}}}(x)}\mathrm{d}F^{\mathrm{I}}(t)\Big)\\
&\le 1+(1+\varepsilon)r^{-1}(a-\varepsilon)^{-1}\limsup\int_0^x\frac{\overline{F^{\mathrm{I}}_+}(x-t)}{\overline{F^{\mathrm{I}}_+}(x)}\mathrm{d}F^{\mathrm{I}}_+(t)\\
&=1+(1+\varepsilon)r^{-1}(a-\varepsilon)^{-1}\limsup\Big(\frac{\overline{(F^{\mathrm{I}}_+)^{*2}}(x)}{\overline{F^{\mathrm{I}}_+}(x)}-1\Big)\\
&=1+(1+\varepsilon)r^{-1}(a-\varepsilon)^{-1}(l^*(F^{\mathrm{I}}_+)-1),
\end{aligned}$$

which implies

$$\lambda\le(1+\varepsilon)r^{-1}(a-\varepsilon)^{-1}(l^*(F^{\mathrm{I}}_+)-1)+\varepsilon,$$

not depending on R. By $a^{-1}(l^*(F_+^{\mathrm{I}})-1)<1$, we can choose ε small enough, such that

$$r\lambda \le (1+\varepsilon)(a-\varepsilon)^{-1}\big(l^*(F_+^{\mathrm{I}})-1\big)+\varepsilon<1.$$

Thus, fixing R, we obtain from (2.8)–(2.10) and $F^{\mathrm{I}}\in\mathcal{L}$ that for sufficiently large x,

$$\begin{aligned}\limsup\frac{\overline{W}(x)}{\overline{F^{\mathrm{I}}}(x)} &\le \sum_{n=1}^{\infty} r^n \limsup \frac{\overline{L^{*n}}(x-R)}{\overline{L}(x-R)}\cdot\frac{\overline{L}(x-R)}{\overline{F^{\mathrm{I}}}(x-R)}\cdot\frac{\overline{F^{\mathrm{I}}}(x-R)}{\overline{F^{\mathrm{I}}}(x)}\\ &\le C_2(1+\varepsilon)(a-\varepsilon)^{-1}\sum_{n=1}^{\infty} r^{n-1}\lambda^n<\infty. \end{aligned}\tag{2.11}$$

By (2.1), we get that $\overline{W}(x)\ge\overline{G}(x)$, which, together with Lemma 2.2, implies $\limsup\frac{\overline{F^{\mathrm{I}}}(x)}{\overline{W}(x)}<\infty$. This and (2.11) lead to $\overline{W}(x)\approx\overline{F^{\mathrm{I}}}(x)$. Again by Lemma 2.2, we prove (iii).

Finally, each of (i), (ii) and (iii) in Theorem 1.2 implies $W\in\mathcal{OS}$ according to [10, Proposition 2.3(ii)]. This completes the proof of Theorem 1.2.

Remark 2.1 When we deal with (2.1), we do not use [10, Proposition 3.1], which needed some stronger conditions, such as $\sup\Big\{x\ge 1:\sum\limits_{n=0}^{\infty}\lambda_n x^n<\infty\Big\}=\infty$ with $\lambda_n\ge 0,\ n\ge 0$ and $\sum\limits_{n=0}^{\infty}\lambda_n=1$. Indeed, the proof of (i) + (ii) ⇒ (iii) does not depend on λ_n, here, $\lambda_n=(1-p)p^n,\ n\ge 0$. This shows the advantage of the ideas of Zachary [15].

3 Proof of Theorem 1.4

We start this section by a simple lemma below.

Lemma 3.1 *Let X_1 and X_2 be two nonnegative independent random variables with distributions V_1 and V_2, respectively. Suppose that $\overline{V_i}(x)=O(\overline{V}(x)),\ i=1,2$, where V is also a distribution. If $V\in\mathcal{OS}$, then $\overline{V_1*V_2}(x)=O(\overline{V}(x))$.*

We still follow the line of the proof of Proposition 2.1 in [6] to prove Theorem 1.4. For simplicity, somewhere we directly utilize some results in [6]. Without loss of generality, assume that $\xi_n\ge-\beta$ a.s. for all n. By (1.1), there exist indepentlent and identically distributed random variables $\{\eta_n:n\ge 1\}$ with common distribution F, such that for all $n\ge 1$, $\xi_n\le\eta_n$ a.s. Choose a constant $y^*>0$ sufficiently large, such that

$$m\equiv \mathrm{E}\eta_1\mathbf{1}_{\{\eta_1>y^*\}}\le\frac{\alpha}{4}.\tag{3.1}$$

Denote $\overline{F}(y^*)=\varepsilon,\ K_0=m\varepsilon^{-1}+1$. Clearly, $y^*\le m\varepsilon^{-1}<K_0$. For each $i\ge 1$, define random variables $\delta_i=\mathbf{1}_{\{\eta_i>y^*\}}$, $\varphi_i=\xi_i(1-\delta_i)+K_0\delta_i$, $\psi_i=(\eta_i-K_0)\delta_i$. Then, (1.2), (3.1) and $\xi_n\ge-\beta$ a.s. imply

$$\mathrm{E}\varphi_i\le\mathrm{E}\xi_i+(\beta+K_0)\mathrm{E}\delta_i\le-\alpha+\varepsilon+m\le-\frac{\alpha}{2},\tag{3.2}$$

$$\mathrm{E}\psi_i=m-K_0\varepsilon=-\varepsilon<0,\tag{3.3}$$

and that $\{\delta_n:n\ge 1\}$ and $\{\psi_n:n\ge 1\}$ are both sequences of independent and identically distributed random variables. For each $n\ge 0$, denote $S_n^{\varphi}=\sum\limits_{i=1}^{n}\varphi_i,\ S_n^{\psi}=\sum\limits_{i=1}^{n}\psi_i,\ M^{\varphi}=$

$\sup_{n\ge0} S_n^\varphi$, $M^\psi = \sup_{n\ge0} S_n^\psi$. Since $\varphi_i + \psi_i = \xi_i + (\eta_i - \xi_i)\delta_i \ge \xi_i$ a.s., $i \ge 1$, we have

$$M \le \sup_{n\ge0}(S_n^\varphi + S_n^\psi) \le M^\varphi + M^\psi. \tag{3.4}$$

As stated in [6], M^ψ does not depend on random variables $\{(\varphi_i, \delta_i) : i \ge 1\}$ (it can also be directly verified), so M^φ is independent. We firstly estimate the tail distribution of M^ψ. It follows from (3.3), Theorem 1.2 and $F^{\mathrm{I}} \in \mathcal{OS} \subset \mathcal{OL}$ that

$$\mathrm{P}(M^\psi > x) \approx \int_x^\infty \mathrm{P}(\psi_1 > y)\mathrm{d}y \approx \overline{F^{\mathrm{I}}}(x). \tag{3.5}$$

Now we consider $\mathrm{P}(M^\varphi > x)$. As pointed out by Foss et al. [6] that there exists a positive constant s, depending only on F, α and β, such that $\{-\exp\{sS_n^\varphi\} : n \ge 1\}$ is a martingale. By the martingale maximal inequality, we have

$$\begin{aligned}\mathrm{P}\Big(\max_{0\le k\le n} S_k^\varphi > x\Big) &\le \mathrm{P}\Big(\min_{0\le k\le n}(-\mathrm{e}^{sS_k^\varphi}) \le -\mathrm{e}^{sx}\Big)\\ &\le \mathrm{e}^{-sx}\Big(1 + \int_\Omega \mathrm{e}^{sS_n^\varphi}\mathbf{1}_{\big\{\max\limits_{0\le k\le n}\mathrm{e}^{sS_k^\varphi}\le \mathrm{e}^{sx}\big\}}\mathrm{dP}\Big).\end{aligned} \tag{3.6}$$

Since $S_n^\varphi \to -\infty$ a.s., $n \to \infty$, by (3.2), we can use the dominated convergence theorem to obtain that

$$\lim_{n\to\infty}\int_\Omega \mathrm{e}^{sS_n^\varphi}\mathbf{1}_{\big\{\max\limits_{k\le n}\mathrm{e}^{sS_k^\varphi}\le \mathrm{e}^{sy}\big\}}\mathrm{dP} = 0,$$

which, together with (3.6), implies

$$\mathrm{P}(M^\varphi > x) = \lim_{n\to\infty}\mathrm{P}\Big(\max_{0\le k\le n} S_k^\varphi > x\Big) \le \mathrm{e}^{-sx}. \tag{3.7}$$

From $F^{\mathrm{I}} \in \mathcal{OS} \cap \mathcal{DK}^c$, (3.7) and (1.4), we derive that

$$\mathrm{P}(M^\varphi > x) = o(\overline{F^{\mathrm{I}}}(x)). \tag{3.8}$$

It follows from (3.4)–(3.5), (3.8) and Lemma 3.1 that

$$\overline{W}(x) \le \mathrm{P}(M^\psi + M^\varphi > x) = O(\overline{F^{\mathrm{I}}}(x)).$$

Therefore, there exists a positive constant r_0, depending only on F, α and β, such that (1.3) holds. This completes the proof of Theorem 1.4.

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