Chin. Ann. Math.
32B(6), 2011, 863–884
DOI: 10.1007/s11401-011-0679-7

Chinese Annals of Mathematics, Series B

Thompson's Group *F* and the Linear Group $\mathrm{GL}_\infty(\mathbb{Z})$*

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Abstract The authors study the finite decomposition complexity of metric spaces of H, equipped with different metrics, where H is a subgroup of the linear group $\mathrm{GL}_\infty(\mathbb{Z})$. It is proved that there is an injective Lipschitz map $\varphi : (F, d_S) \to (H, d)$, where F is the Thompson's group, d_S the word-metric of F with respect to the finite generating set S and d a metric of H. But it is not a proper map. Meanwhile, it is proved that $\varphi : (F, d_S) \to (H, d_1)$ is not a Lipschitz map, where d_1 is another metric of H.

Keywords Finite decomposition complexity, Thompson's group F, Word-metric, Lipschitz map, Reduced tree diagram
2000 MR Subject Classification 46L07, 46L80

1 Introduction

Inspired by the notion of finite asymptotic dimension of Gromov [1], a geometric concept of finite decomposition complexity is recently introduced by E. Guentner, R. Tessera and G. Yu. Roughly speaking, a metric space has finite decomposition complexity when there exists an algorithm to decompose the space into nice pieces in a certain asymptotic way. The class of groups with finite decomposition complexity includes all linear groups, subgroups of almost connected Lie groups, hyperbolic groups, and elementary amenable groups and is closed under various operations (see [2]).

Thompson's group F was discovered by R. Thompson in the 1960s, in connection with his work on associativity. It is a long-standing open problem to determine whether F is amenable. The study of finite decomposition complexity of F is partially inspired by the question of amenability of F. It is worth noticing that a bounded geometry metric space having finite decomposition complexity has Property A (see [2]), which is a weak form of amenability. It is not known whether F has Property A or not. So the question about the finite decomposition complexity of F is interesting. R. Willett [3] proved that amenable groups satisfy Property A. So the question arises naturally: Do amenable groups have finite decomposition complexity?

The paper is organized as follows. In Section 2, we recall some definitions and basic properties about finite decomposition complexity. In Section 3, we study finite decomposition complexity of metric spaces of H, equipped with different metrics. Finally, in Section 4, using the nice action of generators on the forest diagram, we prove that there is an injective Lipschitz

Manuscript received November 18, 2010. Revised May 11, 2011.
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*Project supported by the National Natural Science Foundation of China (No. 10731020) and the Shanghai Natural Science Foundation of China (No. 09ZR1402000).

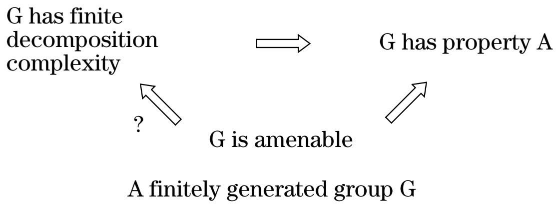

Figure 1 Relationship

map $\varphi : (F, d_S) \to (H, d)$, where H is a subgroup of linear group $\mathrm{GL}_\infty(\mathbb{Z})$, d is a metric for H and d_S is the word-metric of F with respect to the finite generating set S. However, it is not a proper map. Besides, we show that $\varphi : (F, d_S) \to (H, d_1)$ is not a Lipschitz map, where d_1 is another metric for H.

2 Preliminaries

Recall that a collection of subspaces $\{Z_i\}$ of a metric space Z is r-disjoint if for all $i \neq j$ we have $d(Z_i, Z_j) \geq r$. To express the idea that Z is the union of subspaces Z_i and that the collection of these subspaces is r-disjoint, we write

$$Z = \bigsqcup_{r\text{-disjoint}} Z_i.$$

A family of metric spaces $\{Z_i\}$ is bounded if there is a uniform bound on the diameter of the individual Z_i:

$$\sup \operatorname{diam}(Z_i) < \infty.$$

Definition 2.1 *Let X be a metric space. We say that the asymptotic dimension of X does not exceed n and write* $\operatorname{asdim} X \leq n$ *if for every $r > 0$, X can be written as a union of $n+1$ subspaces, each of which can be further decomposed as an r-disjoint union, i.e.,*

$$X = \bigcup_{i=0}^{n} X_i, \quad X_i = \bigsqcup_{r\text{-disjoint}} X_{ij}, \quad \sup_{i,j} \operatorname{diam} X_{i,j} < \infty.$$

If there is a natural number n such that $\operatorname{asdim} X \leq n$, *then we say that X has a finite asymptotic dimension* (*see* [4]).

Definition 2.2 *A metric family $\mathcal{X}$ is r-decomposable over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits a decomposition*

$$X = X_0 \cup X_1, \quad X_i = \bigsqcup_{r\text{-disjoint}} X_{ij},$$

where each $X_{ij} \in \mathcal{Y}$. It is denoted by $\mathcal{X} \xrightarrow{r} \mathcal{Y}$.

Definition 2.3 (1) *Let $\mathcal{D}_0$ be the collection of bounded families: $\mathcal{D}_0 = \{\mathcal{X} : \mathcal{X}$ is bounded$\}$.*

(2) *Let α be an ordinal greater than 0, and let $\mathcal{D}_\alpha$ be the collection of metric families decomposable over $\bigcup_{\beta<\alpha} \mathcal{D}_\beta$:*

$$\mathcal{D}_\alpha = \{\mathcal{X} : \forall\ r > 0,\ \exists\ \beta < \alpha,\ \exists\ \mathcal{Y} \in \mathcal{D}_\beta,\ \text{such that } \mathcal{X} \xrightarrow{r} \mathcal{Y}\}.$$

We have two immediate observations:

(i) For any $\beta < \alpha$, $\mathcal{D}_\beta \subseteq \mathcal{D}_\alpha$;

(ii) $\mathrm{asdim} X = 1$ if and only if $X \in \mathcal{D}_1$ exactly, i.e., $X \in \mathcal{D}_1$ and $X \overline{\in} \mathcal{D}_0$.

Moreover, by [2], we have known that X has a finite asymptotic dimension if and only if X belongs to $\mathcal{D}_n$ for some $n \in \mathbb{N}$.

Definition 2.4 *Let $\mathfrak{U}$ be a collection of metric families. A metric family $\mathcal{X}$ is decomposable over $\mathfrak{U}$ if for every $r > 0$, there exists a metric family $\mathcal{Y} \in \mathfrak{U}$ and an r-decomposition of $\mathcal{X}$ over $\mathcal{Y}$. The collection $\mathfrak{U}$ is stable under decomposition if every metric family which decomposes over $\mathfrak{U}$ actually belongs to $\mathfrak{U}$.*

Definition 2.5 *The collection $\mathcal{D}$ of metric families with finite decomposition complexity is the minimal collection of metric families containing bounded families and is stable under decomposition. We abbreviate membership in $\mathcal{D}$ by saying that a metric family in $\mathcal{D}$ has FDC.*

Proposition 2.1 (see [2, Theorem 2.3.2]) *A metric family $\mathcal{X}$ has finite decomposition complexity if and only if there exists a countable ordinal α such that $\mathcal{X} \in \mathcal{D}_\alpha$.*

Definition 2.6 *Let G be a countable discrete group. A length function $l : G \to \mathbb{R}_+$ on G is a function satisfying that for all $g, f \in G$,*

(1) *$l(g) = 0$ if and only if g is the identity element of G;*

(2) *$l(g^{-1}) = l(g)$;*

(3) *$l(gf) \le l(g) + l(f)$.*

If we replace condition (1) by

(1)′ $l(1_G) = 0$, where 1_G is the identity element of G,

then we say that l is a pseudo-length function for G.

A (pseudo-)length function l is called proper if for all $C > 0$, $l^{-1}([0, C]) \subset G$ is a finite set.

Definition 2.7 *Let G be a finitely generated discrete group and S be a generating set for G. The word-length function for G with respect to S of g is the length of the shortest word representing g in elements of the generating set S. The associated left-invariant word-metric is $d_{S,l}(g, h) = l_S(g^{-1}h)$ and the right-invariant word-metric is $d_{S,r}(g, h) = l_S(hg^{-1})$.*

Recall that a metric space has bounded geometry if for every $r > 0$, there exists an $N = N(r)$ such that every ball of radius r contains at most N points.

Definition 2.8 *If $f : X \to Y$ is a map of metric spaces, it is said to be*

(1) *Bornologous if for all $R > 0$, there exists an $S > 0$, such that $d(x_1, x_2) < R$ implies $d(f(x_1), f(x_2)) < S$.*

(2) *Effectively Proper if for all $R > 0$, there exists an $S > 0$, such that for all $x \in X$, $f^{-1}(B(f(x), R)) \subseteq B(x, S)$.*

A coarse embedding is an effectively proper, bornologous map. A coarse embedding f is a coarse equivalence if it admits a coarse embedding $g : Y \to X$ and there exists $K > 0$, such that

$$d(x, gf(x)) \le K \quad \text{and} \quad d(y, fg(y)) \le K$$

for all $x \in X$ and $y \in Y$. Two metric spaces X and Y are coarsely equivalent if there is a coarse equivalence $f : X \to Y$.

Lemma 2.1 (see [3, Proposition 2.3.3]) *Let G be a countable discrete group. Then there exists a left-invariant metric d_l on G, such that (G, d_l) is a bounded geometry space. Moreover, if d'_l is another metric on G with these properties, then the identity map $(G, d_l) \to (G, d'_l)$ is a coarse equivalence. Similarly, there exists a right-invariant metric d_r on G, such that (G, d_r) is a bounded geometry space. Moreover, if d'_r is another metric on G with these properties, then the identity map $(G, d_r) \to (G, d'_r)$ is a coarse equivalence.*

Lemma 2.2 (Coarse Invariant) (see [2]) *Finite decomposition complexity is invariant under a coarse equivalence, i.e., if X and Y are coarsely equivalent, then X has FDC if and only if Y has FDC.*

As a consequence, we say that a discrete group has finite decomposition complexity if it is a metric space having finite decomposition complexity equipped with a left-invariant metric induced by a proper length function.

Recall that a linear group is any subgroup of the invertible matrices over some field. Let $\mathrm{GL}_n(\mathbb{Z})$ be the general linear group of degree n over $\mathbb{Z}$, and $\mathrm{SL}_n(\mathbb{Z})$ be the special linear group of degree n over $\mathbb{Z}$.

Tessera et al. [2] proved that for every $n \in \mathbb{N}$, $\mathrm{GL}_n(\mathbb{Z})$ has finite decomposition complexity (FDC). In the similar way, we can obtain the following lemma.

Lemma 2.3 *For every $n \in \mathbb{N}$, let l be a proper length function for $\mathrm{GL}_n(\mathbb{Z})$ and d_r be the associated right-invariant metric. Then $(\mathrm{GL}_n(\mathbb{Z}), d_r)$ has finite decomposition complexity (FDC).*

Lemma 2.4 (see [2]) *The collection of countable groups having finite decomposition complexity is closed under the formation of subgroups, products, extensions, free amalgamated products, HNN extensions and direct unions.*

3 Linear Group $\mathrm{GL}_\infty(\mathbb{Z})$

Let $\mathrm{GL}_\infty(\mathbb{Z}) = \bigcup\limits_{n=1}^{\infty} \mathrm{GL}_n(\mathbb{Z})$ and

$$H = \left\{ h = \mathrm{diag}(h_{0,1}, h_{1,1}, h_{1,2}, \cdots, h_{k,1}, \cdots, h_{k,2^k}, \cdots) \;\middle|\; \begin{array}{l} h_{i,j} \in \mathrm{SL}_2(\mathbb{Z}), 1 \le j \le 2^i \text{ and only} \\ \text{finitely many } h_{i,j} \ne \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right\}.$$

Note that H is a linear group.

Proposition 3.1 *Let H be the group defined above. Then H has FDC.*

Proof Since $\mathrm{GL}_\infty(\mathbb{Z})$ is the direct union of $\{\mathrm{GL}_n(\mathbb{Z})\}_{n \ge 1}$ and by Lemma 2.4, $\mathrm{GL}_\infty(\mathbb{Z})$ has FDC. It is easy to see that H is a subgroup of $\mathrm{GL}_\infty(\mathbb{Z})$. Therefore, H has FDC.

Now we define a pseudo-length function $\widetilde{l}$ for $\mathrm{SL}_2(\mathbb{Z})$ as follows:

$$\forall A \in \mathrm{SL}_2(\mathbb{Z}), \quad \widetilde{l}(A) = \log \max \{\|A\|, \|A^{-1}\|\},$$

where $\|A\|$ is the norm of A. Note that $\widetilde{l}$ is a proper length function.

Note that if $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right)$, then

$$\max_{i,j} |a_{ij}| \leq \|A\| \leq \sum_{i,j} |a_{ij}|.$$

It follows that

$$\forall A \in \mathrm{SL}_2(\mathbb{Z}), \quad \text{either } \widetilde{l}(A) = 0 \text{ or } \widetilde{l}(A) > \frac{1}{2}.$$

Now define a pseudo-length function l_1 for H:

$$\forall h \in H, \quad l_1(h) = \sum_{k=0}^{\infty} (k+1)(\widetilde{l}(h_{k,1}) + \cdots + \widetilde{l}(h_{k,2^k})). \tag{3.1}$$

It is not hard to see that $\{h \in H \mid l_1(h) = 0\}$ is an infinite set. Thus l_1 is not proper.

Define another pseudo-length function l for H:

$$\forall h \in H, \quad l(h) = \sum_{k=0}^{\infty} (2^{-k})(\widetilde{l}(h_{k,1}) + \cdots + \widetilde{l}(h_{k,2^k})). \tag{3.2}$$

Let $d(g,h) = l(hg^{-1})$ be the right-invariant pseudo-metric induced by l.

Let 1_n denote the identity matrix of size n and 1_∞ denote the infinite identity matrix

$$1_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \cdots$$

Proposition 3.2 *The group H, equipped with the right-invariant pseudo-metric d_1 induced by l_1, has FDC.*

Proof For every $k \geq 0$, let

$$H_k = \{h \in H \mid \widetilde{l}(h_{i,j}) = 0, \ \forall i > k\}$$

and

$$G_k = \left\{ h \in H \,\middle|\, h_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \forall i > k \right\}.$$

It is easy to see that G_k is a subgroup of $\mathrm{GL}_m(\mathbb{Z})$, where $m = 2^{k+1} - 2$. Observe that l_1 is a proper pseudo-length function for G_k. By Lemma 2.3, (G_k, d_1) has FDC. Define a map

$$\begin{aligned} \psi : (H_k, d_1) &\to (G_k, d_1), \\ h &= \mathrm{diag}(h_{0,1}, h_{1,1}, h_{1,2}, \cdots, h_{k,1}, \cdots, h_{k,2^k}, \cdots) \\ &\to \psi(h) = \mathrm{diag}(h_{0,1}, h_{1,1}, h_{1,2}, \cdots, h_{k,1}, \cdots, h_{k,2^k}, 1_\infty). \end{aligned}$$

Clearly, ψ is an isometry. Since FDC is a coarse invariant and (G_k, d_1) has FDC, (H_k, d_1) has FDC. For every $r > 0$, there exists $k > 2r$, so that $H = \bigsqcup_{r\text{-disjoint}} H_k h$. Indeed, if $H_k g \neq H_k h$, then $hg^{-1} \overline{\in} H_k$. By the definition of H_k, there is an $i > k$, such that $\widetilde{l}(h_{i,j} g_{i,j}^{-1}) \neq 0$ for some $1 \leq j \leq 2^i$. Then we have $\widetilde{l}(h_{i,j} g_{i,j}^{-1}) > \frac{1}{2}$. Therefore, $d_1(g,h) > \frac{1}{2}(i+1) > \frac{1}{2}(k+1) > r$. Since (H_k, d_1) has FDC, it is readily verified that $\{H_k h\}_h$ has FDC. Therefore, (H, d_1) has FDC.

4 Thompson's Group *F*

The valence of a vertex of a graph is the number of edges incident to the vertex.

An ordered rooted binary tree is a tree S such that

(1) S has a root v_0;

(2) if S contains vertices other than v_0, then v_0 has valence 2;

(3) if v is a vertex in S with valence greater than 1, then there are exactly two edges $e_{v,L}, e_{v,R}$ which contain v and are not contained in the geodesic from v_0 to v.

The edge $e_{v,L}$ is called a left edge of v and $e_{v,R}$ is called a right edge of v.

For every $x, y \in \mathbb{Z}$, let $\gcd(x, y)$ be the greatest common divisor of x and y. Let a be a nonnegative integer and let b, c, d be positive integers, such that $a \leq b$, $c \leq d$, $[\frac{a}{b}, \frac{c}{d}] \subset [0, 1]$ and $\gcd(a, b) = 1 = \gcd(c, d)$, with $[\frac{a}{b}, \frac{c}{d}]$ being an integral subsimplex of $[0, 1]$ if $ad - bc = -1$. The left part of $[\frac{a}{b}, \frac{c}{d}]$ is $[\frac{a}{b}, \frac{a+c}{b+d}]$ and the right part of $[\frac{a}{b}, \frac{c}{d}]$ is $[\frac{a+c}{b+d}, \frac{c}{d}]$. The left and right parts of $[\frac{a}{b}, \frac{c}{d}]$ are integral subsimplices of $[0, 1]$. The tree of integral subsimplices of $[0, 1]$ is the tree T with vertices being the integral subsimplices of $[0, 1]$ and with edges the pairs (I, J), where I and J are integral subsimplices of $[0, 1]$ and I is either the left part of J or the right part of J. An edge (I, J) of T is a left edge if I is the left part of J and is a right edge if I is the right part of J.

We define a caret to be a vertex of the tree together with two downward-oriented edges, which we refer to as the left and right edges of the caret. Every caret has the form of the rooted tree in Figure 2. We call v_1 is the left child of v and v_2 is the right child of v.

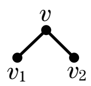

Figure 2 A caret

Label the vertex set $V(T)$ of T by the following inductive method: label the root vertex by $T_{0,1}$. Assume that a vertex v of T is labeled by $T_{i,j}$. Then label the left child v_1 of v by $T_{i+1,2j-1}$ and label the right child v_2 of v by $T_{i+1,2j}$. Throughout this paper, we view $T_{i,j}$ as both a vertex of a tree and an integral subsimplex of $[0, 1]$.

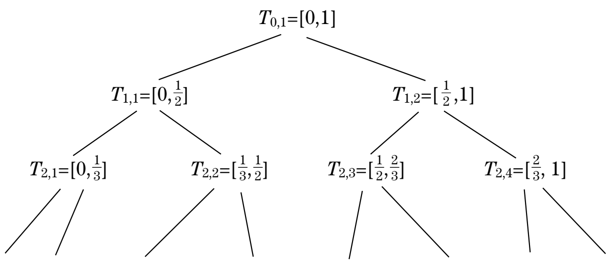

Figure 3 The tree T of integral subsimplices of [0,1]

We present a brief introduction to Thompson's group F and refer the interested readers to [5–7] for more detailed discussions. Thompson's group F has been studied for several decades.

F is the set of orientation-preserving piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable except at finitely many dyadic rational numbers (i.e., rational numbers of the form: $\frac{m}{2^n}, m, n \in \mathbb{Z}_+$) and such that on intervals of differentiability the derivatives are powers of 2.

Elements of F can be viewed as pairs of finite binary rooted trees, each with the same number of carets, called tree diagrams. A binary forest is a sequence $(T_0, T_1, \cdots)$ of finite binary trees. A binary forest is bounded if only finitely many of the trees are nontrivial. The forest diagram, which represents an element of F as a pair of bounded binary forests is another useful diagram representation for F.

A tree diagram (forest diagram) is reduced if it does not have any opposing pairs of carets.

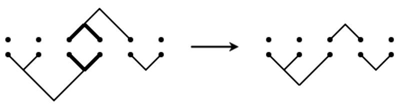

Figure 4 An example of an unreduced forest diagram and a reduced forest diagram representing the same element in F

An exposed caret in a forest is a caret whose children are both leaves (see Figure 5).

Figure 5 Exposed carets

Remark 4.1 (see [5]) We can translate between tree diagrams and forest diagrams in the following way: given a reduced tree diagram, we remove the right stalk of the tree to get the corresponding reduced forest diagram (see Figure 6).

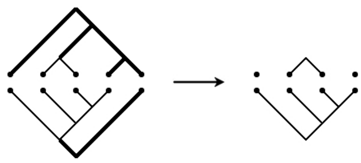

Figure 6 A reduced tree diagram being translated into a reduced forest diagram

Let $x_0, x_1, x_2, \cdots$ be the elements of F with reduced tree diagrams in Figure 7 and reduced forest diagrams in Figure 8. These elements generate the group F. Since $x_{n+1} = x_0^{-1} x_n x_0$ for $n \geq 1$, F is finitely generated by $\{x_0, x_1\}$.

Thompson's group F can also be described as the group with the following infinite presentation:

$$\langle x_0, x_1, \cdots, x_n, \cdots \mid x_n x_k = x_k x_{n+1},\ \forall k < n \rangle.$$

Lemma 4.1 (see [5]) *There is a canonical bijection between* F *and the set of reduced forest diagrams* (*or reduced tree diagrams*).

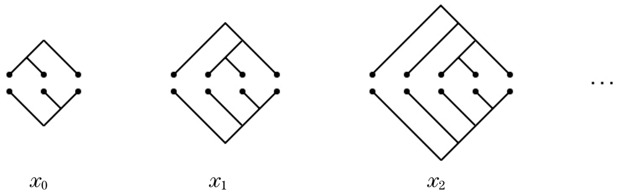

Figure 7 Reduced tree diagrams for an infinite generating set

Figure 8 Reduced forest diagrams for an infinite generating set

The action of the generators $\{x_0, x_1, \cdots x_n, \cdots\}$ on forest diagrams is particularly nice.

Lemma 4.2 (see [5, Proposition 2.3.1]) *Let $\mathfrak{f}$ be a forest diagram for some $f \in F$. Then a forest diagram for $x_n f$ can be obtained by attaching a caret to the roots of trees n and $(n+1)$ in the top forest of $\mathfrak{f}$. The forest diagram given for $x_n f$ may not be reduced, even if we started with a reduced forest diagram. In particular, the caret that was created could oppose a caret in the bottom forest. In this case, left-multiplication by x_n effectively "cancels" the bottom caret.*

By Lemma 4.2 and the translation between tree diagrams and forest diagrams, we immediately obtain the following lemma.

Lemma 4.3 *For any $f \in F$, let $\binom{R_f}{S_f}$ be the reduced tree diagrams for f and $\binom{R_{x_n f}}{S_{x_n f}}$ be the reduced tree diagrams for $x_n f$. Let $L(R_f)$ be the set of leaves of R_f. Then there are three cases:*

(1) *The number of leaves in R_f is the same as the number of leaves in $R_{x_n f}$, i.e., $|L(R_{x_n f})| = |L(R_f)|$;*

(2) $|L(R_{x_n f})| = |L(R_f)| + 1$;

(3) $|L(R_{x_n f})| = |L(R_f)| - 1$.

Let $M\big(\big[\frac{a}{b}, \frac{c}{d}\big], \big[\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\big]\big) = \big(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\big)\big(\begin{smallmatrix}a & c\\ b & d\end{smallmatrix}\big)^{-1}$. Then

$$M\Big(\Big[\frac{a}{b}, \frac{c}{d}\Big], \Big[\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\Big]\Big)^{-1} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}^{-1} = M\Big(\Big[\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\Big], \Big[\frac{a}{b}, \frac{c}{d}\Big]\Big).$$

It is easy to see that

$$\begin{aligned} M\Big(\Big[\frac{a}{b}, \frac{c}{d}\Big], \Big[\frac{\alpha_1}{\beta_1}, \frac{\gamma_1}{\delta_1}\Big]\Big) M\Big(\Big[\frac{\alpha_2}{\beta_2}, \frac{\gamma_2}{\delta_2}\Big], \Big[\frac{a}{b}, \frac{c}{d}\Big]\Big) &= \begin{pmatrix}\alpha_1 & \gamma_1\\ \beta_1 & \delta_1\end{pmatrix}\begin{pmatrix}a & c\\ b & d\end{pmatrix}^{-1}\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\alpha_2 & \gamma_2\\ \beta_2 & \delta_2\end{pmatrix}^{-1} \\ &= \begin{pmatrix}\alpha_1 & \gamma_1\\ \beta_1 & \delta_1\end{pmatrix}\begin{pmatrix}\alpha_2 & \gamma_2\\ \beta_2 & \delta_2\end{pmatrix}^{-1} \\ &= M\Big(\Big[\frac{\alpha_2}{\beta_2}, \frac{\gamma_2}{\delta_2}\Big], \Big[\frac{\alpha_1}{\beta_1}, \frac{\gamma_1}{\delta_1}\Big]\Big). \end{aligned}$$

Remark 4.2 Note that $\widetilde{l}\,(M(T_{i,j},T_{k,l}))\le 2n+4$, where $n=\max\{i,k\}$.

Indeed, let $T_{i,j}=\left[\frac{a}{b},\frac{c}{d}\right]$ and $T_{k,l}=\left[\frac{\alpha}{\beta},\frac{\gamma}{\delta}\right]$. Then it is not hard to see that $\max\{|a|,|b|,|c|,|d|\}$ $\le 2^i\le 2^n$ and $\max\{|\alpha|,|\beta|,|\gamma|,|\delta|\}\le 2^k\le 2^n$. It follows that

$$\|M(T_{i,j},T_{k,l})\|=\left\|\begin{pmatrix}\alpha&\gamma\\ \beta&\delta\end{pmatrix}\begin{pmatrix}a&c\\ b&d\end{pmatrix}^{-1}\right\|\le 2^{2n+4}$$

and

$$\|M(T_{i,j},T_{k,l})^{-1}\|=\|M(T_{k,l},T_{i,j})\|\le 2^{2n+4}.$$

Therefore, $\widetilde{l}(M(T_{i,j},T_{k,l}))\le 2n+4$.

Define a map $\varphi:F\to H$. For every $f\in F$, let $\binom{R_f}{S_f}$ be the reduced tree diagram for f. If $T_{i,j}=\left[\frac{a}{b},\frac{c}{d}\right]$ is a leaf of R_f and $T_{k,l}=\left[\frac{\alpha}{\beta},\frac{\gamma}{\delta}\right]$ is the corresponding leaf in S_f, which is denoted by $f(T_{i,j})$. Then

$$M(T_{i,j},f(T_{i,j}))=M\Big(\Big[\frac{a}{b},\frac{c}{d}\Big],\Big[\frac{\alpha}{\beta},\frac{\gamma}{\delta}\Big]\Big)=\begin{pmatrix}\alpha&\gamma\\ \beta&\delta\end{pmatrix}\begin{pmatrix}a&c\\ b&d\end{pmatrix}^{-1}.$$

Since $T_{i,j}$ and $f(T_{i,j})$ are integral subsimplices of $[0,1]$, $M\left(T_{i,j},f(T_{i,j})\right)\in\mathrm{SL}_2(\mathbb{Z})$. Define

$$\varphi(f)_{i,j}=\begin{cases}M\left(T_{i,j},f(T_{i,j})\right), & T_{i,j}\text{ is a leaf of }R_f,\\ 1_2, & \text{otherwise.}\end{cases}$$

Let $\varphi(f)=\mathrm{diag}(\varphi(f)_{0,1},\varphi(f)_{1,1},\varphi(f)_{1,2},\cdots,\varphi(f)_{k,1},\cdots,\varphi(f)_{k,2^k},\cdots)$. It is easy to see that $\varphi(f)\in H$.

Example 4.1 Figure 9 is the reduced tree diagram for x_0. Then we obtain

$$\varphi(x_0)=\mathrm{diag}(1_2,1_2,\varphi(x_0)_{1,2},\varphi(x_0)_{2,1},\varphi(x_0)_{2,2},1_\infty),$$

where $\varphi(x_0)_{1,2}=M\left(T_{1,2},T_{2,4}\right)=M\left(\left[\frac{1}{2},\frac{1}{1}\right],\left[\frac{2}{3},\frac{1}{1}\right]\right)=\begin{pmatrix}2&1\\ 3&1\end{pmatrix}\begin{pmatrix}1&1\\ 2&1\end{pmatrix}^{-1}$, $\varphi(x_0)_{2,1}=M\left(T_{2,1},T_{1,1}\right)=M\left(\left[\frac{0}{1},\frac{1}{3}\right],\left[\frac{0}{1},\frac{1}{2}\right]\right)=\begin{pmatrix}0&1\\ 1&2\end{pmatrix}\begin{pmatrix}0&1\\ 1&3\end{pmatrix}^{-1}$ and $\varphi(x_0)_{2,2}=M\left(T_{2,2},T_{2,3}\right)=M\left(\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{1}{2},\frac{2}{3}\right]\right)=\begin{pmatrix}1&2\\ 2&3\end{pmatrix}\begin{pmatrix}1&1\\ 3&2\end{pmatrix}^{-1}$.

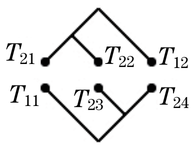

Figure 9 The reduced tree diagram for x_0

Proposition 4.1 *Let $\varphi:F\to H$ be the map defined above. Then φ is injective.*

Proof For any $f,g\in F$ such that $f\ne g$, we are going to prove that $\varphi(f)\ne\varphi(g)$. Let $\binom{R_f}{S_f}$ and $\binom{R_g}{S_g}$ be the reduced tree diagrams for f and g respectively. By Lemma 4.1, $\binom{R_f}{S_f}\ne\binom{R_g}{S_g}$.

Case 1 If $R_f \neq R_g$, then there is an exposed caret c, such that

(1) c is in exactly one of R_f and R_g, i.e., if c is in R_f, then c is not in R_g, and if c is not in R_f, then c is in R_g.

(2) if c is not in R_f, then the root vertex $T_{i,j}$ of c is either a leaf or not a vertex of R_f.

(3) if c is not in R_g, then the root vertex $T_{i,j}$ of c is either a leaf or not a vertex of R_g.

Assume that c is an exposed caret in R_g. Then $T_{i,j}$ is either a leaf or not a vertex of R_f. Therefore, $T_{i+1,2j-1}$ and $T_{i+1,2j}$ are not leaves of R_f. It follows that $\varphi(f)_{i+1,2j-1} = 1_2 = \varphi(f)_{i+1,2j}$. Since c is an exposed caret in R_g, $T_{i+1,2j-1}$ and $T_{i+1,2j}$ are leaves of R_g. It follows that either $\varphi(g)_{i+1,2j-1} \neq 1_2$ or $\varphi(g)_{i+1,2j} \neq 1_2$. Indeed, if $\varphi(g)_{i+1,2j-1} = 1_2 = \varphi(g)_{i+1,2j}$, then $T_{i+1,2j-1}$ and $T_{i+1,2j}$ in R_g correspond to $T_{i+1,2j-1}$ and $T_{i+1,2j}$ in S_g. Thus we obtain an opposing caret in $\binom{R_g}{S_g}$, which gives a contradiction. Therefore, $\varphi(f) \neq \varphi(g)$.

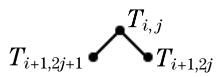

Figure 10 The caret c

Case 2 If $R_f = R_g$, then $S_f \neq S_g$. Let $L(R_f)$ and $L(R_g)$ be the sets of leaves in R_f and R_g respectively. There exists $T_{k,l} \in L(R_f) = L(R_g)$ corresponding to different leaves in S_f and S_g, i.e., $f(T_{k,l}) \neq g(T_{k,l})$. Thus

$$\varphi(f)_{k,l} = M\left(T_{k,l}, f(T_{k,l})\right) \neq M\left(T_{k,l}, g(T_{k,l})\right) = \varphi(g)_{k,l}.$$

It follows that $\varphi(f) \neq \varphi(g)$.

Let $V(T)$ be the vertex set of T, and define a weight function $w : V(T) \to \mathbb{R}$ by $w(T_{i,j}) = 2^{-i}$.

Lemma 4.4 *Let R be a subtree of T with the root vertex $T_{i,j}$, and $T_{i_1,j_1}, T_{i_2,j_2}, \cdots, T_{i_n,j_n}$ be the leaves of R. Then $\sum\limits_{k=1}^{n} w(T_{i_k,j_k}) = w(T_{i,j})$.*

Proof We are going to prove it by induction on n. If $n = 1$, then R is a trivial tree and $T_{i_1,j_1} = T_{i,j}$, and thus the result is true for $n = 1$. Suppose that the result is true for $n \leq m$. Now assume that $n = m + 1$. There is an exposed caret c_1 as in Figure 11. By the definition of the weight function, $w(T_{i_k,j_k}) = w(T_{i_{k+1},j_{k+1}}) = \frac{1}{2}w(v)$.

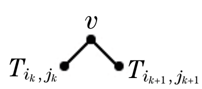

Figure 11 The caret c_1

Deleting caret c_1 from R, we obtain a subtree R' of T with the root vertex $T_{i,j}$. It has m leaves and its leaves are $T_{i_1,j_1}, \cdots, T_{i_{k-1},j_{k-1}}, v, T_{i_{k+2},j_{k+2}}, \cdots, T_{i_n,j_n}$. By assumption, $w(T_{i_1,j_1}) + \cdots + w(T_{i_{k-1},j_{k-1}}) + w(v) + w(T_{i_{k+2},j_{k+2}}) + \cdots + w(T_{i_n,j_n}) = w(T_{i,j})$. Since $w(v) = w(T_{i_k,j_k}) + w(T_{i_{k+1},j_{k+1}})$,

$$\sum_{k=1}^{n} w(T_{i_k,j_k}) = w(T_{i,j}).$$

Lemma 4.5 *Assume that R is a subtree of T, and let R_1 be the subtree R with the root vertex $T_{i,j}$, and R_2 be the subtree R with the root vertex $T_{k,l}$, that is, R_1 and R_2 have the same tree structure with different root vertices. Let $T_{i_1,j_1}, T_{i_2,j_2}, \cdots, T_{i_n,j_n}$ be the leaves of R_1 in order, and $T_{k_1,l_1}, T_{k_2,l_2}, \cdots, T_{k_n,l_n}$ be the leaves of R_2 in order. Then*

$$\forall\, 1 \le m \le n, \quad M(T_{i_m,j_m}, T_{k_m,l_m}) = M(T_{i,j}, T_{k,l}).$$

Proof We will prove it by induction on n. Clearly, the result is true for $n = 1$. If $n = 2$, then we obtain the picture of R_1 and R_2 as Figure 12.

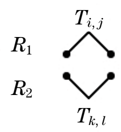

Figure 12 The tree of R_1 and R_2

Assume that $T_{i,j} = \left[\frac{a}{b}, \frac{c}{d}\right]$ and $T_{k,l} = \left[\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right]$. Then

$$T_{i_1,j_1} = \left[\frac{a}{b}, \frac{a+c}{b+d}\right], \quad T_{i_2,j_2} = \left[\frac{a+c}{b+d}, \frac{c}{d}\right],$$
$$T_{k_1,l_1} = \left[\frac{\alpha}{\beta}, \frac{\alpha+\gamma}{\beta+\delta}\right], \quad T_{k_2,l_2} = \left[\frac{\alpha+\gamma}{\beta+\delta}, \frac{\gamma}{\delta}\right].$$

We immediately have

$$\forall 1 \le m \le 2, \quad M(T_{i_m,j_m}, T_{k_m,l_m}) = M(T_{i,j}, T_{k,l}).$$

Suppose that the result is true for $n \le m$. Now assume that $n = m + 1$. There is an exposed caret c_2 with the root vertex v_1 of R_1. Let T_{i_t,j_t} and $T_{i_{t+1},j_{t+1}}$ be the leaves of caret c_2 of R_1. Then T_{k_t,l_t} and $T_{k_{t+1},l_{t+1}}$ are the leaves of caret c_2 with the root vertex v_2 of R_2. Note that

$$M(T_{i_t,j_t}, T_{k_t,l_t}) = M(T_{i_{t+1},j_{t+1}}, T_{k_{t+1},l_{t+1}}) = M(v_1, v_2).$$

Delete caret c_2 from R, we have a subtree R'. Let R'_1 be the subtree R' with the root vertex $T_{i,j}$ and R'_2 be the subtree R' with the root vertex $T_{k,l}$. Then $T_{i_1,j_1}, \cdots, T_{i_{t-1},j_{t-1}}, v_1, T_{i_{t+1},j_{t+1}}, \cdots, T_{i_n,j_n}$ are the leaves of R'_1, and $T_{k_1,l_1}, \cdots, T_{k_{t-1},l_{t-1}}, v_2, T_{k_{t+1},l_{t+1}}, \cdots, T_{i_n,j_n}$ are the leaves of R'_2. By assumption, we have

$$M(v_1, v_2) = M(T_{i,j}, T_{k,l})$$

and

$$\forall 1 \le m \le n, m \ne t \text{ and } m \ne t+1, \quad M(T_{i_m,j_m}, T_{k_m,l_m}) = M(T_{i,j}, T_{k,l}).$$

Therefore,

$$\forall 1 \le m \le n, \quad M(T_{i_m,j_m}, T_{k_m,l_m}) = M(T_{i,j}, T_{k,l}).$$

A map $f : X \to Y$ of metric spaces is called a Lipschitz map if there exists a constant $\lambda > 0$, such that

$$d(f(x), f(y)) \le \lambda d(x, y), \quad \forall x, y \in X.$$

Proposition 4.2 *Let $S = \{x_0, x_1\}$ be the finite generating set for Thompson group F, d_S be the left-invariant word-metric with respect to S, and d be the right-invariant pseudo-metric for H induced by l which is defined in* (2.2). *Then $\varphi : (F, d_S) \to (H, d)$ is a Lipschitz map.*

Proof For every $f, g \in F$, let $\begin{pmatrix} R_f \\ S_f \end{pmatrix}$ and $\begin{pmatrix} R_g \\ S_g \end{pmatrix}$ be the reduced tree diagrams for f and g respectively. Then $\begin{pmatrix} S_f \\ R_f \end{pmatrix}$ and $\begin{pmatrix} S_g \\ R_g \end{pmatrix}$ are the reduced tree diagrams for f^{-1} and g^{-1} respectively. Let

$$d(\varphi(f), \varphi(g)) = l(\varphi(g)\varphi(f)^{-1}),$$

where

$$\varphi(g)\varphi(f)^{-1} = \mathrm{diag}(\varphi(g)_{0,1}\varphi(f)_{0,1}^{-1}, \varphi(g)_{1,1}\varphi(f)_{1,1}^{-1}, \cdots, \varphi(g)_{k,1}\varphi(f)_{k,1}^{-1}, \cdots, \varphi(g)_{k,2^k}\varphi(f)_{k,2^k}^{-1}, \cdots).$$

If $T_{i,j}$ is a leaf in both R_f and R_g, then

$$\begin{aligned} \varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} &= M\left(T_{i,j}, g(T_{i,j})\right) M\left(T_{i,j}, f(T_{i,j})\right)^{-1} \\ &= M\left(T_{i,j}, g(T_{i,j})\right) M\left(f(T_{i,j}), T_{i,j}\right) \\ &= M\left(f(T_{i,j}), g(T_{i,j})\right). \end{aligned}$$

Therefore,

$$\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = \begin{cases} M\left(f(T_{i,j}), g(T_{i,j})\right), & T_{i,j} \in L(R_f) \text{ and } T_{i,j} \in L(R_g), \\ M\left(T_{i,j}, g(T_{i,j})\right), & T_{i,j} \overline{\in} L(R_f) \text{ and } T_{i,j} \in L(R_g), \\ M\left(f(T_{i,j}), T_{i,j}\right), & T_{i,j} \overline{\in} L(R_g) \text{ and } T_{i,j} \in L(R_f), \\ 1_2, & \text{otherwise.} \end{cases}$$

First we will show that if $d_S(f, g) = 1$, then $d(\varphi(f), \varphi(g)) \leq 13$.

Since $d_S(f, g) = 1$, $l_S(g^{-1}f) = l_S(f^{-1}g) = 1$. It follows that $g^{-1}f \in \{x_0, x_0^{-1}, x_1, x_1^{-1}\}$. Let $S_1, S_2, \cdots, S_n$ be ordered rooted binary subtrees of T.

(1) Suppose that $g^{-1}f = x_0$. Then $g^{-1} = x_0 f^{-1}$.

Case 1 The number of leaves in R_f is equal to the number of leaves in R_g, i.e., $|L(R_f)| = |L(R_g)|$. f has the form of reduced tree diagram of Figure 13. By Lemma 4.2 and the translation between tree diagrams and forest diagrams, we obtain the reduced tree diagram for g as Figure 14.

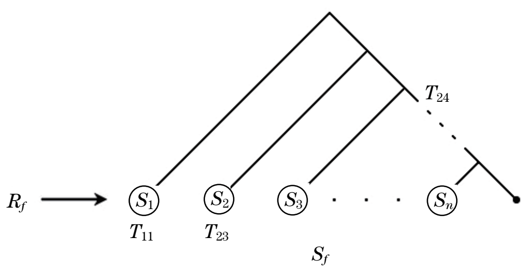

Figure 13 The reduced tree diagram for f

If $T_{i,j} \in L(R_f) = L(R_g)$, $\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = M\left(f(T_{i,j}), g(T_{i,j})\right)$. By Lemma 4.5, we have

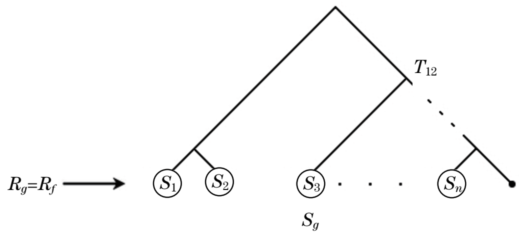

Figure 14 The reduced tree diagram for g

(i) If $f(T_{i,j})$ is a leaf of S_1, i.e., $f(T_{i,j}) \in L(S_1)$, then

$$\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = M\left(T_{1,1}, T_{2,1}\right) = M\left(\left[\frac{0}{1}, \frac{1}{2}\right], \left[\frac{0}{1}, \frac{1}{3}\right]\right) = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It follows that $\widetilde{l}(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}) \leq 2$.

(ii) If $f(T_{i,j}) \in L(S_2)$, then

$$\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = M\left(T_{2,3}, T_{2,2}\right) = M\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right) = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ -5 & 4 \end{pmatrix}.$$

It follows that $\widetilde{l}(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}) \leq 4$.

(iii) Otherwise,

$$\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = M\left(T_{2,4}, T_{1,2}\right) = M\left(\left[\frac{2}{3}, \frac{1}{1}\right], \left[\frac{1}{2}, \frac{1}{1}\right]\right) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}\right) \leq 2$.

If $T_{i,j} \overline{\in} L(R_f) = L(R_g)$, $\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = 1_2$.

By Lemma 4.4, we have

$$d(\varphi(f), \varphi(g)) = l(\varphi(g)\varphi(f)^{-1}) = \sum w(T_{i,j})\widetilde{l}(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}) \leq 2 + 4 + 2 = 8.$$

Case 2 $\left|L(R_f)\right| < \left|L(R_g)\right|$. f has the form of the reduced tree diagram of Figure 15. Then we obtain the reduced tree diagram for g as Figure 16.

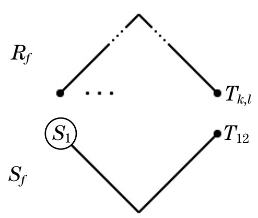

Figure 15 The reduced tree diagram for f

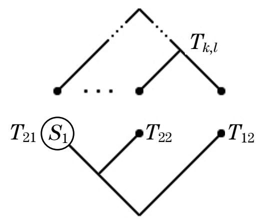

Figure 16 The reduced tree diagram for g

(i) If $T_{i,j} \in L(R_f) \cap L(R_g)$, then $f(T_{i,j}) \in L(S_1)$.

$$\begin{aligned}\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} &= M\left(f(T_{i,j}), g(T_{i,j})\right) = M\left(T_{1,1}, T_{2,1}\right)\\ &= M\left(\left[\frac{0}{1}, \frac{1}{2}\right], \left[\frac{0}{1}, \frac{1}{3}\right]\right) = \begin{pmatrix}0 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.\end{aligned}$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}\right) \le 2$.

(ii) $T_{k,l} \overline{\in} L(R_g)$ and $T_{k,l} \in L(R_f)$. So $\varphi(g)_{k,l}\varphi(f)_{k,l}^{-1} = M\left(f(T_{k,l}), T_{k,l}\right) = M\left(T_{1,2}, T_{k,l}\right)$. By Remark 4.2,

$$\widetilde{l}(\varphi(g)_{k,l}\varphi(f)_{k,l}^{-1}) \le 2k + 4.$$

(iii) $T_{k+1,2l-1}, T_{k+1,2l} \overline{\in} L(R_f)$ and $T_{i,j} \in L(R_g)$. Then we have

$$\varphi(g)_{k+1,2l-1}\varphi(f)_{k+1,2l-1}^{-1} = M\left(T_{k+1,2l-1}, g(T_{k+1,2l-1})\right) = M\left(T_{k+1,2l-1}, T_{2,2}\right)$$

and

$$\varphi(g)_{k+1,2l}\varphi(f)_{k+1,2l}^{-1} = M\left(T_{k+1,2l}, g(T_{k+1,2l})\right) = M\left(T_{k+1,2l}, T_{1,2}\right).$$

It follows that

$$\widetilde{l}(\varphi(g)_{k+1,2l-1}\varphi(f)_{k+1,2l-1}^{-1}) \le 2(k+1) + 4 \quad \text{and} \quad \widetilde{l}(\varphi(g)_{k+1,2l}\varphi(f)_{k+1,2l}^{-1}) \le 2(k+1) + 4.$$

Therefore,

$$\begin{aligned}d(\varphi(f), \varphi(g)) &= \sum w(T_{i,j})\widetilde{l}(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1})\\ &\le 2 + 2^{-k}(2k+4) + 2^{-(k+1)}(2(k+1)+4) + 2^{-(k+1)}(2(k+1)+4)\\ &\le 2 + 3 + 3 + 3 = 11.\end{aligned}$$

Case 3 $\left|L(R_f)\right| > \left|L(R_g)\right|$. f has the form of the reduced tree diagram of Figure 17. Then we obtain the reduced tree diagram for g as Figure 18.

(i) If $T_{i,j} \in L(R_f) \cap L(R_g)$, then

$$\begin{aligned}\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} &= M\left(f(T_{i,j}), g(T_{i,j})\right) = M\left(T_{2,4}, T_{1,2}\right)\\ &= M\left(\left[\frac{2}{3}, \frac{1}{1}\right], \left[\frac{1}{2}, \frac{1}{1}\right]\right) = \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}^{-1} = \begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}.\end{aligned}$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}\right) \le 2$.

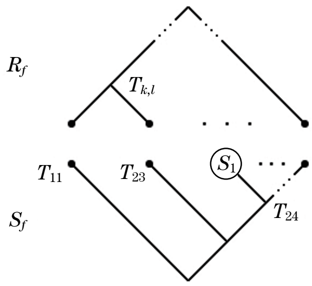

Figure 17 The reduced tree diagram for f

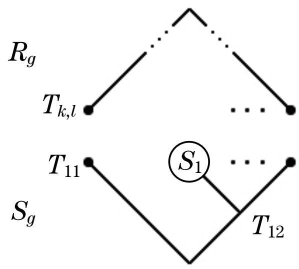

Figure 18 The reduced tree diagram for g

(ii) If $T_{k,l} \overline{\in} L(R_f)$ and $T_{k,l} \in L(R_g)$, then

$$\varphi(g)_{k,l}\varphi(f)_{k,l}^{-1} = M\left(T_{k,l}, g(T_{k,l})\right) = M\left(T_{k,l}, T_{1,1}\right).$$

It follows that

$$\widetilde{l}(\varphi(g)_{k,l}\varphi(f)_{k,l}^{-1}) \le 2k + 4.$$

(iii) $T_{k+1,2l-1}, T_{k+1,2l} \overline{\in} L(R_g)$ and $T_{k+1,2l-1}, T_{k+1,2l} \in L(R_f)$. Then we have

$$\varphi(g)_{k+1,2l-1}\varphi(f)_{k+1,2l-1}^{-1} = M\left(f(T_{k+1,2l-1}), T_{k+1,2l-1}\right) = M\left(T_{1,1}, T_{k+1,2l-1}\right)$$

and

$$\varphi(g)_{k+1,2l}\varphi(f)_{k+1,2l}^{-1} = M\left(f(T_{k+1,2l}), T_{k+1,2l}\right) = M\left(T_{2,3}, T_{k+1,2l}\right).$$

It follows that

$$\widetilde{l}(\varphi(g)_{k+1,2l-1}\varphi(f)_{k+1,2l-1}^{-1}) \le 2(k+1) + 4 \quad \text{and} \quad \widetilde{l}(\varphi(g)_{k+1,2l}\varphi(f)_{k+1,2l}^{-1}) \le 2(k+1) + 4.$$

Therefore,

$$\begin{aligned} d(\varphi(f), \varphi(g)) &= \sum w(T_{i,j})\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}\right) \\ &\le 2 + 2^{-k}(2k+4) + 2^{-(k+1)}(2(k+1)+4) + 2^{-(k+1)}(2(k+1)+4) \\ &\le 2 + 3 + 3 + 3 = 11. \end{aligned}$$

(2) Suppose that $g^{-1}f = x_0^{-1}$. Then $f^{-1} = x_0 g^{-1}$. By the result of (1), $d(\varphi(g), \varphi(f)) \le 11$.

(3) Suppose that $g^{-1}f = x_1$. Then $g^{-1} = x_1 f^{-1}$.

Case 1 The number of leaves in R_f is equal to the number of leaves in R_g, i.e., $\left|L(R_f)\right| = \left|L(R_g)\right|$. f has the form of the reduced tree diagram of Figure 19. Then we obtain the reduced tree diagram for g in Figure 20.

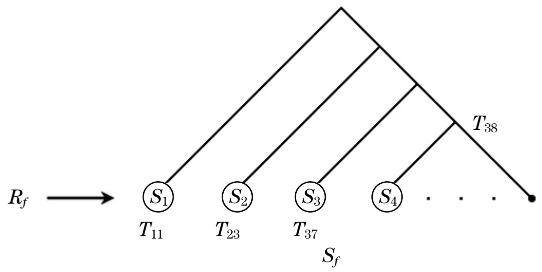

Figure 19 The reduced tree diagram for f

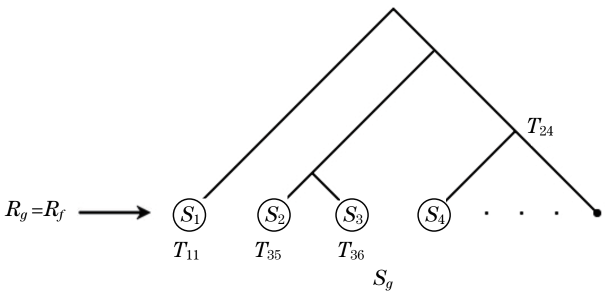

Figure 20 The reduced tree diagram for g

If $T_{i,j} \in L(R_f) = L(R_g)$, $\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j} = M\left(f(T_{i,j}), g(T_{i,j})\right)$.

(i) If $f(T_{i,j}) \in L(S_1)$, then

$$\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j} = M\left(T_{1,1}, T_{1,1}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j}\right) = 0$.

(ii) If $f(T_{i,j}) \in L(S_2)$, then

$$\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j} = M\left(T_{2,3}, T_{3,5}\right) = M\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{1}{2}, \frac{3}{5}\right]\right) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}.$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j}\right) \leq 4$.

(iii) If $f(T_{i,j}) \in L(S_3)$, then

$$\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j} = M\left(T_{3,7}, T_{3,6}\right) = M\left(\left[\frac{2}{3}, \frac{3}{4}\right], \left[\frac{3}{5}, \frac{2}{3}\right]\right) = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & 5 \\ -11 & 9 \end{pmatrix}.$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j}\right) \leq 5$.

(iv) Otherwise,

$$\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j} = M\left(T_{3,8}, T_{2,4}\right) = M\left(\left[\frac{3}{4}, \frac{1}{1}\right], \left[\frac{2}{3}, \frac{1}{1}\right]\right) = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}\right) \le 2$.

If $T_{i,j}\overline{\in}L(R_f) = L(R_g)$, $\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = 1_2$.

Therefore,

$$d(\varphi(f), \varphi(g)) = \sum w(T_{i,j})\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}\right) \le 4 + 5 + 2 = 11.$$

Case 2 $\left|L(R_f)\right| < \left|L(R_g)\right|$. f has the form of the reduced tree diagram of Figure 21. Then we obtain the reduced tree diagram for g as Figure 22.

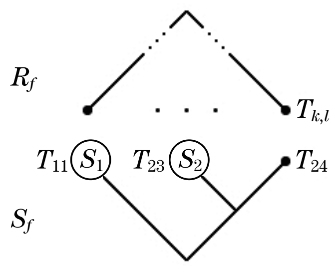

Figure 21 The reduced tree diagram for f

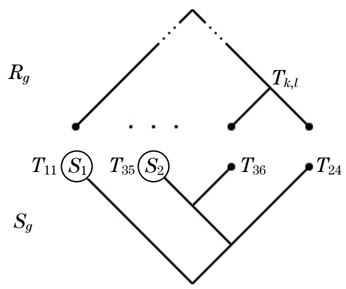

Figure 22 The reduced tree diagram for g

(i) If $T_{i,j} \in L(R_f) \cap L(R_g)$, $\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = M\left(f(T_{i,j}), g(T_{i,j})\right)$.

(a) If $f(T_{i,j}) \in L(S_1)$, then

$$\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = M\left(T_{1,1}, T_{1,1}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}\right) = 0$.

(b) If $f(T_{i,j}) \in L(S_2)$, then

$$\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1} = M\left(T_{2,3}, T_{3,5}\right) = M\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{1}{2}, \frac{3}{5}\right]\right) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}.$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1}\right) \le 4$.

(ii) $T_{k,l}\overline{\in}L(R_g)$ and $T_{k,l} \in L(R_f)$. So $\varphi(g)_{k,l}\varphi(f)_{k,l}^{-1} = M\left(f(T_{k,l}), T_{k,l}\right) = M\left(T_{2,4}, T_{k,l}\right)$. Then

$$\widetilde{l}(\varphi(g)_{k,l}\varphi(f)_{k,l}^{-1}) \le 2k + 4.$$

(iii) $T_{k+1,2l-1}, T_{k+1,2l} \overline{\in} L(R_f)$ and $T_{k+1,2l-1}, T_{k+1,2l} \in L(R_g)$. Then we have

$$\varphi(g)_{k+1,2l-1}\varphi(f)^{-1}_{k+1,2l-1} = M\left(T_{k+1,2l-1}, g(T_{k+1,2l-1})\right) = M\left(T_{k+1,2l-1}, T_{3,6}\right)$$

and

$$\varphi(g)_{k+1,2l}\varphi(f)^{-1}_{k+1,2l} = M\left(T_{k+1,2l}, g(T_{k+1,2l})\right) = M\left(T_{k+1,2l}, T_{2,4}\right).$$

It follows that

$$\widetilde{l}(\varphi(g)_{k+1,2l-1}\varphi(f)^{-1}_{k+1,2l-1}) \leq 2(k+1)+4 \quad \text{and} \quad \widetilde{l}(\varphi(g)_{k+1,2l}\varphi(f)^{-1}_{k+1,2l}) \leq 2(k+1)+4.$$

Therefore,

$$\begin{aligned} d(\varphi(f),\varphi(g)) &= \sum w(T_{i,j})\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j}\right) \\ &\leq 4 + 2^{-k}(2k+4) + 2^{-(k+1)}(2(k+1)+4) + 2^{-(k+1)}(2(k+1)+4) \\ &\leq 4+3+3+3 = 13. \end{aligned}$$

Case 3 $\left|L(R_f)\right| > \left|L(R_g)\right|$. f has the form of the reduced tree diagram of Figure 23. Then we obtain the reduced tree diagram for g as Figure 24.

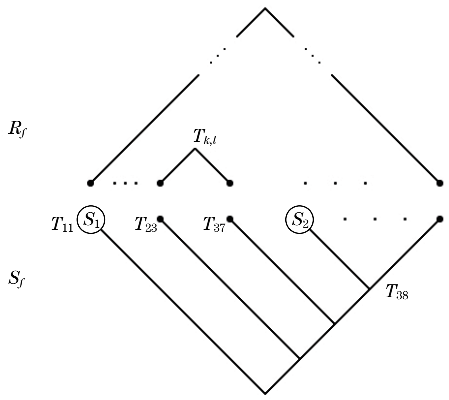

Figure 23 The reduced tree diagram for f

(i) If $T_{i,j} \in L(R_f) \cap L(R_g)$, then $\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j} = M\left(f(T_{i,j}), g(T_{i,j})\right)$.

(a) If $f(T_{i,j}) \in L(S_1)$, then

$$\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j} = M\left(T_{1,1}, T_{1,1}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $\widetilde{l}(\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j}) = 0$.

(b) If $f(T_{i,j}) \in L(S_2)$, then

$$\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j} = M\left(T_{3,8}, T_{2,4}\right) = M\left(\left[\frac{3}{4}, \frac{1}{1}\right], \left[\frac{2}{3}, \frac{1}{1}\right]\right) = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

It follows that $\widetilde{l}\left(\varphi(g)_{i,j}\varphi(f)^{-1}_{i,j}\right) \leq 2$.

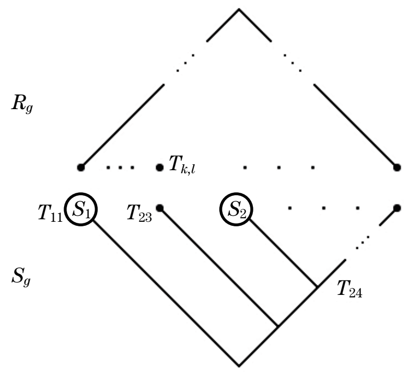

Figure 24 The reduced tree diagram for g

(ii) $T_{k,l}\overline{\in}L(R_f)$ and $T_{k,l}\in L(R_g)$. Then

$$\varphi(g)_{k,l}\varphi(f)_{k,l}^{-1}=M\left(T_{k,l},g(T_{k,l})\right)=M\left(T_{k,l},T_{2,3}\right).$$

It follows that

$$\widetilde{l}(\varphi(g)_{k,l}\varphi(f)_{k,l}^{-1})\le 2k+4.$$

(iii) $T_{k+1,2l-1},T_{k+1,2l}\overline{\in}L(R_g)$ and $T_{k+1,2l-1},T_{k+1,2l}\in L(R_f)$. Then we have

$$\varphi(g)_{k+1,2l-1}\varphi(f)_{k+1,2l-1}^{-1}=M\left(f(T_{k+1,2l-1}),T_{k+1,2l-1}\right)=M\left(T_{2,3},T_{k+1,2l-1}\right)$$

and

$$\varphi(g)_{k+1,2l}\varphi(f)_{k+1,2l}^{-1}=M\left(f(T_{k+1,2l}),T_{k+1,2l}\right)=M\left(T_{3,7},T_{k+1,2l}\right).$$

It follows that

$$\widetilde{l}(\varphi(g)_{k+1,2l-1}\varphi(f)_{k+1,2l-1}^{-1})\le 2(k+1)+4$$

and

$$\widetilde{l}(\varphi(g)_{k+1,2l}\varphi(f)_{k+1,2l}^{-1})\le 2(k+1)+4.$$

Therefore,

$$\begin{aligned}d(\varphi(f),\varphi(g))&=\sum w(T_{i,j})\widetilde{l}(\varphi(g)_{i,j}\varphi(f)_{i,j}^{-1})\\&\le 2+2^{-k}(2k+4)+2^{-(k+1)}(2(k+1)+4)+2^{-(k+1)}(2(k+1)+4)\\&\le 2+3+3+3=11.\end{aligned}$$

(4) Suppose that $g^{-1}f=x_1^{-1}$. Then $f^{-1}=x_1g^{-1}$. By the result of (3), $d(\varphi(g),\varphi(f))\le 13$. Now we will show that if $d_S(f,g)=n$, then $d(\varphi(f),\varphi(g))\le 13n$.

Since $l_S(f^{-1}g)=n$, $f^{-1}g=x_{i_1}x_{i_2}\cdots x_{i_n}$, where $x_{i_k}\in S=\{x_0,x_1\}$. We have $g=fx_{i_1}x_{i_2}\cdots x_{i_n}$. It follows that

$$\begin{aligned}d(\varphi(g),\varphi(f))&\le d(\varphi(fx_{i_1}x_{i_2}\cdots x_{i_n}),\varphi(fx_{i_1}x_{i_2}\cdots x_{i_{n-1}}))+\cdots\\&\quad+d(\varphi(fx_{i_1}x_{i_2}),\varphi(fx_{i_1}))+d(\varphi(fx_{i_1}),\varphi(f))\\&\le 13n.\end{aligned}$$

Therefore,

$$d(\varphi(f), \varphi(g)) \leq 13 d_S(f, g).$$

Lemma 4.6 (see [8, Theorem 3.1]) Let $S = \{x_0, x_1\}$ be the finite generating set for Thompson's group F, and for every $f \in F$, $|f|_S$ is the word-length with respect to S. Let $\binom{R_f}{S_f}$ be the reduced tree diagram for f, and $N(f)$ be the number of carets in R_f (or S_f). Then

$$N(f) - 2 \leq |f|_S \leq 4N(f) - 4.$$

Definition 4.1 *Let $f : X \to Y$ be a map of metric spaces. If for every bounded set $B \subseteq Y$, $f^{-1}(B)$ is a bounded set of X, then we say that f is a proper map.*

Proposition 4.3 *Let $S = \{x_0, x_1\}$ be the finite generating set for Thompson's group F, d_S be the left-invariant word-metric with respect to S, and d be the right-invariant pseudo-metric for H induced by l which is defined in* (2.2)*. Then $\varphi : (F, d_S) \to (H, d)$ is not a proper map.*

Proof It suffices to show that there exist $\{f_n\} \subseteq F$ such that $|f_n|_S \to \infty$ and $l(\varphi(f_n)) \leq 7$.

Define a map $\psi : F \to F$ as follows: for every $f \in F$, let $\binom{R_f}{S_f}$ be the reduced tree diagram for f. Then define $\psi^n(f)$ as the element of F with the reduced tree diagram in Figure 25.

Now let

$$f_0 = x_0, \quad f_n = \psi^n(x_0), \quad \forall n \geq 1.$$

Then we obtain the reduced tree diagram for f_n (see Figure 26).

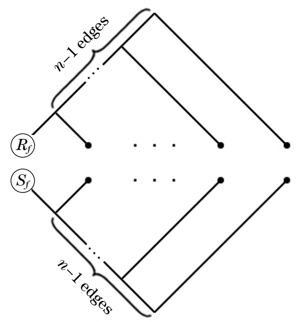

Figure 25 The reduced tree diagram of $\psi^n(f)$

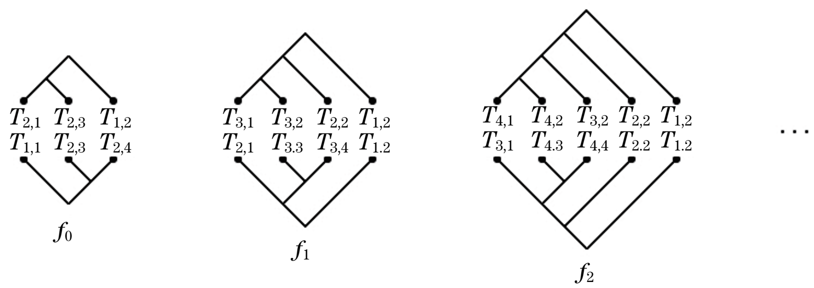

Figure 26 The reduced tree diagram of f_n

Let $\binom{R_{f_n}}{S_{f_n}}$ be the reduced tree diagram for f_n, $N(f_n)$ be the number of carets in R_{f_n}. Then $N(f_n) = n+2$. Thus by Lemma 4.6, $|f_n|_S \geq N(f_n) - 2 = n$. Note that

$$T_{n+2,1}, T_{n+2,2}, T_{n+1,2}, T_{n,2}, \cdots, T_{1,2}$$

are the leaves of R_{f_n} and correspond to the leaves of S_{f_n} respectively.

$$T_{n+1,1}, T_{n+2,3}, T_{n+2,4}, T_{n,2}, \cdots, T_{1,2}.$$

Therefore,

$$l\left(\varphi(f_n)\right) = 2^{-(n+2)}(\widetilde{l}(\varphi(f_n)_{n+2,1}) + \widetilde{l}(\varphi(f_n)_{n+2,2})) + 2^{-(n+1)}\widetilde{l}(\varphi(f_n)_{n+1,2}).$$

By the Remark 4.2,

$$\begin{aligned}
\varphi(f_n)_{n+2,1} &= (M(T_{n+2,1}, T_{n+1,1})) \leq 2(n+2)+4, \\
\varphi(f_n)_{n+2,2} &= (M(T_{n+2,2}, T_{n+2,3})) \leq 2(n+2)+4, \\
\varphi(f_n)_{n+1,2} &= (M(T_{n+1,2}, T_{n+2,4})) \leq 2(n+2)+4.
\end{aligned}$$

So $l\left(\varphi(f_n)\right) \leq 7$.

Proposition 4.4 *Let $S = \{x_0, x_1\}$ be the finite generating set for Thompson's group F, d_S be the left-invariant word-metric with respect to S, and d_1 be the right-invariant pseudo-metric for H induced by l_1 which is defined in* (2.1)*. Then $\varphi : (F, d_S) \to (H, d_1)$ is not a bornologous map. Therefore, it is not a Lipschitz map.*

Proof We will show that for any $\lambda > 0$, there exist f and g, such that $d_S(f,g) = 1$ and $d_1(\varphi(f), \varphi(g)) > \lambda$.

For any $\lambda > 0$, there exists an $n (> \lambda + 1)$. Let $f = x_0^n$, $g = x_0^{n-1}$. Then $d_S(f,g) = 1$. Let $\binom{R_f}{S_f}$ and $\binom{R_g}{S_g}$ be the reduced tree diagrams for f and g respectively.

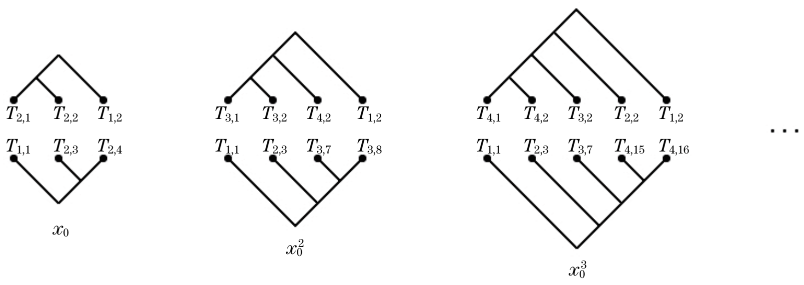

Figure 27 The reduced tree diagrams of x_0^n $(n \geq 1)$

Note that

$$\forall\ 2 \leq i \leq n, \quad T_{i,2} \in L(R_f) \cap L(R_g)$$

and

$$f(T_{i,2}) = T_{n+3-i, 2^{n+3-i}-1}, \quad g(T_{i,2}) = T_{n+2-i, 2^{n+2-i}-1}$$

Since $T_{n,2^n-1} = \left[\frac{n-1}{n}, \frac{n}{n+1}\right]$, we have

$$\begin{aligned}
\varphi(g)_{i,2}\varphi(f)_{i,2}^{-1} &= M\left(f(T_{i,2}), g(T_{i,2})\right) \\
&= M\left(T_{n+3-i,2^{n+3-i}-1}, T_{n+2-i,2^{n+2-i}-1}\right) \\
&= M\Big(\Big[\frac{n+2-i}{n+3-i}, \frac{n+3-i}{n+4-i}\Big], \Big[\frac{n+1-i}{n+2-i}, \frac{n+2-i}{n+3-i}\Big]\Big) \\
&= \begin{pmatrix} n+1-i & n+2-i \\ n+2-i & n+3-i \end{pmatrix} \begin{pmatrix} n+2-i & n+3-i \\ n+3-i & n+4-i \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore, $\widetilde{l}\left(\varphi(g)_{i,2}\varphi(f)_{i,2}^{-1}\right) \geq 1$.

$$\begin{aligned}
d_1\left(\varphi(f), \varphi(g)\right) = l_1\left(\varphi(g)\varphi(f)^{-1}\right) &= \sum_{k=0}^{\infty}(k+1)(\widetilde{l}(\varphi(g)_{k,1}\varphi(f)_{k,1}^{-1}) + \cdots + \widetilde{l}(\varphi(g)_{k,2^k}\varphi(f)_{k,2^k}^{-1})) \\
&> \sum_{i=2}^{n}\widetilde{l}\left(\varphi(g)_{i,2}\varphi(f)_{i,2}^{-1}\right) \\
&\geq n-1 > \lambda.
\end{aligned}$$

References

[1] Gromov, M., Asymptotic invariants of infinite groups, Geometric Group Theory, London Math. Soc., Lecture Notes, Vol. 2, Ser. 182, Cambridge Univ. Press, Cambridge, 1993.

[2] Tessera, R., Guentner, E. and Yu, G., A notion of geometric complexity and its application to topological rigidity, 2010. arXiv:1008.0884v1

[3] Willett, R., Some Notes on Property A, Limits of Graphs in Group Theory and Computer Science, EPFL Press, Lausanne, 2009, 191–281.

[4] Bell, G. and Dranishnikov, A., Asymptotic dimension in Będlewo, *Topol. Proc.*, **38**, 2011, 209–236.

[5] Belk, J. M., Thompson's Group F, PhD Thesis, Cornell University, Ithaca, New York, 2004. arXiv: math.GR/ 0708.3609v1

[6] Fordham, S. B., Minimal Length Elements of Thompson's Group F, PhD Thesis, Brigham Young University, Provo, Utah, 1995.

[7] Floyd, W. J., Cannon, J. W. and Parry, W. R., Introductory notes on Richard Thompson's groups, *L'Enseign. Math.* (2), **42**(3–4), 1996, 215–256.

[8] Cleary, S. and Taback, J., Combinatorial properties of Thompson's group F, *Trans. Amer. Math. Soc.*, **356**(7), 2004, 2825–2849.