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Completeness of the System of Root Vectors of 2 × 2 Upper Triangular Infinite-Dimensional Hamiltonian Operators in Symplectic Spaces and Applications*

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Abstract The authors investigate the completeness of the system of eigen or root vectors of the 2×2 upper triangular infinite-dimensional Hamiltonian operator H_0. First, the geometrical multiplicity and the algebraic index of the eigenvalue of H_0 are considered. Next, some necessary and sufficient conditions for the completeness of the system of eigen or root vectors of H_0 are obtained. Finally, the obtained results are tested in several examples.

Keywords 2×2 upper triangular infinite-dimensional Hamiltonian operator, Eigenvector, Root vector, Completeness
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1 Introduction

Using the simulation theory between structural mechanics and optimal control, Zhong proposed the separation of variables based on Hamiltonian systems, which provides a unified and analytical approach to elasticity and related fields (see [1–4]). This method extends the traditional separation of variables, and brings some important problems such as the invertibility and spectral theory of Hamiltonian operators (see [5–13]).

As known, the feasibility of the method completely depends on the completeness of the system of eigen or root vectors of the associated Hamiltonian operators. However, the completeness concerning Hamiltonian operators has not been systematically discussed. On the other hand, we find that a great number of practical problems can be described as upper triangular Hamiltonian forms (for instance, see the examples in Section 3). So, in this paper, we study the completeness of the system of eigen or root vectors of upper triangular Hamiltonian operators.

Throughout this paper, an operator or operator matrix is always linear (not necessarily bounded). Note that for the eigenvalue λ of an operator T, the set $E(\lambda; T)$ consists of all associated eigenvectors. Λ is a countable index set. For the positive integer k, the set $N^k(\lambda; T) =$

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$\{x \in \mathcal{D}(T^k) \mid (\lambda I - T)^k x = 0\}$. In the following, we present some basic notions and auxiliary lemmas.

Definition 1.1 *Let X be a Hilbert space and $H : \mathcal{D}(H) \subseteq X \times X \to X \times X$,*

$$H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \tag{1.1}$$

be a densely defined closed operator. If A is a densely defined closed operator, and B, C are self-adjoint operators, then H is called an infinite-dimensional Hamiltonian operator or simply Hamiltonian operator. In particular, we say that the Hamiltonian operator H is upper triangular if $C = 0$, and is denoted by H_0. Note that for an operator T, T^* represents its adjoint operator.

Definition 1.2 *Let X be a Hilbert space over the complex field $\mathbb{C}$, and the function $(\cdot\,,\cdot)$ be an inner product on the product space $X \times X$. Then, the function $(\cdot\,, J\cdot\,)$ is called a symplectic form generated by the inner product $(\cdot\,,\cdot)$, where*

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

with I being the identity operator on X.

A linear space equipped with a symplectic form is called a symplectic space. In the present paper, the symplectic form on the symplectic space $X \times X$ is always defined by the function $(\cdot\,, J\cdot\,)$.

Definition 1.3 *The symplectic orthogonal system $\{u_k, v_k \mid k \in \Lambda\}$ is said to be complete in the symplectic space $X \times X$, if for each $u \in X \times X$, there exists a unique constant sequence $\{c_k, d_k \mid k \in \Lambda\}$ such that*

$$u = \sum_{k \in \Lambda} c_k u_k + d_k v_k,$$

which converges in the norm of the space $X \times X$, where Λ represents a directed countable index set that consists of some integeral numbers in natural order.

Lemma 1.1 (see [14]) *Let λ and μ be the eigenvalues of the Hamiltonian operator H given by (1.1), and the associated eigenvectors be $u^0 = (x^0\ y^0)^{\mathrm{T}}$ and $v^0 = (f^0\ g^0)^{\mathrm{T}}$, respectively. Assume that $u^1 = (x^1\ y^1)^{\mathrm{T}}$ and $v^1 = (f^1\ g^1)^{\mathrm{T}}$ are the first-order root vectors associated with the pairs (λ, u^0) and (μ, v^0), respectively. If $\lambda + \overline{\mu} \neq 0$, then $(u^0, Jv^0) = 0$, $(u^0, Jv^1) = 0$ and $(u^1, Jv^1) = 0$.*

Lemma 1.2 (see [8]) *The point spectrum, consisting of all eigenvalues, of the upper triangular infinite-dimensional Hamiltonian operator*

$$H_0 = \begin{pmatrix} A & B \\ 0 & -A^* \end{pmatrix} \tag{1.2}$$

is given by

$$\sigma_p(H_0) = \sigma_p(A) \cup \sigma_p^1(-A^*),$$

where

$$\sigma_p^1(-A^*) = \{\lambda \in \mathbb{C} \mid \lambda \in \sigma_p(-A^*),\ \mathcal{R}(B_\lambda) \cap \mathcal{R}(\lambda I - A) \neq \emptyset\},$$
$$B_\lambda = B|_{(\mathcal{N}(\lambda I + A^*) \cap \mathcal{D}(B)) \setminus \{0\}}.$$

Obviously, $\sigma_p^1(-A^*) \subseteq \sigma_p(-A^*)$.

2 Main Results

In this section, we give the main results of this paper and their proofs. Note that the upper triangular infinite-dimensional Hamiltonian operators arisen below are always defined by (1.2), and for the definition of the algebraic index of an eigenvalue, the reader is referred to [14].

Theorem 2.1 *Let the eigenvalues of the operators* A *and* $-A^*$ *be all simple. Assume that* $\sigma_p(A) \cap \sigma_p(-A^*) = \emptyset$, *and* $(x, y) \neq 0$ *for* $x \in E(\lambda; A)$ *and* $y \in E(-\overline{\lambda}; -A^*)$. *Then the following statements hold:*

(i) $\lambda \in \sigma_p(H_0)$, *and the geometrical multiplicity and the algebraic index of the eigenvalue* λ *of the Hamiltonian operator* H_0 *are both one.*

(ii) *If* $-\overline{\lambda} \in \sigma_p^1(-A^*)$, *then* $-\overline{\lambda} \in \sigma_p(H_0)$, *and the geometrical multiplicity and the algebraic index of the eigenvalue* $-\overline{\lambda}$ *of the Hamiltonian operator* H_0 *are both one.*

Proof (i) The fact that $\lambda \in \sigma_p(H_0)$ and $u = (x\ 0)^{\mathrm{T}} \in E(\lambda; H)$ follows immediately from $\sigma_p(A) \subseteq \sigma_p(H_0)$ and $x \in E(\lambda; A)$. By $\sigma_p(A) \cap \sigma_p(-A^*) = \emptyset$ and Lemma 1.2, we have $E(\lambda; H_0) \cup \{0\} = \operatorname{span}\{u\}$ since every eigenvalue of the operator A is simple, i.e., the geometrical multiplicity of the eigenvalue λ of the operator H_0 is one.

In order to prove the algebraic index of the eigenvalue λ of H_0 being one, it suffices to show that the operator H_0 does not have the first-order root vector associated with the pair (λ, u). Now, suppose that H_0 has the first-order root vector $u^1 = (x^1\ w^1)^{\mathrm{T}}$ associated with the pair (λ, u), i.e.,

$$\begin{cases} Ax^1 + Bw^1 = \lambda x^1 + x, \\ -A^* w^1 = \lambda w^1. \end{cases} \tag{2.1}$$

The relation $\sigma_p(A) \cap \sigma_p(-A^*) = \emptyset$ implies $\lambda \notin \sigma_p(-A^*)$, so $w^1 = 0$. Thus, by the first equality in (2.1), we obtain

$$Ax^1 = \lambda x^1 + x.$$

Note that $(Ax^1, y) = (x^1, A^* y) = (x^1, \overline{\lambda} y) = (\lambda x^1, y)$. Therefore, the relation

$$(Ax^1, y) = (\lambda x^1, y) + (x, y)$$

indicates $(x, y) = 0$, which is a contradiction to the assumption $(x, y) \neq 0$. This proves the assertion (i).

(ii) If $-\overline{\lambda} \in \sigma_p^1(-A^*)$, then $-\overline{\lambda} \in \sigma_p(H_0)$ by Lemma 1.2. Note that $y \in E(-\overline{\lambda}; -A^*)$ and $-\overline{\lambda}$ is simple. Then, there exists a unique vector z for y such that

$$\begin{cases} Az + By = -\overline{\lambda} z, \\ -A^* y = -\overline{\lambda} y, \end{cases}$$

i.e., $v = (z\ y)^{\mathrm{T}} \in E(-\overline{\lambda}; H_0)$. If there exists another vector $\widetilde{z}\ (\neq z)$ such that $\widetilde{v} = (\widetilde{z}\ y)^{\mathrm{T}} \in E(-\overline{\lambda}; H_0)$, then $(A + \overline{\lambda})(z - \widetilde{z}) = 0$. By $z - \widetilde{z} \neq 0$, we get $-\overline{\lambda} \in \sigma_p(A)$, which contradicts the fact $\sigma_p(A) \cap \sigma_p(-A^*) = \emptyset$. Thus, $E(-\overline{\lambda}; H_0) \cup \{0\} = \operatorname{span}\{v\}$, i.e., the geometrical multiplicity of the eigenvalue $-\overline{\lambda}$ is one.

On the other hand, suppose that $v^1 = (z^1\ y^1)^{\mathrm{T}}$ is the first-order root vectors of H_0 associated with the pair $(-\overline{\lambda}, v)$, i.e.,

$$\begin{cases} Az^1 + By^1 = -\overline{\lambda} z^1 + z, \\ -A^* y^1 = -\overline{\lambda} y^1 + y. \end{cases} \tag{2.2}$$

Taking the inner product of the second equality by x in (2.2), we deduce $(y, x) = 0$, a contradiction. Therefore, the algebraic index of the eigenvalue $-\overline{\lambda}$ of H_0 is one.

Theorem 2.2 *Let the eigenvalues of the operators A and $-A^*$ be all simple, and the operator A possess countable eigenvalues $\{\lambda_k \mid k \in \Lambda\}$ with $x_k \in E(\lambda_k; A)$. Assume that $\sigma_p(A) = \overline{\sigma_p(A^*)}$, $\sigma_p(A) \cap \sigma_p(-A^*) = \emptyset$, $(x_k, y_k) \neq 0$ and $(By_k, y_j) = 0$ $(k \neq j)$ for $y_k \in E(-\overline{\lambda_k}; -A^*)$ $(k, j \in \Lambda)$. If $\sigma_p^1(-A^*) = \sigma_p(-A^*)$, then the system of eigenvectors of the upper triangular Hamiltonian operator H_0 is complete in the symplectic space $X \times X$ if and only if $\{y_k \mid k \in \Lambda\}$ is a base in the symplectic space X.*

Proof By $\sigma_p(A) = \overline{\sigma_p(A^*)}$, it follows that $\sigma_p(-A^*) = -\overline{\sigma_p(A)} = \{-\overline{\lambda}_k \mid k \in \Lambda\}$. Since $\sigma_p^1(-A^*) = \sigma_p(-A^*)$, $\sigma_p(H_0) = \{\lambda_k, -\overline{\lambda}_k \mid k \in \Lambda\}$. By assumptions and the proof of Theorem 2.1, we see that there exists a unique vector z_k, such that $v_k = (z_k\ y_k)^{\mathrm{T}} \in E(-\overline{\lambda_k}; H_0)$ for each $k \in \Lambda$, and $\{u_k, v_k \mid k \in \Lambda\}$ is a system of eigenvectors of H_0, where $u_k = (x_k\ 0)^{\mathrm{T}} \in E(\lambda_k; H_0)$. By Theorem 2.1, the algebraic multiplicities of the eigenvalues $\lambda_k, -\overline{\lambda}_k$ are all one, so H_0 does not have root vectors.

Sufficiency. From $\sigma_p(A) \cap \sigma_p(-A^*) = \emptyset$, it follows that

$$\lambda_k + \overline{\lambda}_j \neq 0, \quad \lambda_k + \overline{\mu}_j \begin{cases} = 0, & k = j, \\ \neq 0, & k \neq j, \end{cases} \quad \mu_k + \overline{\mu}_j \neq 0, \quad k, j \in \Lambda,$$

where $\mu_j = -\overline{\lambda}_j$. Then, by Lemma 1.1, it can be readily seen that for $k, j \in \Lambda$,

$$\begin{cases} (u_k, Ju_j) = 0, \\ (u_k, Jv_j) = \begin{cases} (x_k, y_k), & k = j, \\ (x_k, y_j) = 0, & k \neq j, \end{cases} \\ (v_k, Jv_j) = (z_k, y_j) - (y_k, z_j) = 0. \end{cases} \tag{2.3}$$

Noting that $v_k = (z_k\ y_k)^{\mathrm{T}} \in E(-\overline{\lambda}_k; H_0)$, we have $(By_k, y_j) = (\lambda_j + \overline{\lambda}_k)(z_k, y_j)$. So,

$$(z_k, y_j) = 0, \quad k \neq j. \tag{2.4}$$

In the following, we prove that the symplectic orthogonal system $\{u_k, v_k \mid k \in \Lambda\}$ is complete in the symplectic space $X \times X$, i.e., there exists a unique constant sequence $\{c_k, d_k \mid k \in \Lambda\}$, such that for each $\Delta = (f\ g)^{\mathrm{T}} \in X \times X$,

$$\Delta = \sum_{k \in \Lambda} c_k u_k + d_k v_k. \tag{2.5}$$

For $k \in \Lambda$, set

$$\begin{cases} c_k = \dfrac{(\Delta, Jv_k)}{(u_k, Jv_k)} = \dfrac{-(g, z_k) + (f, y_k)}{(x_k, y_k)}, \\ d_k = \dfrac{(\Delta, Ju_k)}{(v_k, Ju_k)} = \dfrac{(g, x_k)}{(y_k, x_k)}. \end{cases} \tag{2.6}$$

Then,

$$\sum_{k\in\Lambda} c_k u_k + d_k v_k = \sum_{k\in\Lambda} \begin{pmatrix} \dfrac{-(g,z_k)+(f,y_k)}{(x_k,y_k)}x_k + \dfrac{(g,x_k)}{(y_k,x_k)}z_k \\ \dfrac{(g,x_k)}{(y_k,x_k)}y_k \end{pmatrix}. \tag{2.7}$$

Since the system $\{y_k \mid k \in \Lambda\}$ of vectors is a base in X, there exists a unique constant sequence $\{d_k(g) \mid k \in \Lambda\}$ such that

$$g = \sum_{k\in\Lambda} d_k(g) y_k$$

for each $g \in X$. Taking the inner product of the above relation by x_k on the right-hand side, we clearly have $d_k(g) = \frac{(g,x_k)}{(y_k,x_k)}$, which shows that the second component of the right-hand side of (2.7) is exactly the expression of g in terms of the base $\{y_k \mid k \in \Lambda\}$, i.e.,

$$g = \sum_{k\in\Lambda} \frac{(g,x_k)}{(y_k,x_k)} y_k. \tag{2.8}$$

For $j \neq k$, by (2.3) and (2.4), it is clear that

$$\Big(\frac{-(g,z_k)}{(x_k,y_k)}x_k + \frac{(g,x_k)}{(y_k,x_k)}z_k, y_j\Big) = 0, \quad j \in \Lambda.$$

For $j = k$, by (2.3) and (2.8), we have

$$\Big(\frac{-(g,z_k)}{(x_k,y_k)}x_k + \frac{(g,x_k)}{(y_k,x_k)}z_k, y_j\Big) = -(g,z_j) + \frac{(g,z_k)}{(y_k,x_k)}(y_k,z_j) = 0, \quad j \in \Lambda.$$

Thus,

$$\frac{-(g,z_k)}{(x_k,y_k)}x_k + \frac{(g,x_k)}{y_k,x_k)}z_k = 0, \quad k \in \Lambda,$$

since $\{y_j \mid j \in \Lambda\}$ is a base. Write the first component given by the right-hand side of (2.7) as Υ. Then

$$\Upsilon = \sum_{k\in\Lambda} \frac{(f,y_k)}{(x_k,y_k)} x_k.$$

Note that the relation (2.3) shows that $\{x_k \mid k \in \Lambda\}$ and $\{y_k \mid k \in \Lambda\}$ are biorthogonal. So, $\{x_k \mid k \in \Lambda\}$ is also a base in X. Thus, $\Upsilon = f$. Therefore, there exists a constant sequence $\{c_k, d_k \mid k \in \Lambda\}$, such that the expansion (2.5) is valid for each $\Delta = (f\ g)^{\mathrm{T}} \in X \times X$.

On the other hand, we assume that there is another constant sequence $\{\widehat{c}_k, \widehat{d}_k \mid k \in \Lambda\}$ such that the expansion (2.5) is valid. Then, we have

$$\sum_{k\in\Lambda} (c_k - \widehat{c}_k)u_k + (d_k - \widehat{d}_k)v_k = 0.$$

Taking the inner product by Jv_k and Ju_k on the right-hand side, respectively, we have $c_k = \widehat{c}_k$ and $d_k = \widehat{d}_k$ $(k \in \Lambda)$. Therefore, $\{u_k, v_k \mid k \in \Lambda\}$ is complete in the symplectic space $X \times X$.

Necessity. Assume that the system $\{u_k, v_k \mid k \in \Lambda\}$ of eigenvectors of the Hamiltonian operator H_0 is complete in the symplectic space $X \times X$. Then there exists a unique constant sequence $\{c_k, d_k \mid k \in \Lambda\}$, such that the equality (2.5) holds. Taking the inner product of the relation (2.5) by Jv_k and Ju_k on the right-hand side, respectively, we deduce that c_k and d_k $(k \in \Lambda)$ are determined by (2.6). Thus,

$$g = \sum_{k\in\Lambda} d_k y_k = \sum_{k\in\Lambda} \frac{(g, x_k)}{(y_k, x_k)} y_k.$$

Thus, by the arbitrariness of g and the uniqueness of d_k, the system $\{y_k \mid k \in \Lambda\}$ of vectors is a base in the symplectic space X. Therefore, the proof is completed.

Remark 2.1 It follows from Theorem 2.1 that the algebraic indexes of the eigenvalues λ_k and $-\overline{\lambda}_k$ $(k \in \Lambda)$ of the Hamiltonian operator H_0 in Theorem 2.2 are all one. If $\sigma_p^1(-A^*) \neq \sigma_p(-A^*)$, i.e., there exists a $-\overline{\lambda}_{k_0} \in \sigma_p(-A^*)$ but $-\overline{\lambda}_{k_0} \notin \sigma_p^1(-A^*)$, then $\sigma_p(H_0) = \{\lambda_{k_0}, \lambda_k, -\overline{\lambda}_k \mid k \in \Lambda \setminus k_0\}$. Thus, by Lemma 1.1, the coefficient of the eigenvector u_{k_0} associated with λ_{k_0} in the corresponding eigenvector expansion cannot be computed by using the symplectic orthogonality.

Theorem 2.3 *Let $\sigma_p(A) = \sigma_p(-A^*)$, and the operators A and $-A^*$ both possess countable simple eigenvalues $\{\lambda_k \mid k \in \Lambda\}$ with $x_k \in E(\lambda_k; A)$ and $y_k \in E(\lambda_k; -A^*)$. Then, $\sigma_p(H_0) = \{\lambda_k \mid k \in \Lambda\}$, and $u_k = (x_k\ 0)^{\mathrm{T}} \in E(\lambda_k; H_0)$. Further, assume that for some $k_0 \in \Lambda$, there exists a $\widetilde{k}_0 \in \Lambda$ such that $(x_{k_0}, y_{\widetilde{k}_0}) \neq 0$ and $(x_{\widetilde{k}_0}, y_{k_0}) \neq 0$. Then, the following statements hold:*

(i) *If $\lambda_{k_0} \in \sigma_p^1(-A^*)$, then the geometrical multiplicity and the algebraic index of the eigenvalue λ_{k_0} of the Hamiltonian operator H_0 are two and one, respectively.*

(ii) *If $\lambda_{k_0} \notin \sigma_p^1(-A^*)$, then the geometrical multiplicity of the eigenvalue λ_{k_0} of the Hamiltonian operator H_0 is one. Further, if H_0 has the root vector associated with the pair (λ_{k_0}, u_{k_0}), then it only has the first-order root vector, i.e., the algebraic index of the eigenvalue λ_{k_0} of the Hamiltonian operator H_0 is two.*

Proof The assertion that $\sigma_p(H_0) = \{\lambda_k \mid k \in \Lambda\}$ and $u_k = (x_k\ 0)^{\mathrm{T}} \in E(\lambda_k; H_0)$ follows from the fact that $\sigma_p(A) = \sigma_p(-A^*)$ and $x_k \in E(\lambda_k; A)$. From $Ax_{k_0} = \lambda_{k_0} x_{k_0}$, we have $(Ax_{k_0}, y_{\widetilde{k}_0}) = (\lambda_{k_0} x_{k_0}, y_{\widetilde{k}_0})$, which shows $\lambda_{k_0} + \overline{\lambda}_{\widetilde{k}_0} = 0$ by $A^* y_{\widetilde{k}_0} = -\lambda_{\widetilde{k}_0} y_{\widetilde{k}_0}$ and $(x_{k_0}, y_{\widetilde{k}_0}) \neq 0$.

(i) If $\lambda_{k_0} \in \sigma_p^1(-A^*)$, then at least there exists a vector z_{k_0} for $y_{k_0} \in E(\lambda_{k_0}; -A^*)$ such that $v_{k_0} = (z_{k_0}\ y_{k_0})^{\mathrm{T}} \in E(\lambda_{k_0}; H_0)$. Note that the eigenvalues of A and $-A^*$ are all simple. Thus, it is easy to check that the eigenspace associated with the eigenvalue λ_{k_0} is $E(\lambda_{k_0}; H_0) \cup \{0\} = \operatorname{span}\{u_{k_0}, v_{k_0}\}$, i.e., the geometrical multiplicity of the eigenvalue λ_{k_0} of H_0 is two.

Suppose that $u_{k_0}^1 = (x_{k_0}^1\ w_{k_0}^1)^{\mathrm{T}}$ is the first-order root vector associated with the pair (λ_{k_0}, U_{k_0}), where $U_{k_0} = au_{k_0} + bv_{k_0} \in E(\lambda_{k_0}; H_0)$, and a, b are arbitrary numbers but not both zero, i.e.,

$$\begin{cases} Ax_{k_0}^1 + Bw_{k_0}^1 = \lambda_{k_0} x_{k_0}^1 + ax_{k_0} + bz_{k_0}, \\ -A^* w_{k_0}^1 = \lambda_{k_0} w_{k_0}^1 + by_{k_0}. \end{cases} \tag{2.9}$$

When $b = 0$, obviously, $a \neq 0$, and by the second equality in (2.9), we see that $w_{k_0}^1 \in E(\lambda_{k_0}; -A^*) \cup \{0\}$. If $w_{k_0}^1 = 0$, then by the first equality in (2.9), we obtain

$$(Ax_{k_0}^1, y_{\widetilde{k}_0}) = (\lambda_{k_0} x_{k_0}^1, y_{\widetilde{k}_0}) + a(x_{k_0}, y_{\widetilde{k}_0}), \tag{2.10}$$

which implies that $(x_{k_0}, y_{\widetilde{k}_0}) = 0$, since $(Ax_{k_0}^1, y_{\widetilde{k}_0}) = (x_{k_0}^1, A^* y_{\widetilde{k}_0}) = (x_{k_0}^1, -\lambda_{\widetilde{k}_0} y_{\widetilde{k}_0})$ and $\lambda_{\widetilde{k}_0} + \overline{\lambda}_{k_0} = 0$. A contradiction occurs. If $w_{k_0}^1 \neq 0$, without loss of generality, we set $w_{k_0}^1 = y_{k_0} \in$

$E(\lambda_{k_0};-A^*)$. Taking the inner product of the first relation by $y_{\widetilde{k}_0}$ in (2.9), we have

$$(Ax^1_{k_0},y_{\widetilde{k}_0})+(By_{k_0},y_{\widetilde{k}_0})=(\lambda_{k_0}x^1_{k_0},y_{\widetilde{k}_0})+a(x_{k_0},y_{\widetilde{k}_0}). \tag{2.11}$$

According to $v_{k_0}=(z_{k_0}\ y_{k_0})^{\mathrm{T}}\in E(\lambda_{k_0};H_0)$, we deduce $(By_{k_0},y_{\widetilde{k}_0})=0$. So (2.11) becomes (2.10), the same contradiction occurs. Similarly, when $b\neq 0$, we can obtain $(y_{k_0},x_{\widetilde{k}_0})=0$, a contradiction. Thus, the operator H_0 does not have the first-order root vector associated with the pair (λ_{k_0},U_{k_0}). Since U_{k_0} is arbitrary eigenvector of λ_{k_0}, the algebraic index of the eigenvalue λ_{k_0} of the Hamiltonian operator H_0 is one.

(ii) Note that the eigenvalues of A are all simple. By Lemma 1.2, if $\lambda_{k_0}\notin\sigma^1_p(-A^*)$, then the geometrical multiplicity of the eigenvalue λ_{k_0} of the operator H_0 is one.

If H_0 has the root vector associated with the pair (λ_{k_0},u_{k_0}), then $N^1(\lambda_{k_0};H_0)\subsetneq N^2(\lambda_{k_0};H_0)$. It is easy to know $\dim N^2(\lambda_{k_0};H_0)=2$ by $\dim N(\lambda_{k_0};H_0)=1$. Assume that $u^1_{k_0}=(x^1_{k_0}\ w^1_{k_0})^{\mathrm{T}}$ is any first-order root vector associated with the pair (λ_{k_0},u_{k_0}). Then $w^1_{k_0}\neq 0$ from the proof of (i). Thus $w^1_{k_0}\in E(\lambda_{k_0};-A^*)$, and we take $w^1_{k_0}=y_{k_0}$, i.e., $u^1_{k_0}=(x^1_{k_0}\ y_{k_0})^{\mathrm{T}}$. So, $N^2(\lambda_{k_0};H_0)=\mathrm{span}\{u_{k_0},u^1_{k_0}\}$. Suppose that the algebraic index of the eigenvalue λ_{k_0} of the operator H_0 is not two. Then $N^2(\lambda_{k_0};H_0)\subsetneq N^3(\lambda_{k_0};H_0)$. Let $u^2_{k_0}\in N^3(\lambda_{k_0};H_0)$ and $u^2_{k_0}\notin N^2(\lambda_{k_0};H_0)$. Obviously, we have $0\neq(H_0-\lambda_{k_0})u^2_{k_0}\in N^2(\lambda_{k_0};H_0)$. Thus, $(H_0-\lambda_{k_0})u^2_{k_0}=au_{k_0}+bu^1_{k_0}$, where a,b are arbitrary numbers, and $b\neq 0$. Then

$$\begin{cases}Ax^2_{k_0}+Bw^2_{k_0}=\lambda_{k_0}x^2_{k_0}+ax_{k_0}+bx^1_{k_0},\\ -A^*w^1_{k_0}=\lambda_{k_0}w^1_{k_0}+by_{k_0}.\end{cases} \tag{2.12}$$

In a way similar to the proof of (i), we can obtain the contradiction $(y_{k_0},x_{\widetilde{k}_0})=0$. Therefore, $N^1(\lambda_{k_0};H_0)\subsetneq N^2(\lambda_{k_0};H_0)=N^3(\lambda_{k_0};H_0)=N^4(\lambda_{k_0};H_0)=\cdots$, i.e., the algebraic index of the eigenvalue λ_{k_0} of the Hamiltonian operator H_0 is two.

Theorem 2.4 *Let $\sigma_p(A)=\sigma_p(-A^*)$, and the operators A and $-A^*$ both possess countable simple eigenvalues $\{\lambda_k\mid k\in\Lambda\}$ with $x_k\in E(\lambda_k;A)$ and $y_k\in E(\lambda_k;-A^*)$. Assume that for each index $k\in\Lambda$, there exists a unique index $\widetilde{k}\in\Lambda$ such that $(x_k,y_{\widetilde{k}})\neq 0$. Then, the following statements hold:*

(i) *If $\sigma^1_p(-A^*)=\sigma_p(-A^*)$, and the series*

$$\sum_{k\in\Lambda}\Big(\frac{-(g,z_{\widetilde{k}})-\frac{(g,x_{\widetilde{k}})}{(y_k,x_{\widetilde{k}})}((z_k,y_{\widetilde{k}})-(y_k,z_{\widetilde{k}}))}{(x_k,y_{\widetilde{k}})}x_k+\frac{(g,x_{\widetilde{k}})}{(y_k,x_{\widetilde{k}})}z_k\Big)$$

is convergent, then the system of eigenvectors of the Hamiltonian operator H_0 is complete in the symplectic space $X\times X$ if and only if $\{y_k\mid k\in\Lambda\}$ is a base in the symplectic space X, where $v_k=(z_k\ y_k)^{\mathrm{T}}\in E(\lambda_k;H_0)$.

(ii) *If $\sigma^1_p(-A^*)\neq\sigma_p(-A^*)$, H_0 has the root vector $u^1_k=(z^1_k,\ y_k)^{\mathrm{T}}$ associated with the pair (λ_k,u_k) for $k\in\Lambda_2$, and the series*

$$\sum_{k\in\Lambda}\Big(\frac{-(g,z_{\widetilde{k}})-\frac{(g,x_{\widetilde{k}})}{(y_k,x_{\widetilde{k}})}((z_k,y_{\widetilde{k}})-(y_k,z_{\widetilde{k}}))}{(x_k,y_{\widetilde{k}})}x_k+\frac{(g,x_{\widetilde{k}})}{(y_k,x_{\widetilde{k}})}z_k\Big)$$

is convergent, where $z_k=z^1_k$ for $k\in\Lambda_2$, then the system of root vectors of the Hamiltonian operator H_0 is complete in the symplectic space $X\times X$ if and only if $\{y_k\mid k\in\Lambda\}$ is a base in *the symplectic space X, where $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 = \{k \in \Lambda \mid \lambda_k \in \sigma_p^1(-A^*)\}$, $\Lambda_2 = \{k \in \Lambda \mid \lambda_k \notin \sigma_p^1(-A^*)\}$, $u_k = (x_k\ 0)^{\mathrm{T}} \in E(\lambda_k; H_0)$, and $v_k = (z_k\ y_k)^{\mathrm{T}} \in E(\lambda_k; H_0)$ $(k \in \Lambda_1)$.*

Proof Theorem 2.3 shows that $\sigma_p(H_0) = \{\lambda_k \mid k \in \Lambda\}$, and $u_k = (x_k\ 0)^{\mathrm{T}} \in E(\lambda_k; H_0)$. Since there exists a unique index $\widetilde{k} \in \Lambda$ such that $(x_k, y_{\widetilde{k}}) \neq 0$ for each $k \in \Lambda$, there exists a unique index $\widetilde{k} \in \Lambda$ such that $\lambda_k + \overline{\lambda}_{\widetilde{k}} = 0$ for each $k \in \Lambda$ from the proof of Theorem 2.3, which also shows $(x_{\widetilde{k}}, y_k) \neq 0$. Hence, if k runs over the index set Λ, then $\widetilde{k}$ also runs over the index set Λ.

(i) By the assumptions and the proof of Theorem 2.3(i), the geometrical multiplicity and the algebraic multiplicity of the eigenvalue λ_{k_0} of the Hamiltonian operator H_0 are both two. So, there exists a vector z_k for $y_k \in E(\lambda_k; -A^*)$ such that $v_k = (z_k\ y_k)^{\mathrm{T}} \in E(\lambda_k; H_0)$. Thus, $\{u_k, v_k \mid k \in \Lambda\}$ is a system of eigenvectors of H_0. Moreover, H_0 does not have a system of root vectors by Theorem 2.3(i).

Sufficiency. From $\lambda_k + \overline{\lambda}_{\widetilde{k}} = 0$ $(k, \widetilde{k} \in \Lambda)$, it follows that for $k, j \in \Lambda$,

$$\lambda_k + \overline{\lambda}_j \begin{cases} = 0, & j = \widetilde{k}, \\ \neq 0, & j \neq \widetilde{k}. \end{cases}$$

Then, by Lemma 1.1, it can be readily seen that

$$\begin{cases} (u_k, Ju_j) = 0, \\ (u_k, Jv_j) = \begin{cases} (x_k, y_{\widetilde{k}}), & j = \widetilde{k}, \\ (x_k, y_j) = 0, & j \neq \widetilde{k}, \end{cases} \\ (v_k, Jv_j) = \begin{cases} (z_k, y_{\widetilde{k}}) - (y_k, z_{\widetilde{k}}), & j = \widetilde{k}, \\ (z_k, y_j) - (y_k, z_j) = 0, & j \neq \widetilde{k}. \end{cases} \end{cases} \tag{2.13}$$

Now, we prove that the system $\{u_k, v_k \mid k \in \Lambda\}$ of eigenvectors is complete in the corresponding symplectic space. For each $\Delta = (f\ g)^{\mathrm{T}} \in X \times X$, set

$$\begin{cases} c_k = \dfrac{(\Delta, Jv_{\widetilde{k}})}{(u_k, Jv_{\widetilde{k}})} - d_k \dfrac{(v_k, Jv_{\widetilde{k}})}{(u_k, Jv_{\widetilde{k}})}, \\ d_k = \dfrac{(\Delta, Ju_{\widetilde{k}})}{(v_k, Ju_{\widetilde{k}})}, \quad k \in \Lambda. \end{cases} \tag{2.14}$$

Then,

$$\begin{aligned} &\sum_{k \in \Lambda} c_k u_k + d_k v_k \\ &= \sum_{k \in \Lambda} \begin{pmatrix} \dfrac{-(g, z_{\widetilde{k}}) + (f, y_{\widetilde{k}}) - \frac{(g, x_{\widetilde{k}})}{(y_k, x_{\widetilde{k}})}\big((z_k, y_{\widetilde{k}}) - (y_k, z_{\widetilde{k}})\big)}{(x_k, y_{\widetilde{k}})} x_k + \dfrac{(g, x_{\widetilde{k}})}{(y_k, x_{\widetilde{k}})} z_k \\ \dfrac{(g, x_{\widetilde{k}})}{(y_k, x_{\widetilde{k}})} y_k \end{pmatrix}. \end{aligned} \tag{2.15}$$

Since the system $\{y_k \mid k \in \Lambda\}$ of vectors is a base in X, by (2.13), we have

$$g = \sum_{k \in \Lambda} \frac{(g, x_{\widetilde{k}})}{(y_k, x_{\widetilde{k}})} y_k, \tag{2.16}$$

which shows that the second component of the right-hand side of (2.15) is exactly g. In addition, the relation (2.13) implies that $\{x_k \mid k \in \Lambda\}$ is also a base in X, so $f = \sum\limits_{k\in\Lambda} \frac{(f,y_{\widetilde{k}})}{(x_k,y_{\widetilde{k}})} x_k$. Thus, by assumption, the first component given in (2.15) is convergent, and is written as Υ. By (2.13) and (2.16),

$$
\begin{aligned}
(\Upsilon - f, y_{\widetilde{j}}) &= -(g, z_{\widetilde{j}}) - \frac{(g, x_{\widetilde{j}})}{(y_j, x_{\widetilde{j}})}((z_j, y_{\widetilde{j}}) - (y_j, z_{\widetilde{j}})) + \sum_{k\in\Lambda} \frac{(g, x_{\widetilde{k}})}{(y_k, x_{\widetilde{k}})}(z_k, y_{\widetilde{j}}) \\
&= -(g, z_{\widetilde{j}}) + \frac{(g, x_{\widetilde{j}})}{(y_j, x_{\widetilde{j}})}(y_j, z_{\widetilde{j}}) + \sum_{\substack{k\in\Lambda \\ k\neq j}} \frac{(g, x_{\widetilde{k}})}{(y_k, x_{\widetilde{k}})}(z_k, y_{\widetilde{j}}) \\
&= -(g, z_{\widetilde{j}}) + \frac{(g, x_{\widetilde{j}})}{(y_j, x_{\widetilde{j}})}(y_j, z_{\widetilde{j}}) + \sum_{\substack{k\in\Lambda \\ k\neq j}} \frac{(g, x_{\widetilde{k}})}{(y_k, x_{\widetilde{k}})}(y_k, z_{\widetilde{j}}) \\
&= -(g, z_{\widetilde{j}}) + \frac{(g, x_{\widetilde{j}})}{(y_j, x_{\widetilde{j}})}(y_j, z_{\widetilde{j}}) + \Big((g, z_{\widetilde{j}}) - \frac{(g, x_{\widetilde{j}})}{(y_j, x_{\widetilde{j}})}(y_j, z_{\widetilde{j}})\Big) \\
&= 0, \quad \widetilde{j} \in \Lambda,
\end{aligned}
$$

which deduces that $f = \Upsilon$ since $\{y_j \mid j \in \Lambda\}$ is a base. Therefore, there exists a constant sequence $\{c_k, d_k \mid k \in \Lambda\}$, such that for each $\Delta = (f\ g)^{\mathrm{T}} \in X \times X$,

$$
\Delta = \sum_{k\in\Lambda} c_k u_k + d_k v_k. \tag{2.17}
$$

On the other hand, the uniqueness of constant sequence $\{c_k, d_k \mid k \in \Lambda\}$ can be proved in the similar way as that of Theorem 2.2. By Definition 1.3, the proof of sufficiency is completed.

Necessity. The proof is similar to that of Theorem 2.2, and we omit it here.

(ii) For $k \in \Lambda_1$, by Theorem 2.3(i) and its proof, the geometrical multiplicity and the algebraic multiplicity of the eigenvalue λ_k of the Hamiltonian operator H_0 are both two, and the independent eigenvectors of λ_k are

$$
u_k = (x_k\ 0)^{\mathrm{T}}, \quad v_k = (z_k\ y_k)^{\mathrm{T}}. \tag{2.18}
$$

For $k \in \Lambda_2$, by assumptions and Theorem 2.3(ii), the geometrical multiplicity and the algebraic multiplicity of the eigenvalue λ_k of the Hamiltonian operator H_0 are one and two, respectively, and the eigenvector and root vector of λ_k are

$$
u_k = (x_k\ 0)^{\mathrm{T}}, \quad u_k^1 = (z_k^1\ y_k)^{\mathrm{T}}, \tag{2.19}
$$

respectively.

Write $v_k = u_k^1$ for $k \in \Lambda_2$. By Lemma 1.1, we have for $k, j \in \Lambda$,

$$
\begin{cases}
(u_k, Ju_j) = 0, \quad k, j \in \Lambda, \\
(u_k, Jv_j) = \begin{cases} (x_k, y_{\widetilde{k}}), & j = \widetilde{k}, \\ (x_k, y_j) = 0, & j \neq \widetilde{k}, \end{cases} \\
(v_k, Jv_j) = \begin{cases} (z_k, y_{\widetilde{k}}) - (y_k, z_{\widetilde{k}}), & j = \widetilde{k}, \\ (z_k, y_j) - (y_k, z_j) = 0, & j \neq \widetilde{k}, \end{cases}
\end{cases}
$$

where $z_k = z_k^1$ for $k \in \Lambda_2$,

To prove the completeness of the system $\{u_k, v_k \mid k \in \Lambda\}$ of root vectors of the Hamiltonian operator H_0 in the symplectic space $X \times X$, for each $\Delta = (f\ g)^{\mathrm{T}} \in X \times X$, set

$$\begin{cases} c_k = \dfrac{(\Delta, Jv_{\widetilde{k}})}{(u_k, Jv_{\widetilde{k}})} - d_k \dfrac{(v_k, Jv_{\widetilde{k}})}{(u_k, Jv_{\widetilde{k}})}, \\ d_k = \dfrac{(\Delta, Ju_{\widetilde{k}})}{(v_k, Ju_{\widetilde{k}})}, \quad k \in \Lambda. \end{cases} \tag{2.20}$$

The rest of the proof is analogous to that of (i).

3 Applications

In this section, we present some examples illustrating results of the previous section. We always assume that $X = L^2[0,1]$.

Example 3.1 Consider the boundary value problem

$$\begin{cases} \dfrac{\partial^4 u}{\partial x^4} - \dfrac{\partial^2 u}{\partial y^2} = 0, & 0 < x < 1,\ 0 < y < h, \\ u(0,y) = u(1,y) = 0, \quad u_x''(0,y) = u_x''(1,y) = 0, & 0 \le y \le h, \\ u(x,0) = \varphi_1(x), \quad u(x,h) = \varphi_2(x), & 0 \le x \le 1. \end{cases}$$

Set $p = \frac{\partial^2 u}{\partial x^2}$, $q = \frac{1}{2}(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y})$. Then the above equation can be written as the following upper triangular infinite-dimensional Hamiltonian system:

$$\frac{\partial}{\partial y}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & -2\frac{\partial^2}{\partial x^2} \\ 0 & -\frac{\partial^2}{\partial x^2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}.$$

The corresponding upper triangular Hamiltonian operator is

$$H_0 = \begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x^2} & -2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{pmatrix},$$

where

$$\mathcal{D}(A) = \mathcal{D}(B) = \mathcal{D}(A^*) = \{u \in X \mid u, u' \text{ are absolutely continuous}, u(0) = u(1) = 0, u', u'' \in X\}.$$

It can be readily seen that

$$\sigma_p(A) = \{-(k\pi)^2 \mid k \in \Lambda\}, \quad \sigma_p(-A^*) = \{(k\pi)^2 \mid k \in \Lambda\}, \quad \Lambda = \{1, 2, \cdots\}$$

and $(x_k, y_k) = \frac{1}{2}$, where $x_k = \sin(k\pi x)$ and $y_k = \sin(k\pi x)$. Obviously, $\sigma_p(A) = \overline{\sigma_p(A^*)}$, $\sigma_p(A) \cap \sigma_p(-A^*) = \emptyset$, and $(By_k, y_j) = 0$ $(k \ne j)$. Furthermore, for each y_k, there exists a vector $z_k = \sin(k\pi x)$ such that $Az_k + By_k = (k\pi)^2 z_k$, i.e., $(k\pi)^2 \in \sigma_p^1(-A^*)$, which shows that $\sigma_p^1(-A^*) = \sigma_p(-A^*)$. Then, the assumptions of Theorem 2.2 are satisfied. Also, $\{y_k \mid k \in \Lambda\}$ is an orthogonal base in X. Therefore, the system of eigenfunctions of the Hamiltonian operator H_0 is complete in the symplectic space $X \times X$.

Example 3.2 Consider the boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} - 2\mathrm{i}\dfrac{\partial u}{\partial y} - u = 0, & 0 < x < 1,\ 0 < y < h, \\ u(0,y) = u(1,y), \quad u'_x(0,y) = u'_x(1,y), & 0 \le y \le h, \\ u(x,0) = \varphi_1(x), \quad u(x,h) = \varphi_2(x), & 0 \le x \le 1. \end{cases} \tag{3.1}$$

The upper triangular infinite-dimensional Hamiltonian system from (3.1) is

$$\frac{\partial}{\partial y}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \mathrm{i}\frac{\partial}{\partial x} + \mathrm{i} & -2\mathrm{i}\frac{\partial}{\partial x} \\ 0 & -\mathrm{i}\frac{\partial}{\partial x} + \mathrm{i} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix},$$

where $p = \frac{\partial u}{\partial x}$, $q = \frac{1}{2}(\mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} + u)$, and

$$H_0 = \begin{pmatrix} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i} & -2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \\ 0 & -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i} \end{pmatrix}$$

with

$$\mathcal{D}(A) = \mathcal{D}(B) = \mathcal{D}(A^*) = \{u \in X \mid u \text{ is absolutely continuous}, u(0) = u(1), u' \in X\}. \tag{3.2}$$

Direct calculations show that

$$\sigma_p(A) = \sigma_p(-A^*) = \{\mathrm{i} - 2k\pi \mid k \in \Lambda\}, \quad \Lambda = \{0, \pm 1, \pm 2, \cdots\}$$

and $(x_k, y_{-k}) = 1$ (i.e., $\widetilde{k} = -k$), where $x_k = \mathrm{e}^{2k\pi\mathrm{i}x}$ and $y_k = \mathrm{e}^{-2k\pi\mathrm{i}x}$. In addition, $\sigma_p^1(-A^*) = \sigma_p(-A^*)$. Therefore, by Theorem 2.4(i), the system of eigenfunctions of the Hamiltonian operator H_0 is complete in the symplectic space $X \times X$, since $\{y_k = \mathrm{e}^{-2k\pi\mathrm{i}x} \mid k \in \Lambda\}$ is an orthogonal base in X. Note that the system of eigenfunctions of H_0 is $\{(\mathrm{e}^{2k\pi\mathrm{i}x} \ \ 0)^{\mathrm{T}}, (\mathrm{e}^{-2k\pi\mathrm{i}x} \ \ \mathrm{e}^{-2k\pi\mathrm{i}x})^{\mathrm{T}} \mid k \in \Lambda\}$.

Example 3.3 Consider the mixed problem of the parabolic differential equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0, & 0 < x < 1,\ 0 < y < h, \\ u(0,y) = u(1,y), \quad u'_x(0,y) = u'_x(1,y), & 0 \le y \le h, \\ u(x,0) = \varphi_1(x), \quad u(x,h) = \varphi_2(x), & 0 \le x \le 1. \end{cases} \tag{3.3}$$

Set $p = \frac{\partial u}{\partial x}$, $q = \mathrm{i}(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})$. Then, we obtain

$$\frac{\partial}{\partial y}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & \mathrm{i}\frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial x} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix},$$

which is an infinite-dimensional Hamiltonian system derived from (3.3). The corresponding upper triangular Hamiltonian operator is given by

$$H_0 = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \\ 0 & \frac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix},$$

where domain of A, B, A^* is given by (3.2).

Direct calculations show that

$$\sigma_p(A) = \sigma_p(-A^*) = \{2k\pi\mathrm{i} \mid k \in \Lambda\}, \quad \Lambda = \{0, \pm 1, \pm 2, \cdots\}$$

and $(x_k, y_k) = 1$ $(k \in \Lambda)$, where $x_0 = y_0 = 1$, $x_k = -2k\pi\mathrm{e}^{2k\pi\mathrm{i}x}$ and $y_k = \mathrm{e}^{2k\pi\mathrm{i}x}$ $(k = \pm 1, \pm 2, \cdots)$. It can be verified that $\sigma_p^1(-A^*) \neq \sigma_p(-A^*)$ and $\Lambda_1 = \{0\}$, $\Lambda_2 = \{2k\pi\mathrm{i} \mid k = \pm 1, \pm 2, \cdots\}$. Clearly, $\sigma_p(H) = \{\lambda_k = 2k\pi\mathrm{i} \mid k \in \Lambda\}$, $u_k = (x_k\ 0)^{\mathrm{T}} \in E(\lambda_k; H_0)$ $(k \in \Lambda)$ and $v_0 = (0\ 1)^{\mathrm{T}} \in E(\lambda_0; H_0)$. Moreover, $u_k^1 = (\mathrm{e}^{2k\pi\mathrm{i}x}\ \mathrm{e}^{2k\pi\mathrm{i}x})^{\mathrm{T}}$ is the first-order root vector associated with the pair (λ_k, u_k) for each $k \in \Lambda_2$. Thus, the assumptions of Theorem 2.4(ii) are satisfied. Since $\{y_k = \mathrm{e}^{2k\pi\mathrm{i}x} \mid k \in \Lambda\}$ is an orthogonal base in X, the system of root vectors of the Hamiltonian operator H_0 is complete in the symplectic space $X \times X$.

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