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Proper Submanifolds in Product Manifolds*

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Abstract The authors obtain various versions of the Omori-Yau's maximum principle on complete properly immersed submanifolds with controlled mean curvature in certain product manifolds, in complete Riemannian manifolds whose k-Ricci curvature has strong quadratic decay, and also obtain a maximum principle for mean curvature flow of complete manifolds with bounded mean curvature. Using the generalized maximum principle, an estimate on the mean curvature of properly immersed submanifolds with bounded projection in N_1 in the product manifold $N_1 \times N_2$ is given. Other applications of the generalized maximum principle are also given.

Keywords Calabi-Chern problem, Omori-Yau maximum principle, Properly immersed submanifold, Mean curvature flow, Stochastic completeness
2000 MR Subject Classification 53C40, 35B50

1 Introduction

There is a well-known Calabi-Chern problem (see [4, 7]) on the extrinsic boundedness properties for minimal submanifolds in n-dimensional Euclidean space.

Many efforts have been made during the past years, and the research on the Calabi-Chern problem has made some progress (see [5, 8, 10, 12, 14] et al.). For $\mathbb{R}^3$, Nadirashvili [12] gave an example of complete immersed bounded minimal surface in $\mathbb{R}^3$.

A more ambitious conjecture is: A complete (non-flat) minimal hypersurface in $\mathbb{R}^n$ has an unbounded projection in every $(n-2)$-dimensional flat subspace. This is not true for immersed minimal surfaces in $\mathbb{R}^3$ by the Jorge-Xavier's example in [10].

On the other hand, Colding and Minicozzi [8] showed that the situation is different for embedded minimal disks in $\mathbb{R}^3$, which is proper, whereas the Nadirashvili's example and the Jorge-Xavier's example are not proper.

Recently, L. J. Alias, G. P. Bessa and M. Dajczer [1] gave an estimate of the mean curvature of cylindrically bounded properly immersed submanifolds in some $N \times \mathbb{R}^l$, and as a consequence of their result, they showed that a complete minimal immersed hypersurface in $\mathbb{R}^n$ $(n \geq 3)$ with bounded projection in a two dimensional subspace cannot be proper.

Inspired by Calabi-Chern problem, it is natural to study complete properly immersed submanifolds in a product manifold. We generalized Alias-Bessa-Dajczer's results as follows.

Theorem 1.1 *Let N_1, N_2 be complete Riemannian manifolds of dimensions n_1, n_2 respectively, and let the radial sectional curvature of N_2 satisfy $\kappa^{\rm rad}_{N_2} \geq -c(1 + \rho_2^2 \log^2(\rho_2 + 2))$,*

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where c is a positive constant, ρ_2 is the distance function from a fixed point on N_2. Let $\psi : M^k \to N_1 \times N_2$ be an isometric immersion of a complete Riemannian manifold of dimension $k > n_2$ with mean curvature vector $\overrightarrow{H}$. Given $q \in M$, $p = \pi_1(\psi(q)) \in N_1$. Let $B_{N_1}(r)$ be the geodesic ball of N_1 centered at p with radius r. Assume that the radial sectional curvature $\kappa_{N_1}^{\mathrm{rad}}$ along the radial geodesics issuing from p is bounded above by a constant $\kappa_{N_1}^{\mathrm{rad}} \le b$ (const.) in $B_{N_1}(r)$. Suppose that

$$\psi(M) \subset B_{N_1}(r) \times N_2$$

for $r < \min\left\{\mathrm{inj}_{N_1}(p), \frac{\pi}{2\sqrt{b}}\right\}$, where we replace $\frac{\pi}{2\sqrt{b}}$ by $+\infty$ if $b \le 0$.

(1) *If $\psi : M^k \to N_1 \times N_2$ is proper, then*

$$\sup_M |\overrightarrow{H}| \ge (k - n_2) C_b(r).$$

(2) *If*

$$\sup_M |\overrightarrow{H}| < (k - n_2) C_b(r),$$

then M is stochastically incomplete, where C_b is defined in the beginning of Section 4.

The analytic tool to prove the above theorem is the Omori-Yau maximum principle. Omori firstly gave a maximum principle on complete Riemannian manifolds (see [14]). Later, Yau refined and simplified the argument in [21] under the assumption on Ricci curvature bounded from below. The curvature assumption could be relaxed to strong quadratic decay of Ricci curvature in [5]. There is a general analytic version of the Omori-Yau maximum principle due to Pigola et al. [16]. Based on them, in this paper, we give various versions of the Omori-Yau maximum principle on complete properly immersed submanifolds in rather general ambient manifolds and certain product manifolds in Section 3. Furthermore, we can obtain the maximum principle for mean curvature flow in this setting.

In the last section, we give several geometric applications of the Omori-Yau maximum principle, including the proof of the above mentioned results.

2 Preliminaries

Let $\psi : M \to N$ be an m-submanifold in Riemannian manifold N of dimension n with the second fundamental form B defined by

$$B(X, Y) = (\overline{\nabla}_X Y)^\perp$$

for $X, Y \in \Gamma(TM)$, where $(\,\cdot\,)^\perp$ denotes the orthogonal projections into the normal bundle NM. The second fundamental form B can be viewed as a cross-section of the vector bundle $\mathrm{Hom}(\odot^2 TM, NM)$ over M, where TM and NM denote the tangent bundle and the normal bundle along M, respectively. Taking the trace of B gives the mean curvature vector $\overrightarrow{H}$ of M in N, a cross-section of the normal bundle, and

$$\overrightarrow{H} \triangleq \mathrm{trace}(B) = \sum_{i=1}^m B(e_i, e_i),$$

where $\{e_i\}$ is a local orthonormal frame field of M.

The ambient manifold N in the present paper is rather general. We may impose some curvature assumptions on N. H. Wu [18] introduced an interesting notion of partial positivity (negativity) on Riemannian manifold N. For any $x \in N$ and any $(k+1)$ orthonormal vectors $\{e_0, e_1, \cdots, e_k\} \in T_xN$, denote

$$\mathrm{Ric}_{-,k}(x) \triangleq \min\Big\{\sum_{i=1}^{k} \kappa(e_0 \wedge e_i)\Big\}$$

and

$$\mathrm{Ric}_{+,k}(x) \triangleq \max\Big\{\sum_{i=1}^{k} \kappa(e_0 \wedge e_i)\Big\},$$

where $\kappa(e_0 \wedge e_i)$ denotes the sectional curvature of the plane spanned by e_0 and e_i.

If $\mathrm{Ric}_{-,k} > 0$ ($\mathrm{Ric}_{+,k} < 0$), then N is called k-positive (k-negative). k-Ricci curvature condition is an intermediate one. The cases $k = 1$ and $k = n-1$ are reduced to sectional curvature and Ricci curvature respectively.

Some results under the conditions on sectional curvature can be generalized to those under k-Ricci curvature condition (see [11, 17]).

Now, we consider certain proper submanifolds in an ambient manifold with strong quadratic decay of k-Ricci curvature, and in certain product manifolds.

A Riemannian manifold M is said to be stochastically complete if for some (and therefore, for any) $(x,t) \in M \times (0, +\infty)$ it holds that $\int_M p(x,y,t)\mathrm{d}y = 1$, where $p(x,y,t)$ is the heat kernel of the Laplacian operator. Otherwise, the manifold M is said to be stochastically incomplete. There is an interesting characterization of stochastic completeness: M is stochastically complete if and only if for any C^2-function u with $u_+ \triangleq \sup u < \infty$, there exists a sequence $\{x_j\}$ such that $u(x_j) > u_+ - \frac{1}{j}$ and $\Delta u(x_j) < \frac{1}{j}$ (see [15]).

3 Various Versions of the Generalized Maximum Principle on Submanifolds

Let us generalize Omori-Yau's maximum principle on complete properly immersed submanifold of Euclidean space to complete proper submanifold with controlled mean curvature of some complete Riemannian manifolds. First we give the following lemma (see [5, 19]).

Lemma 3.1 *Let N be a complete Riemannian manifold of dimension n, and let the radial sectional curvature satisfy $\kappa^{\mathrm{rad}} \geq -cF(\rho)$, where $c > 0$ is a constant, ρ is the distance function from a fixed point x_0 on N, $F : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and $F \geq 1$. Let M be a k-dimensional complete submanifold of N with mean curvature vector $\overrightarrow{H}$, and $\sup\limits_M |\overrightarrow{H}| \leq \sqrt{1 + c(k-1)kF(\rho)}$. Let $x_0, x \in M$, and if x is not on the cut locus of the point x_0 in N, then for $\rho(x) \geq \rho_0$ (ρ_0 is a constant),*

$$\Delta_M \rho(x) \leq 2\sqrt{1 + c(k-1)kF(\rho)}. \tag{3.1}$$

Proof The restriction of ρ on M is a function on M. Then we have for $X, Y \in TM \subset TN$,

$$\begin{aligned}\mathrm{Hess}(\rho)(X,Y) &= XY(\rho) - (\overline{\nabla}_X Y - (\overline{\nabla}_X Y)^{\perp})\rho \\ &= \overline{\mathrm{Hess}}(\rho)(X,Y) + \langle B(X,Y), \overline{\nabla}\rho\rangle.\end{aligned} \tag{3.2}$$

Taking trace in (3.2), we get

$$\Delta\rho(x)=\sum_{i=1}^{k}\overline{\mathrm{Hess}}(\rho)(x)(e_i,e_i)+\langle\overrightarrow{H},\overline{\nabla}\rho\rangle, \tag{3.3}$$

where $\{e_1,\cdots,e_k\}$ is an orthonormal basis on T_xM.

Let $\gamma:[0,\rho]\to N$ be a minimal geodesic from x_0 to x with $\gamma(0)=x_0,\gamma(\rho)=x$. Choose an orthonormal basis $\left\{e_1=\frac{(\dot\gamma)^{\mathrm{T}}}{|(\dot\gamma)^{\mathrm{T}}|},\cdots,e_k,e_{k+1},\cdots,e_{n-1},e_n=\frac{(\dot\gamma)^{\perp}}{|(\dot\gamma)^{\perp}|}\right\}\in T_xN$, such that $\{e_1,\cdots,e_k\}$ are orthonormal vectors tangent to M. Let E_1 be the unit vector orthogonal to $\frac{\partial}{\partial\rho}=\dot\gamma$ on the plane spanned by e_1 and e_n. Let

$$e_1=\cos\theta E_1+\sin\theta\frac{\partial}{\partial\rho},$$

where θ is the angle between e_1 and E_1. Then we have

$$\overline{\mathrm{Hess}}(\rho)(x)(e_1,e_1)=\cos^2\theta\overline{\mathrm{Hess}}(\rho)(x)(E_1,E_1)\le\max\left\{\overline{\mathrm{Hess}}(\rho)(x)(E_1,E_1),0\right\}.$$

By parallel translation along γ, we have an orthonormal frame field $\{E_1(t),e_2(t),\cdots,e_k(t)\}$ along γ. Since free of conjugate points in γ, there is a unique Jacobi field J_i along γ, such that $J_1(0)=0,J_1(\rho)=E_1,J_i(0)=0,J_i(\rho)=e_i,i=2,\cdots,k$. Hence

$$\overline{\mathrm{Hess}}(\rho)(x)(e_i,e_i)=\int_0^\rho(|\dot J_i|^2-\langle R(\dot\gamma,J_i)\dot\gamma,J_i\rangle)\mathrm{d}t.$$

Similarly,

$$\overline{\mathrm{Hess}}(\rho)(x)(E_1,E_1)=\int_0^\rho(|\dot J_1|^2-\langle R(\dot\gamma,J_1)\dot\gamma,J_1\rangle)\mathrm{d}t.$$

Let $f(t)$ be any piecewise smooth function defined on $[0,\rho]$ with $f(0)=0$ and $f(\rho)=1$. Then $\{f(t)E_1(t),f(t)e_2(t),\cdots,f(t)e_k(t)\}$ are piecewise smooth vector fields along γ satisfying $f(0)E_1(0)=0,f(\rho)E_1(\rho)=J_1(\rho)$, $f(0)e_i(0)=0$, $f(\rho)e_i(\rho)=J_i(\rho)$. Using the minimization of Jacobi field, we have

$$\begin{aligned}\sum_{i=1}^{k}\overline{\mathrm{Hess}}(\rho)(x)(e_i,e_i)&\le\sum_{i=2}^{k}\int_0^\rho(|f'e_i|^2-\langle R(\dot\gamma,fe_i)\dot\gamma,fe_i\rangle)\mathrm{d}t\\&\quad+\max\Big\{\int_0^\rho(|f'E_1|^2-\langle R(\dot\gamma,fE_1)\dot\gamma,fE_1\rangle)\mathrm{d}t,\ 0\Big\}\\&\le\int_0^\rho(k(f')^2-(k-1)f^2\kappa^{\mathrm{rad}})\mathrm{d}t\\&\le\int_0^\rho(k(f')^2+c(k-1)F(t)f^2)\mathrm{d}t.\end{aligned}$$

Following the proof of Lemma 2.1 in [5], we obtain

$$\sum_{i=1}^{k}\overline{\mathrm{Hess}}(\rho)(x)(e_i,e_i)\le\sqrt{1+c(k-1)kF(\rho)}\,. \tag{3.4}$$

(3.3) and (3.4) give

$$\Delta\rho(x)\le\sqrt{1+c(k-1)kF(\rho)}+|\overrightarrow{H}|\cdot|\overline{\nabla}\rho|\le2\sqrt{1+c(k-1)kF(\rho)}.$$

Similarly, we have the lemma below.

Lemma 3.2 *Let N be a complete Riemannian manifold of dimension n with $\mathrm{Ric}_{-,k} \geq -cF(\rho)$ and $\mathrm{Ric}_{-,k-1} \geq -cF(\rho)$, where $c > 0$ is a constant, ρ is the distance function from a fixed point x_0 on N, $F : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and $F \geq 1$. Let M be a k-dimensional complete submanifold of N with mean curvature vector $\overrightarrow{H}$, and $\sup_M |\overrightarrow{H}| \leq \sqrt{1 + ckF(\rho)}$. Let $x_0, x \in M$, and if x is not on the cut locus of the point x_0 in N, then for $\rho(x) \geq \rho_0$ (ρ_0 is a constant),*

$$\Delta_M \rho(x) \leq 2\sqrt{1 + ckF(\rho(x))}\,.$$

Proof In a way similar to the proof of the above lemma, we have that if $\overline{\mathrm{Hess}}(\rho)(x)(E_1, E_1) > 0$, then

$$\sum_{i=1}^{k} \overline{\mathrm{Hess}}(\rho)(x)(e_i, e_i) \leq \int_0^\rho (k(f')^2 - f^2 \mathrm{Ric}_{-,k}) \mathrm{d}t;$$

if $\overline{\mathrm{Hess}}(\rho)(x)(E_1, E_1) \leq 0$, then

$$\sum_{i=1}^{k} \overline{\mathrm{Hess}}(\rho)(x)(e_i, e_i) \leq \int_0^\rho ((k-1)(f')^2 - f^2 \mathrm{Ric}_{-,k-1}) \mathrm{d}t.$$

Hence,

$$\sum_{i=1}^{k} \overline{\mathrm{Hess}}(\rho)(x)(e_i, e_i) \leq \int_0^\rho (k(f')^2 + cF(t) f^2) \mathrm{d}t.$$

It follows that

$$\Delta_M \rho(x) \leq 2\sqrt{1 + ckF(\rho(x))}.$$

Then the gradient estimate, as was done by Yau [21] (see also [5]), gives us the following result.

Theorem 3.1 *Let N_1, N_2 be complete Riemannian manifolds of dimensions n_1, n_2 respectively, and let the radial sectional curvature of N_2 satisfy $\kappa_{N_2}^{\mathrm{rad}} \geq -c(1 + \rho_2^2 \log^2(\rho_2 + 2))$, where c is a positive constant, ρ_2 is the distance function from a fixed point on N_2. Let $\psi : M^k \to N_1 \times N_2$ be a proper isometric immersion of a complete Riemannian manifold of dimension k with mean curvature vector $\overrightarrow{H}$ and $\sup_M |\overrightarrow{H}| \leq \sqrt{1 + c(1 + \rho_2^2 \log^2(\rho_2 + 2))}$. Suppose that*

$$\psi(M) \subset B_{N_1}(r) \times N_2.$$

Let f be a C^2-function bounded from above on M. Then for any $\varepsilon > 0$, there exists points $\{x_j\} \subset M$, such that

$$\lim_{j \to \infty} f(x_j) = \sup f, \tag{3.5}$$

$$|\nabla f|(x_j) < \varepsilon, \tag{3.6}$$

$$\Delta f(x_j) < \varepsilon. \tag{3.7}$$

Proof Define $\widetilde{\rho}_2 : N_1 \times N_2 \to [0, +\infty)$ by

$$\widetilde{\rho}_2(x_1, x_2) \triangleq \rho_2 \cdot \pi_2(x_1, x_2) = \rho_2(x_2),$$

where we denote π_2 for the projection to the second factor, and so is for π_1 in the sequel. Since ψ is proper and $\psi(M) \subset B_{N_1}(r) \times N_2$, the function $\phi = \widetilde{\rho}_2 \cdot \psi$ satisfies $\phi(x) \to +\infty$ as $x \to \infty$.

Identifying X with $\mathrm{d}\psi(X)$, we have at $x \in M$ and for every $X \in T_x M$ that

$$\langle \mathrm{grad}\phi, X \rangle = \mathrm{d}\phi(X) = \mathrm{d}\widetilde{\rho}_2(X) = \langle \mathrm{grad}^{N_1 \times N_2} \widetilde{\rho}_2, X \rangle.$$

So

$$\mathrm{grad}^{N_1 \times N_2} \widetilde{\rho}_2 = \mathrm{grad}\phi + (\mathrm{grad}^{N_1 \times N_2} \widetilde{\rho}_2)^{\perp},$$

where $(\mathrm{grad}^{N_1 \times N_2} \widetilde{\rho}_2)^{\perp}$ is perpendicular to $T_x M$.

Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on M and $N_1 \times N_2$ respectively. Then

$$\begin{aligned} \mathrm{Hess}_M(\phi)(X, Y) &= XY\phi - (\nabla_X Y)\phi = XY\widetilde{\rho}_2 - (\nabla_X Y)\widetilde{\rho}_2 \\ &= XY\widetilde{\rho}_2 - (\overline{\nabla}_X Y - B(X, Y))\widetilde{\rho}_2 \\ &= \mathrm{Hess}_{N_1 \times N_2}(\widetilde{\rho}_2)(X, Y) + \langle B(X, Y), \mathrm{grad}^{N_1 \times N_2} \widetilde{\rho}_2 \rangle. \end{aligned} \tag{3.8}$$

Taking trace in (3.8), we then have

$$\Delta\phi(x) = \sum_{i=1}^{k} \mathrm{Hess}_{N_1 \times N_2}(\widetilde{\rho}_2(\psi(x)))(e_i, e_i) + \langle \overrightarrow{H}(x), \mathrm{grad}^{N_1 \times N_2} \widetilde{\rho}_2(\psi(x)) \rangle, \tag{3.9}$$

where $\{e_1, \cdots, e_k\}$ is an orthonormal frame on M.

Letting $\{\frac{\partial}{\partial \rho_1}, E_2, \cdots, E_{n_1}\}$ be an orthonormal basis for $T_{\pi_1(x)} N_1$ and $\{\frac{\partial}{\partial \rho_2}, F_2, \cdots, F_{n_2}\}$ be an orthonormal basis for $T_{\pi_2(x)} N_2$, we choose

$$e_i = a_i \frac{\partial}{\partial \rho_1} + \sum_{j=2}^{n_1} b_{ij} E_j + c_i \frac{\partial}{\partial \rho_2} + \sum_{l=2}^{n_2} d_{il} F_l.$$

Since

$$1 = |e_i|^2 = a_i^2 + \sum_{j=2}^{n_1} b_{ij}^2 + c_i^2 + \sum_{l=2}^{n_2} d_{il}^2,$$

we observe that

$$\sum_{i=1}^{k} \sum_{l=2}^{n_2} d_{il}^2 \le k. \tag{3.10}$$

Then by Lemma 3.1 and using (3.10), we have

$$\begin{aligned} \sum_{i=1}^{k} \mathrm{Hess}_{N_1 \times N_2}(\widetilde{\rho}_2(\psi(x)))(e_i, e_i) &= \sum_{i=1}^{k} \mathrm{Hess}_{N_2}(\rho_2)(\pi_{2*} e_i, \pi_{2*} e_i) \\ &= \sum_{i=1}^{k} \sum_{l=2}^{n_2} d_{il}^2 \mathrm{Hess}_{N_2}(\rho_2)(F_l, F_l) \\ &\le k\sqrt{1 + c(1 + \rho_2^2 \log^2(\rho_2 + 2))}. \end{aligned} \tag{3.11}$$

Using (3.9) and (3.11), we obtain

$$\Delta\phi(x) = \sum_{i=1}^{k} \mathrm{Hess}_{N_1\times N_2}(\widetilde{\rho}_2(\psi(x)))(e_i, e_i) + \langle \overrightarrow{H}(x), \mathrm{grad}^{N_1\times N_2}\widetilde{\rho}_2\rangle$$
$$\leq (k+1)\sqrt{1 + c(1+\phi^2\log^2(\phi+2))}. \tag{3.12}$$

We define an auxiliary function on M

$$g(x) = \frac{f(x) - f(x_0) + 1}{[\log(\phi^2(x)+2)]^{\frac{1}{j}}} \tag{3.13}$$

for any $j > 0$. Then $g(x_0) = \frac{1}{(\log 2)^{\frac{1}{j}}} > 0$. Since $\sup f < +\infty$ and M is proper, we have $\phi(x) \to +\infty$ as $x \to \infty$, so $\limsup_{x\to\infty} g(x) = 0$. Thus g attains a positive supremum at $x_j \in M$.

Let us first prove (3.5). Indeed, if this is not true, then there would exist $\delta > 0$ and $\widehat{x} \in M$, such that

$$f(\widehat{x}) > f(x_j) + \delta$$

for each $j \geq j_0$ sufficiently large.

If $\phi(x_j) \to +\infty$ as $j \to +\infty$ for each j such that $\phi(x_j) > \phi(\widehat{x})$, we have

$$g(x) = \frac{f(\widehat{x}) - f(x_0) + 1}{[\log(\phi^2(\widehat{x})+2)]^{\frac{1}{j}}} > \frac{f(x_j) - f(x_0) + 1 + \delta}{[\log(\phi^2(x_j)+2)]^{\frac{1}{j}}} > g(x_j).$$

This contradicts the definition of x_j.

If $\{x_j\}$ lies in a compact set, then for some subsequence of j, $\{x_j\}$ converges to a point $\overline{x}$, so that $f(\widehat{x}) \geq f(\overline{x}) + \delta$. On the other hand, since $g(x_j) \geq g(\widehat{x})$ for each j, we deduce that

$$f(\overline{x}) - f(x_0) + 1 = \lim_{j\to+\infty} g(x_j) \geq \lim_{j\to+\infty} g(\widehat{x}) = f(\widehat{x}) - f(x_0) + 1,$$

that is $f(\overline{x}) \geq f(\widehat{x})$. This is also a contradiction. Thus we prove (3.5).

Again, if $\{x_j\}$ remains in a compact set, then $x_j \to \overline{x} \in M$ as $j \to +\infty$. At $\overline{x}$, we have

$$f(\overline{x}) = \sup f(x), \quad |\nabla f|(\overline{x}) = 0, \quad \Delta f(\overline{x}) \leq 0.$$

In this case, the sequence $x_j = \overline{x}$ $(\forall j)$ obviously satisfies all the requirements.

We now only need to consider the case when $x_j \to \infty$, then because M is proper, $\phi(x_j) \to +\infty$. Without loss of generality, we can assume that x_j is not on the cut locus of x_0 in N_2 (otherwise, we can use Calabi's trick to remedy it). Then, we can differentiate ϕ at x_j. Since g attains a positive supremum at x_j, we have

$$(\nabla \log g)(x_j) = 0, \quad \Delta(\log g)(x_j) \leq 0.$$

By direct computation, we get

$$\nabla f(x_j) = \frac{2(f(x_j) - f(x_0) + 1)\phi(x_j)\nabla\phi(x_j)}{j(\phi^2(x_j)+2)\log(\phi^2(x_j)+2)}, \tag{3.14}$$

$$\Delta f(x_j) \leq \frac{2(f(x_j) - f(x_0) + 1)}{j}\Big\{\frac{\phi\Delta\phi + |\nabla\phi|^2}{(\phi^2+2)\log(\phi^2+2)}$$
$$+ \frac{2\phi^2|\nabla\phi|^2}{j(\phi^2+2)^2(\log(\phi^2+2))^2} - \frac{2\phi^2|\nabla\phi|^2(1+\log(\phi^2+2))}{(\phi^2+2)^2(\log(\phi^2+2))^2}\Big\}$$
$$\leq \frac{2(f(x_j) - f(x_0) + 1)(\phi\Delta\phi + 1)}{j(\phi^2+2)\log(\phi^2+2)}. \tag{3.15}$$

Hence, (3.14) gives that

$$|\nabla f|(x_j) \leq \frac{2(f(x_j) - f(x_0) + 1)\phi(x_j)}{j(\phi^2(x_j) + 2)\log(\phi^2(x_j) + 2)} \to 0,$$

as $j \to +\infty$. This proves (3.6).

Using (3.12) and (3.15), we obtain

$$\Delta f(x_j) \leq \frac{2(f(x_j) - f(x_0) + 1)}{j(\rho^2 + 2)\log(\rho^2 + 2)} \Big\{ 2\rho\sqrt{1 + ck(1 + \rho^2 \log^2(\rho + 2))} + 1 \Big\}.$$

Letting $j \to \infty$, we prove (3.7).

Using Lemma 3.2, we also have the next theorem.

Theorem 3.2 *Let N be a complete Riemannian manifold of dimension n with $\mathrm{Ric}_{-,k} \geq -c(1 + \rho^2 \log^2(\rho + 2))$ and $\mathrm{Ric}_{-,k-1} \geq -c(1 + \rho^2 \log^2(\rho + 2))$, where $c > 0$ is a constant, and ρ is the distance function from a fixed point on N. Let M be a k-dimensional complete properly immersed submanifold of N with mean curvature vector $\overrightarrow{H}$, and $\sup_M |\overrightarrow{H}| \leq \sqrt{1 + ck\left(1 + \rho^2 \log^2(\rho + 2)\right)}$. Let f be a C^2-function bounded from above on M. Then for any $\varepsilon > 0$, there exists points $\{x_j\} \subset M$, such that*

$$\lim_{j\to\infty} f(x_j) = \sup f, \quad |\nabla f|(x_j) < \varepsilon, \quad \Delta f(x_j) < \varepsilon.$$

Let N be a complete Riemannian manifold of dimensional n, and let the radial sectional curvature satisfy $\kappa^{\mathrm{rad}} \geq -c(1+\rho^2 \log^2(\rho+2))$, where c is a positive constant, ρ is the distance function from a fixed point on N. Let M be a k dimensional complete properly immersed submanifold of N with mean curvature vector $\overrightarrow{H}$, and $\sup_M |\overrightarrow{H}| \leq \sqrt{1 + c(k-1)k(1 + \rho^2 \log^2(\rho + 2))}$. The second named author established a maximum principle for the complete space-like mean curvature flow in pseudo-Euclidean space in [20]. Similarly, we obtain a parabolic maximum principle in our setting. This is similar to the one in [9] under different conditions and by a different method.

We now consider the deformation of a submanifold under the mean curvature flow (to MCF), namely, consider a one-parameter family $F_t = F(\,\cdot\,, t)$ of immersions $F_t : M \to N$ with the corresponding images $M_t = F_t(M)$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} F(x,t) &= \overrightarrow{H}(x,t), \quad x \in M, \\ F(x,0) &= F(x) \end{aligned} \tag{3.16}$$

are satisfied, where $\overrightarrow{H}(x,t)$ is the mean curvature vector of M_t at $F(x,t)$. If $\sup_{M_0} |\overrightarrow{H}| < c_1$ (const.), $\sup_{M_t(t>0)} |\overrightarrow{H}| \leq \sqrt{1 + c(k-1)k(1 + \rho_t^2 \log^2(\rho_t + 2))}$, where ρ_t is the distance from a fixed point $F_t(x_0)$ on M_t, we can choose t sufficiently small, such that ρ_t is still proper. In fact,

$$\begin{aligned} \rho_t(y) &\approx \rho_0(y) + \frac{\mathrm{d}}{\mathrm{d}t}\rho_t(y)|_{t=0} \cdot t = \rho_0(y) + \Big\langle \nabla\rho_t, \frac{\mathrm{d}}{\mathrm{d}t} \Big\rangle\Big|_{t=0} \cdot t \\ &= \rho_0(y) + \langle \nabla\rho_0, \overrightarrow{H} \rangle \cdot t \geq \rho_0(y) - c_1 t. \end{aligned}$$

Let $f(x,t) = f(F(x,t))$ be a smooth function bounded from above on $M_t, 0 \le t \le \varepsilon_0$, which can also be viewed as a function on $M \times [0, \varepsilon_0]$. Let ε_0 be a small number, such that ρ_t is proper for each $t \in [0, \varepsilon_0]$,

Define a function on $M \times [0, \varepsilon_0]$,

$$g(x,t) = \frac{f(x,t) - f(x_0, 0) + 1}{[\log(\rho_t^2(x) + 2)]^{\frac{1}{j}}}$$

for any $j > 0$. It is easy to see that g must attain its supremum at the certain point (x_j, t_j). We have

$$g(x_j, t_j) = \sup_{M \times [0, \varepsilon_0]} g = \sup_{M_{t_j}} g.$$

Thus at (x_j, t_j),

$$\nabla f = \frac{2(f - f(x_0,0) + 1)\rho \nabla \rho}{j(\rho^2 + 2)\log(\rho^2 + 2)}. \tag{3.17}$$

Then we have

$$|\nabla f| \le \frac{2(f - f(x_0,0) + 1)\rho}{j(\rho^2 + 2)\log(\rho^2 + 2)} \to 0,$$

as $j \to \infty$. If t_j is the first time such that g attains a new maximum, then $\frac{\mathrm{d}g}{\mathrm{d}t} \ge 0$, namely,

$$\frac{\mathrm{d}f}{\mathrm{d}t} \ge \frac{2(f - f(x_0,0) + 1)\rho}{j(\rho^2 + 2)\log(\rho^2 + 2)} \frac{\mathrm{d}\rho}{\mathrm{d}t}. \tag{3.18}$$

Since

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \left\langle \nabla \rho, \frac{\mathrm{d}}{\mathrm{d}t} \right\rangle = \langle \nabla \rho, \overrightarrow{H} \rangle \le \sqrt{1 + c(k-1)k(1 + \rho^2 \log^2(\rho + 2))}, \tag{3.19}$$

from (3.18) and (3.19),

$$\frac{\mathrm{d}f}{\mathrm{d}t} \ge 0, \quad \text{as } j \to \infty.$$

We also have $\Delta g \le 0$ at (x_j, t_j). By direct computation and using (3.1), (3.17), we obtain

$$\begin{aligned}\Delta f &\le \frac{2(f - f(x_0,0) + 1)(\rho \Delta \rho + 1)}{j(\rho^2 + 2)\log(\rho^2 + 2)} \\ &\le \frac{2(f - f(x_0,0) + 1)}{j(\rho^2 + 2)\log(\rho^2 + 2)} \{2\rho \sqrt{1 + c(k-1)k(1 + \rho^2 \log^2(\rho + 2))} + 1\} \to 0,\end{aligned}$$

as $j \to \infty$. It is not difficult to see that

$$\lim_{j \to \infty} f(x_j, t_j) = \sup_{M_t (0 \le t \le \varepsilon_0)} f.$$

Hence, we have the following maximum principle.

Theorem 3.3 *Let M_t be complete mean curvature flow in N with*

$$\sup_{M_0} |\overrightarrow{H}| < c_1, \qquad \sup_{M_t (t > 0)} |\overrightarrow{H}| \le \sqrt{1 + c(k-1)k(1 + \rho_t^2 \log^2(\rho_t + 2))}.$$

Let $0 \le t \le \varepsilon_0$, such that ρ_t is proper for each t in this interval. Let f be a smooth function bounded from above on M_t. Then for any $\varepsilon > 0$, there exists a sequence of points $\{x_j\} \subset M_{t_j}$, such that

$$\lim_{j\to\infty} f(x_j, t_j) = \sup_{M_t(0\le t\le\varepsilon_0)} f,$$

and when j is sufficiently large,

$$|\nabla f|(x_j, t_j) < \varepsilon, \quad \frac{\mathrm{d}f}{\mathrm{d}t}(x_j, t_j) \ge -\varepsilon, \quad \Delta f(x_j, t_j) < \varepsilon.$$

Theorem 3.4 *Let M_t be a complete mean curvature flow in N with bounded mean curvature for $0 \le t \le \varepsilon_0$. Let f be a smooth function bounded from above on M_t and satisfy the following evolution equation:*

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) f \le \langle A, \nabla f\rangle$$

for a vector A with uniformly bounded $|A|$. Then,

$$\sup_{M_t} f \le \sup_{M_0} f \tag{3.20}$$

for any $0 \le t \le \varepsilon_0$.

Proof Suppose $\sup\limits_{M_t} f > \sup\limits_{M_0} f$. Then by Theorem 3.3, we have

$$\lim_{j\to\infty} f(x_j, t_j) = \sup_{M_t(0<t\le\varepsilon_0)} f.$$

For any $\delta > 0$, let f satisfy

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) f \le \langle A, \nabla f\rangle - \delta. \tag{3.21}$$

Using Theorem 3.3, we have that $\exists x_j \in M_{t_j}$, such that

$$\Delta f(x_j, t_j) + |\nabla f||A| \le \frac{\delta}{2}.$$

It follows that

$$\frac{\mathrm{d}f}{\mathrm{d}t}\Big|_{(x_j,t_j)} \le -\frac{\delta}{2},$$

which contradicts the conclusion of Theorem 3.3. Hence we prove (3.20).

Let

$$\widetilde{f} = f - \sup_{M_0} f - \delta t - \delta, \quad \forall \delta > 0.$$

Then $\widetilde{f}$ satisfies (3.21). The previous discussion implies

$$\sup_{M_t} \widetilde{f} \le \sup_{M_0} \widetilde{f} = -\delta,$$

namely,

$$f \le \sup_{M_0} f + \delta t.$$

Letting $\delta \to 0$, we have (3.20).

4 Geometric Applications

Using the results on the generalized maximum principle in the last section, we can estimate the mean curvature of k-dimensional properly immersed submanifolds with bounded projection in N_1 in the certain product manifold $N_1^{n_1} \times N_2^{n_2}$ $(k > n_2)$. In the following we denote

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}t), & \text{if } b > 0,\ t < \dfrac{\pi}{2\sqrt{b}}, \\ \dfrac{1}{t}, & \text{if } b = 0, \\ \sqrt{-b}\coth(\sqrt{-b}\,t), & \text{if } b < 0. \end{cases}$$

Theorem 4.1 *Let N_1, N_2 be complete Riemannian manifolds of dimensions n_1, n_2 respectively, and let the radial sectional curvature of N_2 satisfy $\kappa_{N_2}^{\mathrm{rad}} \geq -c(1 + \rho_2^2 \log^2(\rho_2 + 2))$, where c is a positive constant, ρ_2 is the distance function from a fixed point on N_2. Let $\psi : M^k \to N_1 \times N_2$ be an isometric immersion of a complete Riemannian manifold of dimension $k > n_2$ with mean curvature vector $\overrightarrow{H}$. Given $q \in M$, $p = \pi_1(\psi(q)) \in N_1$. Let $B_{N_1}(r)$ be the geodesic ball of N_1 centered at p with radius r. Assume that the radial sectional curvature $\kappa_{N_1}^{\mathrm{rad}}$ along the radial geodesics issuing from p is bounded as $\kappa_{N_1}^{\mathrm{rad}} \leq b$ (const.) in $B_{N_1}(r)$. Suppose that*

$$\psi(M) \subset B_{N_1}(r) \times N_2$$

for $r < \min\left\{\mathrm{inj}_{N_1}(p), \frac{\pi}{2\sqrt{b}}\right\}$, where we replace $\frac{\pi}{2\sqrt{b}}$ by $+\infty$ if $b \leq 0$.

(1) *If $\psi : M^k \to N_1 \times N_2$ is proper, then*

$$\sup_M |\overrightarrow{H}| \geq (k - n_2) C_b(r); \tag{4.1}$$

(2) *If*

$$\sup_M |\overrightarrow{H}| < (k - n_2) C_b(r), \tag{4.2}$$

then M is stochastically incomplete.

Proof Define $\widetilde{\rho}_1 : N_1 \times N_2 \to \mathbb{R}$ by

$$\widetilde{\rho}_1(x_1, x_2) = \rho_1(x_1) = \mathrm{dist}_{N_1}(p, x_1),$$

and $f : M^k \to \mathbb{R}$ by

$$f(x) = \widetilde{\rho}_1(\psi(x)).$$

We shall prove (4.1) by contradiction, namely, suppose

$$\sup_M |\overrightarrow{H}| < (k - n_2) C_b(r).$$

Since $\psi(M) \subset B_{N_1}(r) \times N_2$, we have that $\sup_M f \leq r < +\infty$, so by Theorem 3.1 there exists a sequence $\{x_j\} \subset M$, such that

$$|\nabla f|(x_j) < \frac{1}{j}, \quad \Delta f(x_j) < \frac{1}{j}.$$

In a way similar to (3.9), we get

$$\Delta f(x_j) = \sum_{i=1}^{k} \mathrm{Hess}_{N_1\times N_2}(\widetilde{\rho}_1(\psi(x_j)))(e_i, e_i) + \langle \overrightarrow{H}(x_j), \mathrm{grad}^{N_1\times N_2}\widetilde{\rho}_1(\psi(x_j))\rangle, \tag{4.3}$$

where $\{e_1, \cdots, e_k\}$ is an orthonormal frame on M.

Letting $\{\frac{\partial}{\partial\rho_1}, E_1, \cdots, E_{n_1}\}$ be an orthonormal basis for $T_{\pi_1(x_j)}N_1$ and the normal coordinates $(x_2^1, \cdots, x_2^{n_2})$ on N_2 near $\pi_2(x_j)$, we choose

$$e_i = a_i\frac{\partial}{\partial\rho_1} + \sum_{j=2}^{n_1} b_{ij}E_j + \sum_{l=1}^{n_2} c_{il}\frac{\partial}{\partial x_2^l}.$$

Since

$$1 = |e_i|^2 = a_i^2 + \sum_{j=2}^{n_1} b_{ij}^2 + \sum_{l=1}^{n_2} c_{il}^2,$$

by direct computation and the Hessian comparison theorem, we have

$$\begin{aligned}
\mathrm{Hess}_{N_1\times N_2}(\widetilde{\rho}_1(\psi(x_j)))(e_i, e_i) &= \mathrm{Hess}_{N_1}(\rho_1(x_{j1}))(\pi_{1*}e_i, \pi_{1*}e_i)\\
&= \sum_{j=2}^{n_1} b_{ij}^2\mathrm{Hess}_{N_1}(\rho_1(x_{j1}))(E_j, E_j)\\
&\geq \sum_{j=2}^{n_1} b_{ij}^2 C_b(r) = \Big(1 - a_i^2 - \sum_{l=1}^{n_2} c_{il}^2\Big)C_b(r), \qquad (4.4)\\
|\nabla f|^2(x_j) &= \sum_{i=1}^{k}\langle \mathrm{grad}^{N_1\times N_2}\widetilde{\rho}_1(\psi(x_j)), e_i\rangle^2\\
&= \sum_{i=1}^{k}\langle \mathrm{grad}^{N_1}\rho_1, e_i\rangle^2 = \sum_{i=1}^{k} a_i^2 < \frac{1}{j^2}. \qquad (4.5)
\end{aligned}$$

We observe that at $\pi_2(x_j)$,

$$\sum_{i=1}^{k}\sum_{l=1}^{n_2} c_{il}^2 = \sum_{l=1}^{n_2} |\mathrm{grad}(x_2^l\cdot\psi)|^2 \leq \sum_{l=1}^{n_2} |\mathrm{grad}^{N_2}(x_2^l)|^2 = n_2. \tag{4.6}$$

Thus from (4.3)–(4.4), we have

$$\frac{1}{j} > \Big(k - \sum_{i=1}^{k} a_i^2 - \sum_{i,l} c_{il}^2\Big)C_b(r) - \sup_M |\overrightarrow{H}|.$$

Using (4.5) and (4.6), it follows that

$$\frac{1}{j} + \frac{C_b(r)}{j^2} + \sup_M |\overrightarrow{H}| > (k - n_2)C_b(r). \tag{4.7}$$

Letting $j \to +\infty$, we get

$$\sup_M |\overrightarrow{H}| \geq (k - n_2)C_b(r).$$

This is a contradiction. Hence, (4.1) is proved.

To prove (4.2), we assume that M is stochastically complete. Define $h : N_1 \times N_2 \to \mathbb{R}$ by

$$h(x_1, x_2) = g_b(\rho_1(x_1)),$$

where

$$g_b(s) = \begin{cases} 1 - \cos(\sqrt{b}s), & \text{if } b > 0,\ s < \dfrac{\pi}{2\sqrt{b}}, \\ s^2, & \text{if } b = 0, \\ \cosh(\sqrt{-b}s), & \text{if } b < 0. \end{cases}$$

Then $f = h \cdot \psi$ is a C^∞ bounded function on M, and by [16, Theorem 1.1], we know that there is a sequence $\{x_j\} \subset M$, such that

$$f(x_j) > \sup_M f - \frac{1}{j} \quad \text{and} \quad \Delta f(x_j) < \frac{1}{j}.$$

Similarly as before, we get

$$\begin{aligned} &\mathrm{Hess}_{N_1 \times N_2}(h(\psi(x_j)))(e_i, e_i) \\ &= \mathrm{Hess}_{N_1} g_b(\rho_1(x_{j1}))(\pi_{1*}e_i, \pi_{1*}e_i) \\ &= g_b''(\rho_1(x_{j1}))a_i^2 + g_b'(\rho_1(x_{j1})) \sum_{j=2}^{n_1} b_{ij}^2 \mathrm{Hess}_{N_1}(\rho_1)(x_{j1})(E_j, E_j) \\ &\geq g_b''(\rho_1(x_{j1}))a_i^2 + g_b'(\rho_1(x_{j1}))C_b(\rho_1(x_{j1})) \sum_{j=2}^{n_1} b_{ij}^2 \\ &= g_b''(\rho_1(x_{j1}))a_i^2 + g_b'(\rho_1(x_{j1}))C_b(\rho_1(x_{j1}))\Big(1 - a_i^2 - \sum_{l=1}^{n_2} c_{il}^2\Big) \\ &= g_b'(\rho_1(x_{j1}))C_b(\rho_1(x_{j1}))\Big(1 - \sum_{l=1}^{n_2} c_{il}^2\Big). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{j} > \Delta f(x_j) &= \sum_{i=1}^{k} \mathrm{Hess}_{N_1 \times N_2}(h(\psi(x_j)))(e_i, e_i) + \langle \overrightarrow{H}(x), \mathrm{grad}^{N_1 \times N_2} h(\psi(x_j)) \rangle \\ &\geq g_b'(\rho_1(x_{j1}))C_b(\rho_1(x_{j1}))\Big(k - \sum_{i,l} c_{il}^2\Big) + g_b'(\rho_1(x_{j1}))\langle \mathrm{grad}^{N_1} \rho_1, \overrightarrow{H} \rangle \\ &\geq g_b'(\rho_1(x_{j1}))(C_b(\rho_1(x_{j1}))\Big(k - n_2) - \sup_M |\overrightarrow{H}|\Big). \end{aligned}$$

Letting $\varepsilon \to 0$, $j \to +\infty$, and since $\lim_{j\to\infty} g_b'(\rho_1(x_{j1})) > 0$, we get

$$\sup_M |\overrightarrow{H}| \geq (k - n_2)C_b(r).$$

It is a contradiction.

Remark 4.1 This implies that when $\kappa_{N_1}^{\mathrm{rad}} \leq b$ and $\kappa_{N_2}^{\mathrm{rad}} \geq -c(1 + \rho_2^2 \log^2(\rho_2 + 2))$, where $b, c > 0$ are constants, ρ_2 is the distance function from a fixed point on N_2, there does not exist any complete proper minimal immersion $\psi : M^k \to N_1^{n_1} \times N_2^{n_2}$ $(k > n_2)$ which has bounded projection in N_1. In other words, any k-dimensional complete minimal submanifold in $N_1^{n_1} \times N_2^{n_2}$ $(k > n_2)$ with bounded projection in N_1 cannot be proper.

Remark 4.2 In [1], the authors proved similar results in $N^{n-l}\times\mathbb{R}^l$ by using the generalized Omori-Yau maximum principle due to S. Pigola, M. Rigori and A. Setti's [16]. The above theorem generalizes their results.

Corollary 4.1 (see [1, Corollary 1]) *Let $\psi : M^{n-1} \to \mathbb{R}^n$ be a complete hypersurface with mean curvature $\overrightarrow{H}$. If $\psi(M) \subset B_{\mathbb{R}^2}(r) \times \mathbb{R}^{n-2}$ and $\sup_M |\overrightarrow{H}| < \frac{1}{r}$, then ψ cannot be proper.*

Let us recall that a submanifold $f : \Sigma^n \to \mathbb{R} \times_\rho M^n$ is called two-sided if its normal bundle is trivial, namely, there is a globally defined unit normal vector field. We can then define the smooth angle function $\nu : \Sigma^n \to [-1, 1]$ by $\nu(p) = \langle N(p), \frac{\partial}{\partial t}\rangle$, where N denotes the global normal field.

Theorem 4.2 *Let $f : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a two-sided complete proper hypersurface of constant mean curvature H, where M is a complete n-dimensional Riemannian manifold of constant sectional curvature κ_0. Let $y(t) = \frac{\rho'(t)}{\rho(t)}$. Assume that $\max\{-\kappa_0, 0\} < y'(t) \le a$, $y^2(t) \le b$ (a, b are constants), and the angle function ν does not change the sign. If $f(\Sigma^n) \subset [t_1, t_2] \times M^n$, where $t_1, t_2 \in \mathbb{R}$ are finite, then $f(\Sigma^n)$ is a slice.*

Proof Using the relationship between the curvature tensors of a warped product (see [13]), by direct computation, we obtain the sectional curvature of $\mathbb{R} \times_\rho M^n$

$$\begin{aligned}
\kappa &= R(E_i, E_j, E_i, E_j)\\
&= R^{M^n}((E_i)_{M^n}, (E_j)_{M^n}, (E_i)_{M^n}, (E_j)_{M^n}) - \frac{(\rho')^2}{\rho^2}(\langle (E_i)_{M^n}, (E_i)_{M^n}\rangle\langle (E_j)_{M^n}, (E_j)_{M^n}\rangle\\
&\quad - \langle (E_i)_{M^n}, (E_j)_{M^n}\rangle\langle (E_i)_{M^n}, (E_j)_{M^n}\rangle)\\
&\quad - \frac{\rho''}{\rho}\langle (E_i)_{M^n}, (E_i)_{M^n}\rangle\Big\langle E_j, \frac{\partial}{\partial t}\Big\rangle^2 + \frac{\rho''}{\rho}\langle (E_i)_{M^n}, (E_j)_{M^n}\rangle\Big\langle E_i, \frac{\partial}{\partial t}\Big\rangle\Big\langle E_j, \frac{\partial}{\partial t}\Big\rangle\\
&\quad + \frac{\rho''}{\rho}\langle (E_i)_{M^n}, (E_j)_{M^n}\rangle\Big\langle E_i, \frac{\partial}{\partial t}\Big\rangle\Big\langle E_j, \frac{\partial}{\partial t}\Big\rangle - \frac{\rho''}{\rho}\langle (E_j)_{M^n}, (E_j)_{M^n}\rangle\Big\langle E_i, \frac{\partial}{\partial t}\Big\rangle^2\\
&= \kappa_0 - \frac{(\rho')^2}{\rho^2} - \Big(\frac{\rho''}{\rho} - \frac{(\rho')^2}{\rho^2} + \kappa_0\Big)\Big(\Big\langle E_i, \frac{\partial}{\partial t}\Big\rangle^2 + \Big\langle E_j, \frac{\partial}{\partial t}\Big\rangle^2\Big), \quad i \ne j,
\end{aligned} \tag{4.8}$$

where $\{E_i\}$ is an orthonormal frame on $\mathbb{R} \times_\rho M^n$, $(E_i)_{M^n} = E_i - \langle E_i, \frac{\partial}{\partial t}\rangle\frac{\partial}{\partial t}$.

Since $\max\{-\kappa_0, 0\} < y'(t) \le a, y^2(t) \le b$, we get

$$\kappa \ge \kappa_0 - \frac{(\rho')^2}{\rho^2} - \Big(\frac{\rho''}{\rho} - \frac{(\rho')^2}{\rho^2} + \kappa_0\Big) = -\frac{\rho''}{\rho} = -y'(t) - y^2(t) \ge -a - b.$$

Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on Σ^n and $\mathbb{R} \times_\rho M^n$ respectively, and let u be the height function of Σ^n. Then the gradient of u is

$$\nabla u = \frac{\partial}{\partial t} - \Big\langle \frac{\partial}{\partial t}, N\Big\rangle N = \frac{\partial}{\partial t} - \nu N, \tag{4.9}$$

where N is the global normal field. So we have

$$|\nabla u|^2 = 1 - \nu^2. \tag{4.10}$$

It follows from (4.9) that

$$\overline{\nabla}_{e_i}(\nabla u) = \frac{\rho'(u)}{\rho(u)}\Big(e_i - \Big\langle e_i, \frac{\partial}{\partial t}\Big\rangle\Big) - e_i(\nu)N + \nu A(e_i),$$

where $\{e_i\}$ is an orthonormal frame on Σ. Then we get

$$\begin{aligned}\Delta u &= \langle \nabla_{e_i}(\nabla u), e_i\rangle = \langle (\overline{\nabla}_{e_i}(\nabla u))^\top, e_i\rangle \\ &= \Big\langle \frac{\rho'(u)}{\rho(h)}\Big(e_i - \Big\langle e_i, \frac{\partial}{\partial t}\Big\rangle \frac{\partial}{\partial t}\Big) + \nu A(e_i), e_i\Big\rangle \\ &= y(u)\langle e_i - \langle e_i, \nabla u\rangle \nabla u, e_i\rangle + \nu H \\ &= y(u)(n - |\nabla u|^2) + \nu H. \end{aligned} \tag{4.11}$$

According to Theorem 3.2, using (4.10) and (4.11), for any $k \in \mathbb{N}$, there exists a sequence $\{x_k\} \in \Sigma^n$, such that

$$\lim_{k\to\infty} u(x_k) = \sup\ u < \infty, \tag{4.12}$$

$$|\nabla u|^2(x_k) = 1 - \nu^2(x_k) < \Big(\frac{1}{k}\Big)^2, \tag{4.13}$$

$$\Delta u(x_k) = y(u(x_k))(n - |\nabla u|^2(x_k)) + \nu(x_k)H(x_k) < \frac{1}{k}. \tag{4.14}$$

Inequality (4.13) gives that

$$\lim_{k\to\infty} \nu(x_k) = \mathrm{sgn}\ \nu. \tag{4.15}$$

Using (4.12)–(4.15), we get

$$y(\sup\ u) \le -\frac{1}{n}\mathrm{sgn}\ \nu \cdot H. \tag{4.16}$$

Similarly, applying Theorem 3.2 to $-u$, we obtain that

$$y(\inf\ u) \ge -\frac{1}{n}\mathrm{sgn}\ \nu \cdot H, \tag{4.17}$$

where $\inf u > -\infty$. From (4.16)–(4.17), we get $y(\inf\ u) = y(\sup\ u)$. Since $y'(t) > 0$, we conclude that $\sup u = \inf u$, namely, $f(\Sigma^n)$ is a slice.

Remark 4.3 In [2], the authors gave a similar result under the certain curvature condition of the hypersurface. In our given ambient space, we do not require that the Ricci curvature of the hypersurface Σ^n is bounded from below while we need the proper condition.

Corollary 4.2 *Let $f : \Sigma^n \to M^n \times \mathbb{R}$ be a complete proper hypersurface of constant mean curvature H, where M is a complete n-dimensional Riemannian manifold of constant sectional curvature κ_0. Assume that the angle function ν does not change the sign. If $f(\Sigma^n) \subset M^n \times [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$ are finite, then $f(\Sigma^n)$ is minimal.*

Proof $\rho = 1$, from (4.8), we have that the sectional curvature of $M^n \times \mathbb{R}$ is $\kappa = \kappa_0(1 - \langle E_i, \frac{\partial}{\partial t}\rangle^2 - \langle E_j, \frac{\partial}{\partial t}\rangle^2)$. So $-|\kappa_0| \le \kappa \le |\kappa_0|$. Then by (4.16)–(4.17), it follows that $H = 0$.

Corollary 4.3 *Let M^n be a complete n-dimensional Riemannian manifold of constant sectional curvature and $u : M^n \to \mathbb{R}$ be a smooth function. Let $G(u) = \{(x, u(x)) \in M^n \times \mathbb{R}; x \in$*

$M^n\}$ be a complete entire graph with constant mean curvature. If u is bounded, then the graph $G(u)$ must be minimal.

Proof Actually u is the height function of the graph. Let $\widetilde{\nabla}$ be the Levi-Civita connection on M^n. It is easy to know that the unit normal field of the graph is

$$N = \Big(- \frac{\widetilde{\nabla} u}{\sqrt{1 + |\widetilde{\nabla} u|^2}}, \frac{1}{\sqrt{1 + |\widetilde{\nabla} u|^2}} \Big).$$

It follows that $\nu = \langle N, \frac{\partial}{\partial t} \rangle = \frac{1}{\sqrt{1+|\widetilde{\nabla} u|^2}} > 0$. Then by Corollary 4.2, the graph $G(u)$ is minimal.

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