Chin. Ann. Math.
33B(1), 2012, 17–32
DOI: 10.1007/s11401-011-0694-8

Chinese Annals of Mathematics, Series B

Spacelike Graphs with Parallel Mean Curvature in Pseudo-Riemannian Product Manifolds

Zicheng ZHAO[1]

Abstract The author introduces the w-function defined on the considered spacelike graph M. Under the growth conditions $w = o(\log z)$ and $w = o(r)$, two Bernstein type theorems for M in $\mathbb{R}^{n+m}_m$ are got, where z and r are the pseudo-Euclidean distance and the distance function on M to some fixed point respectively. As the ambient space is a curved pseudo-Riemannian product of two Riemannian manifolds (Σ_1, g_1) and (Σ_2, g_2) of dimensions n and m, a Bernstein type result for $n = 2$ under some curvature conditions on Σ_1 and Σ_2 and the growth condition $w = o(r)$ is also got. As more general cases, under some curvature conditions on the ambient space and the growth condition $w = o(r)$ or $w = o(\sqrt{r})$, the author concludes that if M has parallel mean curvature, then M is maximal.

Keywords Product manifold, Spacelike graph, Parallel mean curvature, Maximal, Bernstein
2000 MR Subject Classification 53C42, 53C50

1 Introduction

Bernstein theorem says that the only entire minimal graph in $\mathbb{R}^{n+1}$ is a hyperplane when $n \leq 7$ (due to Bernstein for $n = 2$, de Giorgi [8] for $n = 3$, Almgren [1] for $n = 4$, and Simons [15] for $n = 5, 6, 7$). Bombieri, de Giorgi and Giusti [2] gave counterexamples when $n > 7$. Chern [7] and Flanders [10] proved independently that the only entire graphic hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature must be minimal.

In Lorentz-Minkowski space, there is also the Bernstein result, which says that the only entire maximal hypersurface in $\mathbb{R}^{n+1}_1$ is a hyperplane (see [3] for $n \leq 4$ and [6] for all n). Jost and Xin [11] extended it to a higher codimension, which is as follows.

Theorem 1.1 (see [11]) *Let M be a spacelike extremal n-submanifold in $\mathbb{R}^{n+m}_m$. If M is closed with respect to the Euclidean topology, then M has to be a linear subspace.*

Besides the hyperboloids, Treibergs [16] constructed many nonlinear examples of complete spacelike hypersurfaces with nonzero constant mean curvature. On the other hand, Xin [18] showed that when a constant mean curvature spacelike hypersurface M in $\mathbb{R}^{n+1}_1$ has a bounded Gauss image, it must be a hyperplane. Later, Xin and Ye [22] improved this result by proving that when the Gauss image lies in a horoball in the hyperbolic space, M will also be a hyperplane. In [19], Xin extended the result in [18] to higher codimension, that is, if an n-dimensional spacelike submanifold M in $\mathbb{R}^{n+m}_m$ has parallel mean curvature and a bounded Gauss image, it

Manuscript received May 27, 2011. Revised September 7, 2011.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: 071018016@fudan.edu.cn

must be an n-plane. By relaxing the boundedness of the Gauss image to a controlled growth, Dong [9] proved that M is still an n-plane. For more details on spacelike submanifolds in pseudo-Euclidean space, please consult Xin's book [17].

To state our results, let us introduce the w-function at first.

Let $N = (\Sigma_1 \times \Sigma_2, g_1 - g_2)$ be the pseudo-Riemannian product,where (Σ_1, g_1) and (Σ_2, g_2) are Riemannian manifolds of dimensions n and m. Let K^{Σ_i}, Ric^{Σ_i}, R^{Σ_i} denote the sectional curvature, Ricci curvature and curvature tensor of Σ_i respectively, $i = 1, 2$. Let $M = \{(x, f(x)) : x \in \Sigma_1\}$ be a spacelike graph over Σ_1, where $f : \Sigma_1 \to \Sigma_2$ is a smooth map. For any $p \in \Sigma_1$, df is a linear map from $T_p\Sigma_1$ to $T_{f(p)}\Sigma_2$. As in [14], we can use singular value decomposition to find the orthonormal basis $\{a_i\}$ for $T_p\Sigma_1$, and $\{a_\alpha\}$ for $T_{f(p)}\Sigma_2$, such that

$$df(a_i) = \lambda_i a_{n+i}.$$

Since M is spacelike, $|\lambda_i| < 1$. Notice that $\lambda_i = 0$ when $i > \min\{n, m\}$. Set

$$\begin{aligned} e_i &= \frac{1}{\sqrt{1-\lambda_i^2}}(a_i + \lambda_i a_{n+i}), \\ e_\alpha &= \frac{1}{\sqrt{1-\lambda_{\alpha-n}^2}}(\lambda_{\alpha-n} a_{\alpha-n} + a_\alpha). \end{aligned}$$

Then $e_i \in T_pM$ and $e_\alpha \in T_p^\perp M$ are Lorentzian bases of N at p. Define the w-function as

$$w = \langle e_1 \wedge \cdots \wedge e_n,\ a_1 \wedge \cdots \wedge a_n \rangle.$$

Obviously, it is independent of the choice of the orthonormal basis of Σ_1 and the orthonormal basis of M. Notice that we also have

$$w = \frac{1}{\sqrt{\prod_i (1-\lambda_i^2)}}. \tag{1.1}$$

In [13], Li and Salavessa defined $\cosh\theta = \frac{1}{\sqrt{\det(g_1 - f^* g_2)}}$, where θ is called the hyperbolic angle. When the ambient space is $\mathbb{R}^{n+m}_m$, Dong [9] defined a function

$$*\Omega = \frac{1}{\sqrt{\det\left(I - \sum\limits_{\alpha=n+1}^{n+m} f^\alpha_{x_i} f^\alpha_{x_j}\right)}}.$$

They are essentially the same as w.

Our definition here is similar to the one in [21], where Xin and Yang defined it for a submanifold in Euclidean space. The role of w here is somewhat like $v = \frac{1}{w}$ in [21] (see also [12]).

When the ambient space is $\mathbb{R}^{n+m}_m$, we can define the w-function in a way parallel to [21], and in this way, we do not need the graph condition.

Let $G^m_{n,m}$ be the pseudo-Grassmannian manifold of all spacelike n-subspaces in $\mathbb{R}^{n+m}_m$. It is a symmetric space of non-compact type. Fix $P_0 \in G^m_{n,m}$, which is spanned by a unit spacelike n-vector $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$. For any $P \in G^m_{n,m}$, spanned by a unit n-vector $e_1 \wedge \cdots \wedge e_n$, we define a function $\widetilde{w}$ on $G^m_{n,m}$ by

$$\widetilde{w}(P) = \langle P, P_0 \rangle = \langle e_1 \wedge \cdots \wedge e_n,\ \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle.$$

For an n-dimensional complete spacelike submanifold M in $\mathbb{R}^{n+m}_m$, we define the generalized Gauss map $\gamma : M \to G^m_{n,m}$ by

$$\gamma(x) = T_x M \in G^m_{n,m}.$$

Then, define $w = \widetilde{w} \circ \gamma$. When M is a graph in this case, we can take $\varepsilon_1, \cdots, \varepsilon_n$ to be an orthonormal basis of $\mathbb{R}^n$. Then it is easy to see that the two definitions are equivalent.

Now we state our first two results as follows.

Theorem 1.2 *Let M be a spacelike submanifold of $\mathbb{R}^{n+m}_m$ with parallel mean curvature, which is closed with respect to the Euclidean topology of $\mathbb{R}^{n+m}_m$. Let $z = \langle X, X \rangle$ be the pseudo-Euclidean distance of $\mathbb{R}^{n+m}_m$, where $X \in \mathbb{R}^{n+m}_m$ is the position vector. Assume $0 \in M$. If the w-function satisfies*

$$\lim_{z \to +\infty} \frac{w(x)}{\log z(x)} = 0,$$

when z is restricted to M, then M must be an n-plane.

Theorem 1.3 *Let M be a spacelike submanifold of $\mathbb{R}^{n+m}_m$ with parallel mean curvature, which is closed with respect to the Euclidean topology of $\mathbb{R}^{n+m}_m$. If the w-function satisfies*

$$\lim_{r \to +\infty} \frac{w(x)}{r(x)} = 0,$$

where $r(x)$ is the distance function of M with respect to some fixed point x_0, then M must be an n-plane.

Under the assumptions that M is an entire graph with parallel mean curvature and $w = o(\rho)$, where $\rho = \sqrt{\Sigma x_i^2}$ is the Euclidean distance of $\mathbb{R}^n$, Dong [9] concluded that M is an n-plane. It is easy to see that $\rho^2 \geq z$ and $\rho \geq r$ (here we take x_0 as $f(0)$ in Theorem 1.3, where 0 is the origin of $\mathbb{R}^n$), so the conditions on w in Theorems 1.2 and 1.3 all imply that $\lim\limits_{\rho \to +\infty} \frac{w}{\rho} = 0$. Consequently, we can get the above two theorems by Dong's result when M is an entire graph. In Section 4, we give their proofs in another way. Our method is also valid when the ambient space is a curved pseudo-Riemannian product manifold.

When N is a curved pseudo-Riemannian product manifold, Salavessa [14] proved, under some condition on the second fundamental form at infinity, that if a spacelike graphic submanifold M has parallel mean curvature, and the Cheeger constant of M is zero, then M is maximal. Li and Salavessa [13] proved the following theorems.

Theorem 1.4 (see [13]) *If M is a complete maximal spacelike graphic surface, and for each $p \in \Sigma_1$, $K^{\Sigma_1}(p) \geq \max\{0, K^{\Sigma_2}(f(p))\}$, then M is totally geodesic.*

Theorem 1.5 (see [13]) *Assume that M is a complete spacelike graph with parallel mean curvature, and for any $p \in \Sigma_1$, $\mathrm{Ric}^{\Sigma_1}(p) \geq 0$, $K^{\Sigma_1}(p) \geq K^{\Sigma_2}(f(p))$. If K^{Σ_1}, K^{Σ_2} and w are all bounded, then M is maximal.*

By considering some special cases of Σ_1 and Σ_2, we can relax the condition on w in Theorem 1.5 to $w = o(r)$ in the following theorem, and also conclude the maximal results. Furthermore, by Theorem 1.4, we can get a Bernstein type result for the first case.

Theorem 1.6 *Let $M = \{(p, f(p)) :\ p \in \Sigma_1\}$ be a complete spacelike graphic submanifold with parallel mean curvature. Assume that the w-function satisfies*

$$\lim_{r\to+\infty} \frac{w(x)}{r(x)} = 0,$$

where $r(x)$ is the distance function of M with respect to some fixed point x_0. Then

(i) *If $n = 2$, that is, Σ_1 is a Riemannian surface, and $K_1^\Sigma \geq 0$, $K_2^\Sigma \leq 0$, M must be totally geodesic.*

(ii) *If $\Sigma_1 = \mathbb{R}^n$, Σ_2 is a Riemannian surface with $K_2^\Sigma \leq 0$, then M must be maximal.*

(iii) *If $N = \mathbb{R}^n \times \mathbb{H}^m$, the pseudo-Riemannian product of the Euclidean space and the hyperbolic space, M must be maximal.*

When Σ_2 is 1-dimensional, Σ_1 is complete, $\mathrm{Ric}^{\Sigma_1} \geq 0$, Li and Salavessa [13, Proposition 2] also proved that M is maximal provided $w = o(r)$.

When the conditions on Σ_1 and Σ_2 are the same as in Theorem 1.5, we can relax the condition on w to $w = o(r^{\frac{1}{2}})$, and get the following theorem.

Theorem 1.7 *Let $M = \{(p, f(p)) :\ p \in \Sigma_1\}$ be a complete spacelike graphic submanifold of $N = (\Sigma_1 \times \Sigma_2, g_1 - g_2)$, which has parallel mean curvature. If*

(i) *K^{Σ_1} and K^{Σ_2} are bounded, and for any $p \in \Sigma_1$, $\mathrm{Ric}^{\Sigma_1}(p) \geq 0$, $K^{\Sigma_1}(p) \geq K^{\Sigma_2}(f(p))$.*

(ii) *the w-function satisfies*

$$\lim_{r\to+\infty} \frac{w(x)}{\sqrt{r(x)}} = 0,$$

where $r(x)$ is the distance function of M with respect to some fixed point x_0, then M must be maximal.

Finally, we point out that our proofs of the theorems depend on various generalized maximal principles, which we give as the lemmas in Section 3. Those are interesting in their own right and would be useful in other problems.

2 Local Formulas

Let N be an $(n+m)$-dimensional pseudo-Riemannian manifold of index m. Let M be an n-dimensional spacelike submanifold of N. We choose a local Lorentzian frame field $e_1, \cdots, e_{n+m}$ in N, such that when restricted to M, $e_1, \cdots, e_n$ is a tangent frame field. Let $\omega_1, \cdots, \omega_{n+m}$ be its dual frame field. We agree with the following range of indices:

$$A, B, C, \cdots = 1, \cdots, n+m; \quad i, j, k, \cdots = 1, \cdots, n; \quad \alpha, \beta, \gamma, \cdots = n+1, \cdots, n+m.$$

Then the pseudo-Riemannian metric of N is given by

$$ds^2 = \sum \omega_i^2 - \sum \omega_\alpha^2 = \sum \epsilon_A \omega_A^2,$$

where $\epsilon_i = 1$, $\epsilon_\alpha = -1$. The structure equations of N are given by

$$\begin{aligned} d\omega_A &= \sum \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= \sum \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum \epsilon_C \epsilon_D K_{ABCD} \omega_C \wedge \omega_D. \end{aligned} \tag{2.1}$$

When restricted to M, $\omega_\alpha = 0$, we may put

$$\omega_{i\alpha} = h^\alpha_{ij}\omega_j, \tag{2.2}$$

where h^α_{ij} are components of the second fundamental form of M in N. The induced Riemannian metric of M is given by $ds^2_M = \sum \omega_i^2$, and the structure equations of M are

$$\begin{aligned} d\omega_i &= \sum \omega_{ij} \wedge \omega_j, \\ d\omega_{ij} &= \sum \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum R_{ijkl}\omega_k \wedge \omega_l. \end{aligned} \tag{2.3}$$

From (2.1) and (2.3), we have the Gauss equation

$$R_{ijkl} = K_{ijkl} - \sum (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}). \tag{2.4}$$

The covariant derivative of h^α_{ij} is defined by

$$\sum h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum h^\alpha_{ik}\omega_{kj} + \sum h^\alpha_{kj}\omega_{ki} - \sum h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.5}$$

The exterior differentiation of (2.2) gives

$$\sum dh^\alpha_{ij} \wedge \omega_j = \sum \Big(- h^\alpha_{ik}\omega_{kj} - h^\alpha_{jk}\omega_{ki} + h^\beta_{ij}\omega_{\beta\alpha} - \frac{1}{2}K_{i\alpha kj}\omega_k \Big) \wedge \omega_j. \tag{2.6}$$

From (2.5) and (2.6), we have the Codazzi equation

$$h^\alpha_{ijk} - h^\alpha_{ikl} = K_{i\alpha jk}. \tag{2.7}$$

The mean curvature vector of M in N is defined by

$$H = \frac{1}{n}h^\alpha_{ii}e_\alpha.$$

If

$$DH = \frac{1}{n}h^\alpha_{iik}\omega_k e_\alpha \equiv 0,$$

M is said to have parallel mean curvature. If $H = 0$, M is called a maximal spacelike submanifold.

Let $a_1, \cdots, a_{n+m}$ be another local Lorentzian frame field, and $\theta_1, \cdots, \theta_{n+m}$ be its dual frame field. Denote θ_{AB} as the connection forms of $\theta_1, \cdots, \theta_{n+m}$. Let $\omega = (\epsilon_B\omega_{AB})$, $\theta = (\epsilon_B\theta_{AB})$ (we do not take the sum with respect to B here), which are $(n+m)\times(n+m)$ matrices. Write

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_{n+m} \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_{n+m} \end{pmatrix}.$$

Then there exists a reversible matrix A, such that

$$e = Aa. \tag{2.8}$$

Take its covariant differentiation. Then

$$\omega \otimes e = (dA + A\theta) \otimes a = (dA + A\theta)A^{-1} \otimes e,$$

so,

$$\omega = (dA)A^{-1} + A\theta A^{-1}.$$

From

$$0 = d(AA^{-1}) = (dA)A^{-1} + AdA^{-1},$$

we get

$$dA^{-1} = -A^{-1}(dA)A^{-1},$$

so

$$\begin{aligned} d\omega &= -dA \wedge dA^{-1} + dA \wedge \theta A^{-1} + A(d\theta)A^{-1} - A\theta dA^{-1} \\ &= (dA)A^{-1} \wedge (dA)A^{-1} + dA \wedge \theta A^{-1} + A(d\theta)A^{-1} + A\theta \wedge A^{-1}(dA)A^{-1}, \\ \omega \wedge \omega &= (dA)A^{-1} \wedge (dA)A^{-1} + A\theta \wedge \theta A^{-1} + dA \wedge \theta A^{-1} + A\theta A^{-1} \wedge (dA)A^{-1}. \end{aligned}$$

Then

$$d\omega - \omega \wedge \omega = A(d\theta - \theta \wedge \theta)A^{-1},$$

that is

$$\epsilon_B \epsilon_C \epsilon_D K_{ABCD} \omega_C \wedge \omega_D = A_{AE}(\epsilon_F \epsilon_G \epsilon_H K^{\theta}_{EFGH} \theta_G \wedge \theta_H) A^{-1}_{FB}.$$

So

$$\begin{aligned} \epsilon_B \epsilon_C \epsilon_D K_{ABCD} &= \epsilon_F \epsilon_G \epsilon_H A_{AE} K^{\theta}_{EFGH} A^{-1}_{FB} \begin{vmatrix} \theta_G(e_C) & \theta_H(e_C) \\ \theta_G(e_D) & \theta_H(e_D) \end{vmatrix} \\ &= \epsilon_F \epsilon_G \epsilon_H A_{AE} K^{\theta}_{EFGH} A^{-1}_{FB} (A_{CG} A_{DH} - A_{CH} A_{DG}), \end{aligned} \tag{2.9}$$

where K^θ is the curvature tensor of N with respect to the basis $\{a_i, a_\alpha\}$.

3 Generalized Maximal Principles

We state some propositions first, which will be used in the proofs of the following lemmas.

Proposition 3.1 (see [11]) *Let $z = \langle X, X \rangle$ be the pseudo-Euclidean distance of $\mathbb{R}^{n+m}_m$, where $X \in \mathbb{R}^{n+m}_m$ is the position vector. Let M be an n-dimensional spacelike submanifold of $\mathbb{R}^{n+m}_m$ with parallel mean curvature. Assume that M is closed with respect to the Euclidean topology of $\mathbb{R}^{n+m}_m$, and $0 \in M$. Then z is a proper function on M.*

Proposition 3.2 (see [11]) *Let M be a space-like submanifold in pseudo-Euclidean space $\mathbb{R}^{n+m}_m$ of index m with parallel mean curvature. Let z be the pseudo-distance function on M. If for some $k > 0$, the set $\{z \le k\}$ is compact, then there is a constant b depending only on the dimension n and the norm of the mean curvature $|H|$, such that for all $x \in M$ with $z \le \frac{k}{2}$,*

$$|\nabla z| \le b(z + 1).$$

Proposition 3.3 (see [4]) *Let M be a complete Riemannian manifold of dimension n with $\mathrm{Ricc}|_x \ge -cF(r)$, where $c > 0$ is constant, r is the distance from a fixed point x_0 to the point x, $F : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and $F \ge 1$. If x is not on the cut locus of the point x_0, then for $r(x) \ge r(x_0)$,*

$$\Delta r(x) \le \sqrt{1 + (n-1)cF(r(x))}.$$

Lemma 3.1 *Let $z = \langle X, X\rangle$ be the pseudo-Euclidean distance of $\mathbb{R}^{n+m}_m$, where $X \in \mathbb{R}^{n+m}_m$ is the position vector. Let M be an n-dimensional spacelike submanifold of $\mathbb{R}^{n+m}_m$ with parallel mean curvature. Assume that M is closed with respect to the Euclidean topology, and $0 \in M$. Then for any C^2-function f defined on M satisfying*

$$\lim_{z(x)\to+\infty} \frac{f(x)}{\log z(x)} = 0, \tag{3.1}$$

when z is restricted to M, there exists a sequence $\{q_k\}$ in M, such that

$$\lim_{k\to\infty} f(q_k) = \sup f, \quad \lim_{k\to\infty} |\nabla f(q_k)| = 0, \quad \lim_{k\to\infty} \Delta f(q_k) \le 0.$$

Proof By Proposition 3.1, we know that $z \ge 0$, and is proper on M. Together with Proposition 3.2, we have

$$|\nabla z(x)| \le c_1(z(x)+1), \quad \forall x \in M, \tag{3.2}$$

where c_1 is a constant depending on n and $|H|$ only. By [20, (2.24)],

$$\Delta z(x) \le c_2 z(x) + c_3, \quad \forall x \in M, \tag{3.3}$$

where c_2, c_3 are also constants depending on n and $|H|$ only.

Let $\{\epsilon_k\}$ be a sequence of positive numbers, such that $\epsilon_k \to 0$ as $k \to \infty$. Let

$$f_k(x) = f(x) - \epsilon_k \log(z(x)+1). \tag{3.4}$$

Then by the condition on f, $f_k \to -\infty$ as $z \to +\infty$. Since z is proper, the set $\{z(x) \le C : x \in M\}$ is compact for any constant $C > 0$, so f_k has a lower bound, say A, on it. Then there is a constant $\widetilde{C} \ge C$ such that $f_k(x) < A$ for $x \in \{z(x) \ge \widetilde{C} : x \in M\}$, so f_k attains its maximum at some point $q_k \in \{z(x) \le \widetilde{C} : x \in M\}$, and thus,

$$\nabla f_k(q_k) = 0, \quad \Delta f_k(q_k) \le 0. \tag{3.5}$$

From (3.2)–(3.5), we have

$$\begin{aligned} \lim_{k\to\infty} |\nabla f(q_k)| &= \lim_{k\to\infty} \epsilon_k \frac{|\nabla z(q_k)|}{z(q_k)+1} = 0, \\ \lim_{k\to\infty} \Delta f(q_k) &\le \lim_{k\to\infty} \epsilon_k \Big(\frac{\Delta z(q_k)}{z(q_k)+1} - \frac{|\nabla z(q_k)|^2}{(z(q_k)+1)^2} \Big) = 0. \end{aligned}$$

If there is a subsequence $\{q_{k_l}\} \ne \{q_k\}$, such that $\lim\limits_{l\to\infty} f(q_{k_l}) = \sup f$, then, by still denoting $\{q_{k_l}\}$ as $\{q_k\}$, our proof is completed. Otherwise, we claim that $\lim\limits_{k\to\infty} f(q_k) = \sup f$. In fact, if this were not true, then for an arbitrary big k_0, we can find $q \in M$ and $\delta \ge 0$, such that

$$f(q) - \delta \ge f(q_k), \quad k \ge k_0. \tag{3.6}$$

Since

$$f_k(q_k) = f(q_k) - \epsilon_k \log(z(q_k)+1) \ge f_k(q) = f(q) - \epsilon_k \log(z(q)+1),$$

we have

$$f(q_k) \geq f(q) + \epsilon_k(\log(z(q_k)+1) - \log(z(q)+1)).$$

If there is a subsequence of $\{q_k\}$ (it may be $\{q_k\}$ itself), which we still denote as $\{q_k\}$, such that $z(q_k) \to +\infty$, then

$$\log(z(q_k)+1) - \log(z(q)+1) > 0$$

for k big enough, and then $f(q_k) > f(q)$, which contradicts (3.6).

If $z(q_k)$ is bounded, then

$$\lim_{k\to\infty} \epsilon_k(\log(z(q_k)+1) - \log(z(q)+1)) = 0,$$

so $f(q_k) \geq f(q)$, which is again a contradiction to (3.6). Thus we complete the proof.

Lemma 3.2 *Let M be a complete Riemannian manifold of dimension n with Ricci curvature bounded below by $-c(r(x)+1)^{2\alpha}$, where $r(x)$ is the distance function of M with respect to some fixed point x_0, and $c > 0$ and $0 \leq \alpha < 1$ are constants. Then for any C^2-function f defined on M satisfying*

$$\lim_{r\to+\infty} \frac{f(x)}{r^{1-\alpha}(x)} = 0,$$

there exists a sequence $\{q_k\}$ in M, such that

$$\lim_{k\to\infty} f(q_k) = \sup f, \quad \lim_{k\to\infty} |\nabla f(q_k)| = 0, \quad \lim_{k\to\infty} \Delta f(q_k) \leq 0.$$

Proof By Proposition 3.3, we get

$$\Delta r(x) \leq \sqrt{1+(n-1)c(1+r(x))^{2\alpha}}, \tag{3.7}$$

when x is not the cut locus of x_0.

Let $\{\epsilon_k\}$ be a sequence of positive numbers, such that $\epsilon_k \to 0$ as $k \to \infty$. Let

$$f_k(x) = f(x) - \epsilon_k(1+r(x))^{1-\alpha}. \tag{3.8}$$

Then by the condition on f, $f_k \to -\infty$ as $r \to +\infty$, so f_k attains its maximum at some point q_k. As in [5], we can assume that f_k is C^2 in a neighborhood of q_k, and thus,

$$\nabla f_k(q_k) = 0, \quad \Delta f_k(q_k) \leq 0. \tag{3.9}$$

From (3.7)–(3.9), we have

$$\begin{aligned}
\lim_{k\to\infty} |\nabla f(q_k)| &= \lim_{k\to\infty} \epsilon_k(1-\alpha)\frac{|\nabla r(q_k)|}{(1+r(q_k))^{\alpha}} = 0, \\
\lim_{k\to\infty} \Delta f(q_k) &\leq \lim_{k\to\infty} \epsilon_k(1-\alpha)\Big(\frac{\Delta r(q_k)}{(1+r(q_k))^{\alpha}} - \alpha\frac{|\nabla r(q_k)|^2}{(1+r(q_k))^{1+\alpha}}\Big) \\
&\leq \lim_{k\to\infty} \epsilon_k(1-\alpha)\frac{\sqrt{1+(n-1)c(1+r(q_k))^{2\alpha}}}{(1+r(q_k))^{\alpha}} = 0.
\end{aligned}$$

If there is a subsequence $\{q_{k_l}\} \neq \{q_k\}$, such that $\lim\limits_{l\to\infty} f(q_{k_l}) = \sup f$, then, by still denoting $\{q_{k_l}\}$ as $\{q_k\}$, our proof is completed. Otherwise, we claim that $\lim\limits_{k\to\infty} f(q_k) = \sup f$. In fact, if this were not true, then for an arbitrary big k_0, we can find $q \in M$ and $\delta \geq 0$, such that

$$f(q) - \delta \geq f(q_k), \quad k \geq k_0. \tag{3.10}$$

Since

$$f_k(q_k) = f(q_k) - \epsilon_k(1 + r(q_k))^{1-\alpha} \geq f_k(q) = f(q) - \epsilon_k(1 + r(q))^{1-\alpha},$$

we have

$$f(q_k) \geq f(q) + \epsilon_k((1 + r(q_k))^{1-\alpha} - (1 + r(q))^{1-\alpha}).$$

If there is a subsequence of $\{q_k\}$ (it may be $\{q_k\}$ itself), which we still denote as $\{q_k\}$, such that $r(q_k) \to +\infty$, then

$$(1 + r(q_k))^{1-\alpha} - (1 + r(q))^{1-\alpha} > 0$$

for k big enough, and then $f(q_k) > f(q)$, which contradicts (3.10).

If $r(q_k)$ is bounded, then

$$\lim_{k\to\infty} \epsilon_k((1 + r(q_k))^{1-\alpha} - (1 + r(q))^{1-\alpha}) = 0,$$

so $f(q_k) \geq f(q)$, which is again a contradiction to (3.10). Thus we complete the proof.

4 Proofs of Main Theorems

Proof of Theorem 1.2 Since M is closed with respect to the Euclidean topology, and has parallel mean curvature, by [11, Theorem 3.3], M is complete. So M is an entire graph (for details, see [20, Section 2]).

The following proof is divided into two steps. In Step 1, we calculate the Laplacian of w, and give it a nonnegative lower bound in terms of w, λ_i, and the components of the second fundamental form. In Step 2, we use Lemma 3.1 to get a sequence $\{q_k\}$ such that $w(q_k) \to \sup w$, $|\nabla w(q_k)| \to 0$, $\Delta w \to 0$ as $k \to +\infty$. Then by careful analysis, we get that $|B|^2(q_k) \to 0$, and thus $H = 0$. Then by Theorem 1.1, we conclude that M is an n-plane.

Step 1 Let

$$w_{i\alpha} = \langle e_1 \wedge \cdots \wedge e_\alpha \wedge \cdots \wedge e_n,\ a_1 \wedge \cdots \wedge a_n\rangle,$$

which is got by substituting e_α for e_i in w. And get $w_{i\alpha j\beta}$ by substituting e_β for e_j in $w_{i\alpha}$. Then

$$w_{i\alpha} = \begin{cases} \lambda_i w, & \alpha = n+i, \\ 0, & \alpha \neq n+i, \end{cases} \qquad w_{i\alpha j\beta} = \begin{cases} \lambda_i\lambda_j w, & \alpha = n+i, \beta = n+j, \\ -\lambda_i\lambda_j w, & \alpha = n+j, \beta = n+i, \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

Now we go on to calculate Δw.

$$\begin{aligned} dw &= \sum_i \langle e_1 \wedge \cdots \wedge (de_i) \wedge \cdots \wedge e_n,\ a_1 \wedge \cdots \wedge a_n\rangle \\ &= -\sum_{i\alpha} \omega_{i\alpha} w_{i\alpha} = -\sum_{ik\alpha} h^\alpha_{ik} w_{i\alpha}\omega_k, \\ D\Big(-\sum_{i\alpha} h^\alpha_{ik} w_{i\alpha}\Big) &= -\sum_{i\alpha} d(h^\alpha_{ik} w_{i\alpha}) - \sum_{il\alpha} h^\alpha_{il} w_{i\alpha}\omega_{lk} \\ &= -\sum_{i\alpha}\Big(dh^\alpha_{ik} + \sum_l h^\alpha_{il}\omega_{lk}\Big) w_{i\alpha} - \sum_{i\alpha} h^\alpha_{ik} dw_{i\alpha}, \end{aligned}$$

$$\begin{aligned}
dw_{i\alpha} &= d\langle e_1\wedge\cdots\wedge e_\alpha\wedge\cdots\wedge e_n,\ a_1\wedge\cdots\wedge a_n\rangle\\
&= -\sum_{\substack{j\neq i\\ \beta}}\omega_{j\beta}\langle e_1\wedge\cdots\wedge e_\alpha\wedge\cdots\wedge e_\beta\wedge\cdots\wedge e_n,\ a_1\wedge\cdots\wedge a_n\rangle\\
&\quad+\sum_{j\neq i}\omega_{ji}\langle e_1\wedge\cdots\wedge e_\alpha\wedge\cdots\wedge e_i\wedge\cdots\wedge e_n,\ a_1\wedge\cdots\wedge a_n\rangle\\
&\quad+\omega_{\alpha i}w-\sum_{\beta}\omega_{\alpha\beta}\langle e_1\wedge\cdots\wedge e_\beta\wedge\cdots\wedge e_n,\ a_1\wedge\cdots\wedge a_n\rangle\\
&= -\sum_{\substack{j\neq i\\ \beta}}\omega_{j\beta}w_{i\alpha j\beta}-\sum_{j\neq i}\omega_{ji}w_{j\alpha}+\omega_{\alpha i}w-\sum_{\beta}\omega_{\alpha\beta}w_{i\beta},
\end{aligned}$$

so

$$D\Big(-\sum_{i\alpha}h^{\alpha}_{ik}w_{i\alpha}\Big)=\sum_{l}\Big(-\sum_{i\alpha}h^{\alpha}_{ikl}w_{i\alpha}+\sum_{\substack{i\neq j\\ \alpha\beta}}h^{\alpha}_{ik}h^{\beta}_{jl}w_{i\alpha j\beta}+\sum_{i\alpha}h^{\alpha}_{ik}h^{\alpha}_{il}w\Big)\omega_l,$$

and

$$\begin{aligned}
\Delta w &= -\sum_{ik\alpha}h^{\alpha}_{ikk}w_{i\alpha}+\sum_{\substack{i\neq j\\ k\alpha\beta}}h^{\alpha}_{ik}h^{\beta}_{jk}w_{i\alpha j\beta}+\sum_{ik\alpha}h^{\alpha}_{ik}h^{\alpha}_{ik}w\\
&= -\sum_{ik\alpha}(h^{\alpha}_{kki}+K_{k\alpha ik})w_{i\alpha}+\sum_{\substack{i\neq j\\ k\alpha\beta}}h^{\alpha}_{ik}h^{\beta}_{jk}w_{i\alpha j\beta}+\sum_{ik\alpha}(h^{\alpha}_{ik})^2w. \qquad (4.2)
\end{aligned}$$

From (4.1), we have

$$\sum_{\substack{i\neq j\\ k\alpha\beta}}h^{\alpha}_{ik}h^{\beta}_{jk}w_{i\alpha j\beta}=\sum_{\substack{i\neq j\\ k}}\lambda_i\lambda_j w(h^{n+i}_{ik}h^{n+j}_{jk}-h^{n+j}_{ik}h^{n+i}_{jk}). \qquad (4.3)$$

When the mean curvature vector is parallel,

$$\sum_{k}h^{\alpha}_{kki}=0. \qquad (4.4)$$

Since $\mathbb{R}^{n+m}_m$ is flat, we also have

$$-\sum_{ik\alpha}K_{k\alpha ik}w_{i\alpha}=0, \qquad (4.5)$$

and from (4.2)–(4.5),

$$\begin{aligned}
\Delta w &\geq w\Big(|B|^2+\sum_{\substack{i\neq j\\ k}}\lambda_i\lambda_j(h^{n+i}_{ik}h^{n+j}_{jk}-h^{n+j}_{ik}h^{n+i}_{jk})\Big)\\
&= w\Big(|B|^2+\sum_{k}\Big(\sum_{i}\lambda_i h^{n+i}_{ik}\Big)^2-\sum_{ik}(\lambda_i h^{n+i}_{ik})^2-2\sum_{\substack{i<j\\ k}}\lambda_i\lambda_j h^{n+i}_{jk}h^{n+j}_{ik}\Big), \qquad (4.6)
\end{aligned}$$

where $|B|^2$ is the square length of the second fundamental form. Let $h^{n+i}_{jk}=0$, when $i>m$. We rewrite $|B|^2$ as

$$|B|^2=\sum_{ik}(h^{n+i}_{ik})^2+\sum_{\substack{i<j\\ k}}((h^{n+i}_{jk})^2+(h^{n+j}_{ik})^2), \qquad (4.7)$$

as in [14]. Then from (4.6)–(4.7) and (1.1),

$$\begin{aligned}
\Delta w \geq\ & w\Big(\sum_{ik}(h_{ik}^{n+i})^2+\sum_{\substack{i<j\\k}}((h_{jk}^{n+i})^2+(h_{ik}^{n+j})^2),\\
&+\sum_k\Big(\sum_i\lambda_i h_{ik}^{n+i}\Big)^2-\sum_{ik}(\lambda_i h_{ik}^{n+i})^2-2\sum_{\substack{i<j\\k}}|\lambda_i\lambda_j h_{jk}^{n+i}h_{ik}^{n+j}|\Big)\\
\geq\ & w\sum_{ik}(1-\lambda_i^2)(h_{ik}^{n+i})^2+w\sum_{\substack{i<j\\k}}(1-|\lambda_i\lambda_j|)((h_{jk}^{n+i})^2+(h_{ik}^{n+j})^2)\\
\geq\ & w\sum_{ik}(1-\lambda_i^2)(h_{ik}^{n+i})^2+\frac{1}{2}w\sum_{\substack{i<j\\k}}(1-\lambda_i^2+1-\lambda_j^2)((h_{jk}^{n+i})^2+(h_{ik}^{n+j})^2)\\
\geq\ & w\sum_{ik}(1-\lambda_i^2)(h_{ik}^{n+i})^2+\frac{1}{2}\sum_{\substack{i<j\\k}}\Big(\frac{\sqrt{1-\lambda_i^2}}{\sqrt{1-\lambda_j^2}}+\frac{\sqrt{1-\lambda_j^2}}{\sqrt{1-\lambda_i^2}}\Big)((h_{jk}^{n+i})^2+(h_{ik}^{n+j})^2)\\
\geq\ & w\sum_{ik}(1-\lambda_i^2)(h_{ik}^{n+i})^2+\sum_{\substack{i<j\\k}}((h_{jk}^{n+i})^2+(h_{ik}^{n+j})^2). \qquad (4.8)
\end{aligned}$$

Step 2 Since $\Delta w\geq 0$, by Lemma 3.1, we can get a sequence $\{q_l\}\subset M$, such that

$$\lim_{l\to\infty}w(q_l)=\sup w,\quad \lim_{l\to\infty}|\nabla w(q_l)|=0,\quad \lim_{l\to\infty}\Delta w(q_l)=0. \qquad (4.9)$$

From (4.8)–(4.9), we have

$$\begin{aligned}
&\lim_{l\to\infty}h_{jk}^{n+i}(q_l)=0\quad\text{for } i\neq j \text{ and } \forall k, \qquad (4.10)\\
&\lim_{l\to\infty}\sum_i w(q_l)(1-\lambda_i^2(q_l))(h_{ii}^{n+i}(q_l))^2=0.
\end{aligned}$$

In the following, we will conclude that $\lim\limits_{l\to\infty}h_{ii}^{n+i}(q_l)=0$. Thus, $H=0$.

(I) If for some I_0,

$$\lim_{l\to\infty}w(q_l)(1-\lambda_{\mathrm{I}_0}^2(q_l))=0,$$

since

$$w(1-\lambda_{\mathrm{I}_0}^2)=\frac{\sqrt{1-\lambda_{\mathrm{I}_0}^2}}{\sqrt{\prod\limits_{i\neq\mathrm{I}_0}(1-\lambda_i^2)}}\geq\frac{\sqrt{1-\lambda_{\mathrm{I}_0}^2}}{\sqrt{1-\lambda_i^2}},\quad i\neq\mathrm{I}_0,$$

$$w(1-\lambda_i^2)=\frac{\sqrt{1-\lambda_i^2}}{\sqrt{\prod\limits_{j\neq i}(1-\lambda_j^2)}}\geq\frac{\sqrt{1-\lambda_i^2}}{\sqrt{1-\lambda_{\mathrm{I}_0}^2}},\quad i\neq\mathrm{I}_0,$$

we have

$$\lim_{l\to\infty}w(q_l)(1-\lambda_i^2(q_l))=+\infty,\quad i\neq\mathrm{I}_0.$$

So

$$\lim_{l\to\infty}h_{ii}^{n+i}(q_l)=0,\quad i\neq\mathrm{I}_0. \qquad (4.11)$$

By (4.10), we have

$$\lim_{l\to\infty}\lambda_i(q_l)h^{n+i}_{i\mathrm{I}_0}(q_l)=0,\quad i\neq \mathrm{I}_0.$$

Since $\lim\limits_{l\to\infty}|\nabla w(q_l)|=0$, i.e., $\lim\limits_{l\to\infty}w(q_l)\sum\limits_i\lambda_i(q_l)h^{n+i}_{ik}(q_l)=0, \forall k$, we can let $k=\mathrm{I}_0$, and together with the above equation, we have

$$\lim_{l\to\infty}\lambda_{\mathrm{I}_0}(q_l)h^{n+\mathrm{I}_0}_{\mathrm{I}_0\mathrm{I}_0}(q_l)=0.$$

Since $\lim\limits_{l\to\infty}\lambda_{\mathrm{I}_0}=1$, we have

$$\lim_{l\to\infty}h^{n+\mathrm{I}_0}_{\mathrm{I}_0\mathrm{I}_0}(q_l)=0. \tag{4.12}$$

Thus, by (4.10)–(4.12), we finally get

$$H=\lim_{l\to\infty}H(q_l)=0,$$

since M has parallel mean curvature.

(II) If for all i,

$$\lim_{l\to\infty}w(q_l)(1-\lambda_i^2(q_l))\neq 0,$$

there exists an $\epsilon_1>0$, and a subsequence $\{q_l^1\}$ of $\{q_l\}$, such that

$$w(q_l^1)(1-\lambda_1^2(q_l^1))>\epsilon_1.$$

If for some $i>1$,

$$\lim_{l\to\infty}w(q_l^1)(1-\lambda_i^2(q_l^1))=0,$$

then by (I), we can get $H=0$. Otherwise, we continue to choose a subsequence $\{q_l^2\}$ of $\{q_l^1\}$, such that for some $\epsilon_2>0$,

$$w(q_l^2)(1-\lambda_2^2(q_l^2))>\epsilon_2.$$

Continuing this process, we finally get a subsequence of $\{q_l\}$, which we still denote as $\{q_l\}$, and $\epsilon_0>0$, such that

$$w(q_l)(1-\lambda_i^2(q_l))>\epsilon_0,\quad \forall i.$$

So, we can conclude that

$$\lim_{l\to\infty}h^{n+i}_{ii}(q_l)=0.$$

Together with (4.10), we again get

$$H=0.$$

Finally, by Theorem 1.1, we conclude that M is an n-plane.

Proof of Theorem 1.3 First M is an entire graph. Then as Step 1 in the proof of Theorem 1.2, we also have (4.8)

$$\Delta w\geq w\sum_{ik}(1-\lambda_i^2)(h^{n+i}_{ik})^2+\sum_{\substack{i<j\\ k}}((h^{n+i}_{jk})^2+(h^{n+j}_{ik})^2).$$

As $K_{ijkl} = 0$, from the Gauss equation, we have

$$\begin{aligned}
\mathrm{Ric}^M(f_i, f_i) &= -\sum_{\alpha}(h^\alpha_{ii}h^\alpha_{jj} - h^\alpha_{ij}h^\alpha_{ji}) \\
&= \sum_{\alpha}\Big(h^\alpha_{ii} - \frac{1}{2}\sum_{j} h^\alpha_{jj}\Big)^2 - \frac{n^2}{4}|H|^2 + \sum_{\substack{j\neq i\\ \alpha}}(h^\alpha_{ij})^2 \\
&\geq -\frac{n^2}{4}|H|^2,
\end{aligned}$$

where $\{f_1, \cdots, f_n\}$ is an orthonormal basis of M that diagonalizes Ric^M. So M and w satisfy the conditions in Lemma 3.2 with $\alpha = 0$. Again, we can get a sequence $\{q_l\}$, such that

$$\lim_{l\to\infty} w(q_l) = \sup w, \quad \lim_{l\to\infty} |\nabla w(q_l)| = 0, \quad \lim_{l\to\infty} \Delta w(q_l) = 0.$$

Then following the proof of Theorem 1.2, we can get that M is maximal. By Theorem 1.1, M is an n-plane.

Proof of Theorem 1.6 We also divide the proof into two steps.

Step 1 We have that (4.2)

$$\Delta w = -\sum_{ik\alpha}(h^\alpha_{kki} + K_{k\alpha ik})w_{i\alpha} + \sum_{\substack{i\neq j\\ k\alpha\beta}} h^\alpha_{ik}h^\beta_{jk}w_{i\alpha j\beta} + \sum_{ik\alpha}(h^\alpha_{ik})^2 w$$

still holds. Now we show that

$$-\sum_{ik\alpha} K_{k\alpha ik}w_{i\alpha} \geq 0.$$

Without loss of generality, we can assume $n \geq m$, and write

$$\begin{aligned}
\begin{pmatrix} e_1 \\ \vdots \\ e_{n+m} \end{pmatrix} &= \begin{pmatrix}
\frac{1}{\sqrt{1-\lambda_1^2}} & & & & & \frac{\lambda_1}{\sqrt{1-\lambda_1^2}} & & \\
& \ddots & & & & & \ddots & \\
& & \frac{1}{\sqrt{1-\lambda_m^2}} & & & & & \frac{\lambda_m}{\sqrt{1-\lambda_m^2}} \\
& & & \ddots & & & & \\
& & & & \frac{1}{\sqrt{1-\lambda_n^2}} & & & \\
\frac{\lambda_1}{\sqrt{1-\lambda_1^2}} & & & & & \frac{1}{\sqrt{1-\lambda_1^2}} & & \\
& \ddots & & & & & \ddots & \\
& & \frac{\lambda_m}{\sqrt{1-\lambda_m^2}} & & & & & \frac{1}{\sqrt{1-\lambda_m^2}}
\end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{n+m} \end{pmatrix} \\
&= A \begin{pmatrix} a_1 \\ \vdots \\ a_{n+m} \end{pmatrix} = \begin{pmatrix} C & D \\ D^{\mathrm{T}} & E \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{n+m} \end{pmatrix}.
\end{aligned}$$

Then

$$A^{-1} = \begin{pmatrix} C & -D \\ -D^{\mathrm{T}} & E \end{pmatrix}.$$

From (2.9), we have

$$
\begin{aligned}
-K_{k\alpha ik} &= \epsilon_F\epsilon_G\epsilon_H A_{kE}K^{\theta}_{EFGH}A^{-1}_{F\alpha}(A_{iG}A_{kH}-A_{iH}A_{kG})\\
&= A_{kj}K^{\theta}_{jlrs}A^{-1}_{l\alpha}(A_{ir}A_{ks}-A_{is}A_{kr})\\
&\quad - A_{k\beta}K^{\theta}_{\beta\gamma\sigma\tau}A^{-1}_{\gamma\alpha}(A_{i\sigma}A_{k\tau}-A_{i\tau}A_{k\sigma})\\
&= \frac{2\lambda_{\alpha-n}}{\sqrt{(1-\lambda_i^2)(1-\lambda_{\alpha-n}^2)}(1-\lambda_k^2)}K^{\theta}_{k(\alpha-n)ki}\\
&\quad + \frac{2\lambda_k^2\lambda_i}{\sqrt{(1-\lambda_i^2)(1-\lambda_{\alpha-n}^2)}(1-\lambda_k^2)}K^{\theta}_{(n+k)\alpha(n+k)(n+i)}. \qquad (4.13)
\end{aligned}
$$

Let Π_{ij} and Π'_{ij} be the two planes spanned by $\{a_i,a_j\}$ and $\{a_{n+i},a_{n+j}\}$ respectively. Notice that

$$
K^{\theta}_{kiki}=R^{\Sigma_1}_{kiki},\quad K^{\theta}_{(n+k)(n+i)(n+k)(n+i)}=-R^{\Sigma_2}_{(n+k)(n+i)(n+k)(n+i)}. \qquad (4.14)
$$

Then by (4.1), (4.13)–(4.14), we have

$$
\begin{aligned}
-\sum_{ik\alpha}K_{k\alpha ik}w_{i\alpha} &= \sum_{ik}\frac{2\lambda_k^2\lambda_i^2}{(1-\lambda_i^2)(1-\lambda_k^2)}K^{\theta}_{(n+k)(n+i)(n+k)(n+i)}w+\frac{2\lambda_i^2}{(1-\lambda_i^2)(1-\lambda_k^2)}K^{\theta}_{kiki}w\\
&= w\Big(\frac{2\lambda_k^2\lambda_i^2}{(1-\lambda_i^2)(1-\lambda_k^2)}(K^{\Sigma_1}(\Pi_{ik})-K^{\Sigma_2}(\Pi'_{ik}))+\frac{2\lambda_i^2}{(1-\lambda_i^2)}\mathrm{Ric}^{\Sigma_1}(a_i)\Big)\\
&\geq 0,
\end{aligned}
$$

since for any $p\in\Sigma_1$, $\mathrm{Ric}^{\Sigma_1}(p)\geq 0$, and $K^{\Sigma_1}(p)\geq K^{\Sigma_2}(f(p))$. Then we also have that (4.8) holds.

Step 2 Let $\{f_1,\cdots,f_n\}$ be an orthonormal basis of M that diagonalizes Ric^M at a given point, and by the Gauss equation, we have

$$
\begin{aligned}
\mathrm{Ric}^M(f_i,f_i) &= \sum_{j\neq i}\Big(K(f_i,f_j,f_i,f_j)-\sum_{\alpha}(h^{\alpha}_{ii}h^{\alpha}_{jj}-h^{\alpha}_{ij}h^{\alpha}_{ji})\Big)\\
&= \sum_{j\neq i}K(f_i,f_j,f_i,f_j)+\sum_{\alpha}\Big(h^{\alpha}_{ii}-\frac{1}{2}\sum_j h^{\alpha}_{jj}\Big)^2-\frac{n^2}{4}|H|^2+\sum_{\substack{j\neq i\\ \alpha}}(h^{\alpha}_{ij})^2. \qquad (4.15)
\end{aligned}
$$

Suppose $f_i=P_{ij}e_j$, where (P_{ij}) is an orthogonal matrix. Then

$$
\begin{aligned}
\sum_{j\neq i}K(f_i,f_j,f_i,f_j) &= \sum_{\substack{j\neq i\\ klrs}}P_{ik}P_{jl}P_{ir}P_{js}K(e_k,e_l,e_r,e_s)\\
&= \sum_{klr}P_{ik}P_{ir}K(e_k,e_l,e_r,e_l). \qquad (4.16)
\end{aligned}
$$

As we get (4.13) in the above, we have

$$
K(e_k,e_l,e_r,e_l)=\frac{R^{\Sigma_1}(a_k,a_l,a_r,a_l)-\lambda_k\lambda_r\lambda_l^2R^{\Sigma_2}(a_{n+k},a_{n+l},a_{n+r},a_{n+l})}{\sqrt{(1-\lambda_k^2)(1-\lambda_r^2)}(1-\lambda_l^2)}. \qquad (4.17)
$$

Case (i)

$$
\begin{aligned}
&R^{\Sigma_1}(a_k,a_l,a_r,a_l)=0,\quad R^{\Sigma_2}(a_{n+k},a_{n+l},a_{n+r},a_{n+l})=0,\quad k\neq r,\\
&R^{\Sigma_1}(a_k,a_l,a_r,a_l)\geq 0,\quad R^{\Sigma_2}(a_{n+k},a_{n+l},a_{n+r},a_{n+l})\leq 0,\quad k=r.
\end{aligned}
$$

Case(ii)

$$R^{\Sigma_1}(a_k,a_l,a_r,a_l)=0,\quad R^{\Sigma_2}(a_{n+k},a_{n+l},a_{n+r},a_{n+l})\begin{cases}=0, & r\neq k,\\ \leq 0, & r=k.\end{cases}$$

Case (iii)

$$R^{\Sigma_1}(a_k,a_l,a_r,a_l)=0,\quad R^{\Sigma_2}(a_{n+k},a_{n+l},a_{n+r},a_{n+l})=\begin{cases}0, & r\neq k,\\ -2, & r=k.\end{cases}$$

So, in all the cases, we can get $K(e_k,e_l,e_r,e_l)\geq 0$ from (4.17). And from (4.16), $K(f_i,f_j,f_i,f_j)\geq 0$, and finally from (4.15), $\mathrm{Ric}^M(f_i,f_i)\geq -\frac{n^2}{4}|H|^2$. Thus M and w satisfy the conditions in Lemma 3.2 with $\alpha=0$, and we can use Lemma 3.2 to get a sequence $\{q_l\}$, such that

$$\lim_{l\to\infty} w(q_l)=\sup w,\quad \lim_{l\to\infty}|\nabla w(q_l)|=0,\quad \lim_{l\to\infty}\Delta w(q_l)=0.$$

Then following the proof of Theorem 1.2, we can get that M is maximal. This completes the proof of Cases (ii) and (iii). Using Theorem 1.4, we can complete the proof of Case (i).

Proof of Theorem 1.7 As in the proof of Theorem 1.6, we have

$$-\sum_{ik\alpha}K_{k\alpha ik}w_{i\alpha}\geq 0.$$

Thus, (4.8) still holds.

Choosing $f_1,\cdots,f_n$ as in the proof of Theorem 1.6, we also have that (4.15)–(4.17) hold. Since K^{Σ_1}, K^{Σ_2} are bounded, we have $R^{\Sigma_1}(a_k,a_l,a_r,a_l)$ and $R^{\Sigma_2}(a_{n+k},a_{n+l},a_{n+r},a_{n+l})$ are bounded too. Then,

$$|K(e_k,e_l,e_r,e_l)|\leq cw^2 \tag{4.18}$$

for some positive constant c. From the condition

$$\lim_{r\to+\infty}\frac{w(x)}{\sqrt{r(x)}}=0,$$

we have

$$w\leq\sqrt{r},\quad \text{when } r\geq r_0 \tag{4.19}$$

for some constant $r_0>0$. Then by (4.15)–(4.16) and (4.18)–(4.19), we get

$$\mathrm{Ric}^M(f_i,f_i)\geq -c(1+r).$$

So M and w satisfy the conditions of Lemma 3.2 with $\alpha=\frac{1}{2}$. Thus we again have a sequence $\{q_l\}$, such that

$$\lim_{l\to\infty} w(q_l)=\sup w,\quad \lim_{l\to\infty}|\nabla w(q_l)|=0,\quad \lim_{l\to\infty}\Delta w(q_l)=0.$$

Following the proof of Theorem 1.2, we complete this proof.

Acknowledgements The author wishes to express his sincere thanks to his thesis advisor, Professor Y. L. Xin, for his guidance and advice on this work. He would also like to thank all the members in our differential geometry seminar for discussions and encouragement.

References

[1] Almgren, F. J. Jr., Some interior regularity theorems for minimal surfaces and an extension of Bernsteins theorem, *Ann. Math.*, **84**, 1966, 277–292.

[2] Bombieri, E., de Giorgi, E. and Giusti, E., Minimal cones and the Bernstein problem, *Invent. Math.*, **7**, 1969, 243–268.

[3] Calabi, E., Examples of Bernstein problems for some nonlinear equations, *Proc. Symp. Pure Math.*, **15**, 1970, 223–230.

[4] Chen, Q. and Xin, Y. L., A generalized maximal principle and its applications in geometry, *Amer. J. Math.*, **114**, 1992, 355–366.

[5] Cheng, S. Y. and Yau, S. T., Differential equations on Riemannian manifolds and geometric applications, *Comm. Pure Appl. Math.*, **28**, 1975, 333–354.

[6] Cheng, S. Y. and Yau, S. T., Maximal spacelike hypersurfaces in the Lorentz-Minkowski space, *Ann. Math.*, **104**, 1976, 407–419.

[7] Chern, S. S., On the Curvature of a Piece of Hypersurface in Euclidean Space, Abh. Math. Sem., Hamburg, **29**, 1964.

[8] de Giorgi, E., Una estensione del teorema di Bernstein, *Ann. Scuola Norm. Sup. Pisa.*, **19**, 1965, 79–85.

[9] Dong, Y. X., Bernstein theorems for space-like graphs with parallel mean curvature and controlled growth, *J. Geom. Phys.*, **58**, 2008, 324–333.

[10] Flanders, H., Remark on mean curvature, *J. London Math. Soc.*, **41**, 1966, 364–366.

[11] Jost, J. and Xin, Y. L., Some aspects of the global geometry of entire spacelike submaniflods, *Results Math.*, **40**, 2001, 233–245.

[12] Jost, J., Xin, Y. L. and Yang, L., The Gauss image of entire graphs of higher codimension and Bernstein type theorems, *Calc. Var. Part. Diff. Eq.*, to appear. arXiv: 1009.3901v1

[13] Li, G. H. and Salavessa, I., Graphic Bernstein results in curved pseudo-Riemannian manifolds, *J. Geom. Phys.*, **59**, 2009, 1306–1313.

[14] Salavessa, I., Spacelike graphs with parallel mean curvature, *Bull. Belg. Math. Soc.*, **15**, 2008, 65–76.

[15] Simons, J., Minimal varieties in Riemannian manifolds, *Ann. Math.*, **88**, 1968, 62–105.

[16] Triebergs, A., Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, *Invent. Math.*, **66**, 1982, 39–56.

[17] Xin, Y. L., Minimal Submanifolds and Related Topics, World Scientific Publishing, Singapore, 2003.

[18] Xin, Y. L., On Gauss image of a spacelike hypersurface with constant mean curvature in Minkowski space, *Comment. Math. Helv.*, **66**, 1991, 590–598.

[19] Xin, Y. L., A rigidity theorem for a space-like graph of higher codimension, *Manuscripta Math.*, **103**, 2000, 191–202.

[20] Xin, Y. L., Mean curvature flow with bounded Gauss image, *Results Math.*, **59**, 2011, 415–436.

[21] Xin, Y. L. and Yang, L., Convex functions on Grassmannian manifolds and Lawson-Osserman problem, *Adv. Math.*, **219**(4), 2008, 1298–1326.

[22] Xin, Y. L. and Ye, R. G., Bernstein-type theorems for space-like surfaces with parallel mean curvature, *J. Reine Angew. Math.*, **489**, 1997, 189–198.