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On the Conditions of Extending Mean Curvature Flow*

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Abstract The authors consider a family of smooth immersions $F(\,\cdot\,,t): M^n \to \mathbb{R}^{n+1}$ of closed hypersurfaces in $\mathbb{R}^{n+1}$ moving by the mean curvature flow $\frac{\partial F(p,t)}{\partial t} = -H(p,t)\cdot\nu(p,t)$ for $t \in [0,T)$. They show that if the norm of the second fundamental form is bounded above by some power of mean curvature and the certain subcritical quantities concerning the mean curvature integral are bounded, then the flow can extend past time T. The result is similar to that in [6–9].

Keywords Mean curvature flow, Moser iteration, Type I singularity
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1 Introduction

Let M^n be a compact n-dimensional hypersurface without boundary, and let $F_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of M^n into $\mathbb{R}^{n+1}$. Consider a smooth one-parameter family of immersions

$$F(\,\cdot\,,t): M^n \to \mathbb{R}^{n+1}$$

satisfying

$$F(\,\cdot\,,0) = F_0(\,\cdot\,)$$

and

$$\frac{\partial F(p,t)}{\partial t} = -H(p,t)\nu(p,t), \quad \forall (p,t) \in M \times [0,T). \tag{1.1}$$

Here $H(p,t)$ and $\nu(p,t)$ denote the mean curvature and a choice of unit normal for the hypersurface $M_t = F(M^n,t)$ at $F(p,t)$. We sometimes also write $x(p,t) = F(p,t)$ and refer to (1.1) as to the mean curvature flow equation. For any compact n-dimensional hypersurface M^n which is smoothly embedded in $\mathbb{R}^{n+1}$ by $F : M^n \to \mathbb{R}^{n+1}$, let us denote by $g = (g_{ij})$ the induced metric, $A = (h_{ij})$ the second fundamental form, $\mathrm{d}\mu = \sqrt{\det\,(g_{ij})}\,\mathrm{d}x$ the volume form, ∇ the induced Levi-Civita connection and Δ the induced Laplacian. Then the mean curvature of M^n is given by $H = g^{ij}h_{ij}$. We use the following notation throughout the whole paper:

$$\|v\|_{L^{p,q}(M\times[0,T))} := \Big(\int_0^T \Big(\int_{M_t} |v|^p\,\mathrm{d}\mu\Big)^{\frac{q}{p}}\mathrm{d}t\Big)^{\frac{1}{q}}$$

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for a function $v(\cdot, t)$ defined on $M \times [0, T)$.

Without any special assumptions on M_0, the mean curvature flow (1.1) will in general develop singularities in finite time, characterized by a blow-up of the second fundamental form $A(\cdot, t)$.

Theorem 1.1 (see [4]) *Suppose that $T < \infty$ is the first singularity time for a compact mean curvature flow. Then $\sup\limits_{M_t} |A|(\cdot, t) \to \infty$ as $t \to T$.*

In [7, 9–10], it was proved that at the first singularity time of the mean curvature flow, certain scaling invariant quantities blow up. Specifically, the results are as follows.

Theorem 1.2 *Suppose that $T < \infty$ is the first singularity time for a compact mean curvature flow. Let p and q be positive numbers satisfying $\frac{n}{p} + \frac{2}{q} = 1$. Then $\|A\|_{L^{p,q}(M\times[0,t))} \to \infty$ as $t \to T$. In particular, for $p = q = n + 2$, one has $\int_0^t \int_{M_s} |A|^{n+2} \mathrm{d}\mu \mathrm{d}s \to \infty$ as $t \to T$.*

Theorem 1.3 *Let $F_t : M^n \to \mathbb{R}^{n+1}$ be a solution to the mean curvature flow of closed hypersurfaces on a finite time interval $[0, T)$. If*

(1) there is a positive constant B such that $h_{ij} \geq -B g_{ij}$ for $(x, t) \in M \times [0, T)$;

(2) $\|H\|_{\alpha,M\times[0,T)} = (\int_0^T \int_{M_t} |H|^\alpha \mathrm{d}\mu \mathrm{d}t)^{\frac{1}{\alpha}} < +\infty$ for some $\alpha \geq n + 2$,

then this flow can be extended over time T.

The proof of Theorem 1.2 was completed by using a blow-up argument, and the proof of Theorem 1.3 was based on the Moser iteration and a blow-up argument.

In [8], N. Q. Le and N. Sesum established the blow-up of the mean curvature H at the first singular time of the mean curvature flow in the case of type I singularities. This result somewhat extends that of Huisken [5] on the blow-up of the second fundamental form at the first singular time of the mean curvature flow. Before stating their results, we first recall the following definition.

Definition 1.1 *We say that the mean curvature flow (1.1) is of type I at the first singular time $T < \infty$, if the blow-up rate of the curvature satisfies an upper bound of the form*

$$\max_{M_t} |A|^2(\cdot, t) \leq \frac{C_0}{T - t}, \quad 0 \leq t < T \tag{1.2}$$

for all $t \in [0, T)$.

In [8], N. Q. Le and N. Sesum proved the following results.

Theorem 1.4 (see [8, Theorem 1.2]) *Assume (1.2) for the mean curvature flow (1.1). If*

$$\max_{M_t} |H|^2(\cdot, t) \leq C_0,$$

then the flow can be extended past time T.

Theorem 1.5 (see [8, Theorem 1.3]) *Assume (1.2) for the mean curvature flow (1.1). If*

$$\int_0^T \int_{M_t} |H|^{n+2} \mathrm{d}\mu \mathrm{d}t < \infty,$$

then the flow can be extended past time T.

In [6], the following theorem was improved by N. Q. Le.

Theorem 1.6 (see [6]) *Assume that for the mean curvature flow (1.1), we have*

$$\int_0^T \int_{M_t} \frac{|A|^{n+2}}{\ln(2 + |A|)} \mathrm{d}\mu \mathrm{d}t < \infty.$$

Then the flow can be extended past time T.

The proof of Theorem 1.6 was based on the Moser iteration and a Gronwall-type argument on $\sup\limits_{x\in M_t}|A|(x,t)$.

A national question is whether we can get a result similar to Theorem 1.6 by improving Theorem 1.3 or Theorem 1.5. Luckily, we get it as follows.

Theorem 1.7 *Let $F_t: \ M^n \longrightarrow \mathbb{R}^{n+1}$ be a solution of the mean curvature flow of closed hypersurfaces on a finite time interval $[0,T)$. If*

(1) $|A|^2(x,t) \le C_*|H|^\tau(x,t) + C_{**}, \ (x,t)\in M\times[0,T)$,

(2) $\int_0^T\int_{M_t}\frac{|H|^{n+2}}{\ln(2+|H|)}\mathrm{d}\mu\mathrm{d}t < \infty$,

where $2 \le \tau < 2+\frac{2}{n+2}$ and C_*, C_{**} are uniform positive constants, then this flow can be extended over time T.

Remark 1.1 In condition (1), if we choose $\tau = 2$, from the second fundamental form bounded from below all the way to T, M_0 being a mean convex hypersurface or being a starshaped hypersurface, we can obtain the condition. So our result recovers Theorem 1.3.

Remark 1.2 Considering the inequality (2.15), one can see that the upper bound of τ is optimal.

The proof of Theorem 1.7 is also based on the Moser iteration and a Gronwall-type argument on $\sup\limits_{x\in M_t}|H|(x,t)$. Our method is from [6–7].

The rest of the paper is organized as follows. In Section 2, we use Sobolev type inequality for mean curvature flow to prove the reverse Hölder inequality, and use the reverse Hölder inequality to prove Harnack inequality by Moser iteration. In Section 3, we prove a critical proposition. The proof of Theorem 1.7 is carried out in Section 4.

2 Reverse Hölder and Harnack Inequalities

In this section, we state a soft version of reverse Hölder inequality (see Proposition 2.1) and a Harnack inequality (see Proposition 2.2) for parabolic inequality (2.2) during the mean curvature flow.

Lemma 2.1 (see [6]) (Sobolev Type Inequality for Mean Curvature Flow) *For all nonnegative Lipschitz functions v, one has*

$$\begin{aligned}&\|v\|^{\beta}_{L^\beta(M\times[0,T))}\\ &\le c_n \max_{0\le t\le T}\|v\|^{\frac{4}{n}}_{L^2(M_t)}\Big(\|\nabla v\|^2_{L^2(M\times[0,T))} + \max_{0\le t\le T}\|v\|^2_{L^2(M_t)}\|H\|^{\frac{2(n+3)}{3}}_{L^{n+3,\frac{2(n+3)}{3}}(M\times[0,T))}\Big),\end{aligned} \tag{2.1}$$

where $\beta := \frac{2(n+2)}{n}$.

We start with the differential inequality

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)H^2 \le 2|A|^2H^2. \tag{2.2}$$

Let $v = H^2$, and let $\eta(t,x)$ be a smooth function with the property that $\eta(0,x) = 0$ for all x.

Proposition 2.1 *Let*

$$C_1 = \Big(1+\|H\|^{\frac{2(n+3)}{3}}_{L^{n+3,\frac{2(n+3)}{3}}(M\times[0,T))}\Big)^{\frac{n}{n+2}}, \quad C_2 = \Big(\int_0^T\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{\tau}{n+3}}, \tag{2.3}$$

$\beta > 1$ be a fixed number, and $q = \frac{n+3}{\tau}$, $2 \le \tau < 2 + \frac{2}{n+2}$ and $|A|^2 \le C_*|H|^\tau + C_{**}$. Then there exists a positive constant $C_a(n, \tau, C_2, C_1, C_*, C_{**})$, such that

$$\|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(M\times[0,T))} \le C_a \Lambda(\beta)^{1+\nu} \Big\| v^\beta \Big(\eta^2 + |\nabla\eta|^2 + 2\eta\Big|\Big(\frac{\partial}{\partial t} - \Delta\Big)\eta\Big|\Big)\Big\|_{L^1(M\times[0,T))}, \tag{2.4}$$

where

$$\nu = \frac{n+2}{2q-(n+2)} = \frac{\tau(n+2)}{2(n+3)-(n+2)\tau} > 0,$$

and $\Lambda(\beta)$ is a positive constant depending on β, such that $\Lambda(\beta) \ge 1$ if $\beta \ge 2$ (e.g., we can choose $\Lambda(\beta) = 100\beta$).

In fact, we can choose

$$C_a(n, \tau, C_2, C_1, C_*, C_{**}) = (2c_{(n,C_*,C_{**})}(C_2+1)C_1)^{1+\nu}. \tag{2.5}$$

This proposition can be proved in a way similar to the proof of Lemma 4.1 in [7] by using the Sobolev type inequality for the mean curvature flow established in Lemma 2.1. We give the proof here.

Proof of Proposition 2.1 We use $\eta^2 v^{\beta-1}$ as a test function in the inequality

$$-\Delta v + \frac{\partial v}{\partial t} \le 2|A|^2 v.$$

It follows that, for any $s \in (0, T]$, we have

$$\int_0^s \int_{M_t} (-\Delta v)\eta^2 v^{\beta-1} \mathrm{d}\mu\mathrm{d}t + \int_0^s \int_{M_t} \frac{\partial v}{\partial t}\eta^2 v^{\beta-1}\mathrm{d}\mu\mathrm{d}t \le \int_0^s \int_{M_t} 2|A|^2\eta^2 v^\beta \mathrm{d}\mu\mathrm{d}t. \tag{2.6}$$

By integrating by parts, we note

$$\int_{M_t} (-\Delta v)\eta^2 v^{\beta-1}\mathrm{d}\mu = \int_{M_t} 2\langle\nabla v, \nabla\eta\rangle\eta v^{\beta-1}\mathrm{d}\mu + (\beta-1)\int_{M_t} \eta^2 v^{\beta-2}|\nabla v|^2 \mathrm{d}\mu. \tag{2.7}$$

Using the evolution of the volume form $\partial_t \mathrm{d}\mu = -H^2\mathrm{d}\mu$ and recalling the properties of η, we get

$$\begin{aligned}\int_0^s \int_{M_t} \frac{\partial v}{\partial t}\eta^2 v^{\beta-1}\mathrm{d}\mu\mathrm{d}t &= \frac{1}{\beta}\int_0^s \int_{M_t} \frac{\partial(v^\beta)}{\partial t}\eta^2\mathrm{d}\mu\mathrm{d}t \\ &= \frac{1}{\beta}\int_{M_t} v^\beta\eta^2\mathrm{d}\mu\Big|_0^s - \frac{1}{\beta}\int_0^s\int_{M_t} v^\beta\partial_t(\eta^2\mathrm{d}\mu)\mathrm{d}t \\ &= \frac{1}{\beta}\int_{M_s} v^\beta\eta^2\mathrm{d}\mu - \frac{1}{\beta}\int_0^s\int_{M_t} v^\beta\Big(2\eta\frac{\partial\eta}{\partial t} - H^2\Big)\mathrm{d}\mu\mathrm{d}t.\end{aligned} \tag{2.8}$$

Therefore, we deduce from (2.6)–(2.8) the following inequality:

$$\begin{aligned}&\int_0^s \int_{M_t} (2\langle\nabla v, \nabla\eta\rangle\eta v^{\beta-1} + (\beta-1)\eta^2 v^{\beta-2}|\nabla v|^2)\mathrm{d}\mu\mathrm{d}t + \frac{1}{\beta}\int_{M_s} v^\beta\eta^2\mathrm{d}\mu \\ &\le \frac{1}{\beta}\int_0^s\int_{M_t} v^\beta 2\eta\frac{\partial\eta}{\partial t}\mathrm{d}\mu\mathrm{d}t + \int_0^s\int_{M_t} 2|A|^2\eta^2 v^\beta\mathrm{d}\mu\mathrm{d}t.\end{aligned} \tag{2.9}$$

As will be seen later, because we can get good control of the quantity $(\frac{\partial}{\partial t}-\Delta)\eta$ for suitable choice of η, it is more convenient to make this term appear on the right-hand side of (2.9). Observe that, integrating by parts yields

$$\begin{aligned}
&\frac{1}{\beta}\int_0^s\int_{M_t} v^\beta 2\eta\frac{\partial\eta}{\partial t}\mathrm{d}\mu\mathrm{d}t\\
=&\frac{1}{\beta}\int_0^s\int_{M_t}\Big(v^\beta 2\eta\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta+v^\beta 2\eta\Delta\eta\Big)\mathrm{d}\mu\mathrm{d}t\\
=&\frac{1}{\beta}\int_0^s\int_{M_t}\Big(v^\beta 2\eta\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta-2\nabla(v^\beta\eta)\nabla\eta\Big)\mathrm{d}\mu\mathrm{d}t\\
=&\frac{1}{\beta}\int_0^s\int_{M_t}\Big(v^\beta 2\eta\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta-2v^\beta|\nabla\eta|^2-2\beta\langle\nabla v,\nabla\eta\rangle\eta v^{\beta-1}\Big)\mathrm{d}\mu\mathrm{d}t\\
\le&\frac{1}{\beta}\int_0^s\int_{M_t} v^\beta 2\eta\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\mathrm{d}\mu\mathrm{d}t-\int_0^s\int_{M_t}2\eta\langle\nabla v,\nabla\eta\rangle v^{\beta-1}\mathrm{d}\mu\mathrm{d}t.
\end{aligned}$$

Then (2.9) implies

$$\begin{aligned}
&\int_0^s\int_{M_t}\Big(4\langle\nabla v,\nabla\eta\rangle\eta v^{\beta-1}+(\beta-1)\eta^2v^{\beta-2}|\nabla v|^2\Big)\mathrm{d}\mu\mathrm{d}t+\frac{1}{\beta}\int_{M_s}v^\beta\eta^2\mathrm{d}\mu\\
\le&\frac{1}{\beta}\int_0^s\int_{M_t}v^\beta 2\eta\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big|\mathrm{d}\mu\mathrm{d}t+\int_0^s\int_{M_t}2|A|^2\eta^2v^\beta\mathrm{d}\mu\mathrm{d}t. \qquad (2.10)
\end{aligned}$$

Using the Cauchy-Schwartz inequality

$$\int_0^s\int_{M_t}4\langle\nabla v,\nabla\eta\rangle\eta v^{\beta-1}\mathrm{d}\mu\mathrm{d}t\ge-2\varepsilon^2\int_0^s\int_{M_t}\eta^2v^{\beta-2}|\nabla v|^2\mathrm{d}\mu\mathrm{d}t-\frac{2}{\varepsilon^2}\int_0^s\int_{M_t}v^\beta|\nabla\eta|^2\mathrm{d}\mu\mathrm{d}t,$$

we get from (2.10) that

$$\begin{aligned}
&\int_0^s\int_{M_t}(\beta-1-2\varepsilon^2)\eta^2v^{\beta-2}|\nabla v|^2\mathrm{d}\mu\mathrm{d}t+\frac{1}{\beta}\int_{M_s}v^\beta\eta^2\mathrm{d}\mu\\
\le&\frac{1}{\beta}\int_0^s\int_{M_t}v^\beta 2\eta\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big|\mathrm{d}\mu\mathrm{d}t+\int_0^s\int_{M_t}2|A|^2\eta^2v^\beta\mathrm{d}\mu\mathrm{d}t\\
&+\frac{2}{\varepsilon^2}\int_0^s\int_{M_t}v^\beta|\nabla\eta|^2\mathrm{d}\mu\mathrm{d}t. \qquad (2.11)
\end{aligned}$$

Choosing $\varepsilon^2=\frac{\beta-1}{4}$ and observing $|\nabla(v^{\frac{\beta}{2}})|^2=\frac{\beta^2}{4}v^{\beta-2}|\nabla v|^2$ yield

$$\begin{aligned}
&2\Big(1-\frac{1}{\beta}\Big)\int_0^s\int_{M_t}\eta^2|\nabla(v^{\frac{\beta}{2}})|^2\mathrm{d}\mu\mathrm{d}t+\int_{M_s}v^\beta\eta^2\mathrm{d}\mu\\
\le&\int_0^s\int_{M_t}v^\beta 2\eta\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big|\mathrm{d}\mu\mathrm{d}t+\beta\int_0^s\int_{M_t}2|A|^2\eta^2v^\beta\mathrm{d}\mu\mathrm{d}t\\
&+\frac{8\beta}{\beta-1}\int_0^s\int_{M_t}v^\beta|\nabla\eta|^2\mathrm{d}\mu\mathrm{d}t.
\end{aligned}$$

Combining the previous estimate with

$$|\nabla(\eta v^{\frac{\beta}{2}})|^2\le 2\eta^2|\nabla(v^{\frac{\beta}{2}})|^2+2v^\beta|\nabla\eta|^2$$

implies

$$\begin{aligned}
&\Big(1-\frac{1}{\beta}\Big)\int_0^s\int_{M_t}|\nabla(\eta v^{\frac{\beta}{2}})|^2\mathrm{d}\mu\mathrm{d}t+\int_{M_s}v^\beta\eta^2\mathrm{d}\mu\\
\le&\int_0^s\int_{M_t}v^\beta 2\eta\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big|\mathrm{d}\mu\mathrm{d}t+\beta\int_0^s\int_{M_t}2|A|^2\eta^2v^\beta\mathrm{d}\mu\mathrm{d}t\\
&+8\Big(\frac{\beta}{\beta-1}+\frac{\beta-1}{\beta}\Big)\int_0^s\int_{M_t}v^\beta|\nabla\eta|^2\mathrm{d}\mu\mathrm{d}t.
\end{aligned}$$

It follows that, for some $\Lambda(\beta)\ge 1$ (say $\Lambda(\beta)=100\beta$ if $\beta\ge 2$), and $|A|^2\le C_*|H|^\tau+C_{**}$, where $2\le\tau<2+\frac{2}{n+2}$, we get

$$\begin{aligned}
&\int_0^s\int_{M_t}|\nabla(\eta v^{\frac{\beta}{2}})|^2\mathrm{d}\mu\mathrm{d}t+\int_{M_s}v^\beta\eta^2\mathrm{d}\mu\\
\le&\Lambda(\beta)\Big\{\int_0^s\int_{M_t}v^\beta\Big(2\eta\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big|+|\nabla\eta|^2\Big)\mathrm{d}\mu\mathrm{d}t\\
&+\int_0^s\int_{M_t}2(C_*|H|^\tau+C_{**})\eta^2v^\beta\mathrm{d}\mu\mathrm{d}t\Big\}\\
\le&\Lambda(\beta)\Big\{\int_0^s\int_{M_t}v^\beta\Big(2\eta\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big|+|\nabla\eta|^2\Big)\mathrm{d}\mu\mathrm{d}t\\
&+2C_*\Big(\int_0^s\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{\tau}{n+3}}\|\eta^2v^\beta\|_{L^{\frac{q}{q-1}}(M\times[0,T))}\\
&+2C_{**}\int_0^s\int_{M_t}\eta^2v^\beta\mathrm{d}\mu\mathrm{d}t\Big\}\\
=:&\,A_0,
\end{aligned}$$

where we choose $q=\frac{n+3}{\tau}$.

Consequently,

$$\max_{0\le s\le T}\int_{M_s}\eta^2v^\beta\mathrm{d}\mu\le A_0 \tag{2.12}$$

and

$$\int_0^T\int_{M_t}|\nabla(\eta v^{\frac{\beta}{2}})|^2\mathrm{d}\mu\mathrm{d}t\le A_0. \tag{2.13}$$

We are now in a position to apply Lemma 2.1 to $\eta v^{\frac{\beta}{2}}$ and get the following estimates:

$$\begin{aligned}
\|\eta^2v^\beta\|^{\frac{n+2}{n}}_{L^{\frac{n+2}{n}}(M\times[0,T))}&=\|\eta v^{\frac{\beta}{2}}\|^{\frac{2(n+2)}{n}}_{L^{\frac{2(n+2)}{n}}(M\times[0,T))}\\
&\le c_n\max_{0\le t\le T}\|\eta v^{\frac{\beta}{2}}\|^{\frac{4}{n}}_{L^2(M_t)}\Big(\|\nabla(\eta v^{\frac{\beta}{2}})\|^2_{L^2(M\times[0,T))}\\
&\quad+\max_{0\le t\le T}\|\eta v^{\frac{\beta}{2}}\|^2_{L^2(M_t)}\|H\|^{\frac{2(n+3)}{3}}_{L^{n+3,\frac{2(n+3)}{3}}(M\times[0,T))}\Big)\\
&\le c_nA_0^{\frac{2}{n}}\Big(A_0+A_0\|H\|^{\frac{2(n+3)}{3}}_{L^{n+3,\frac{2(n+3)}{3}}(M\times[0,T))}\Big)\\
&=c_nA_0^{\frac{n+2}{n}}\Big(1+\|H\|^{\frac{2(n+3)}{3}}_{L^{n+3,\frac{2(n+3)}{3}}(M\times[0,T))}\Big).
\end{aligned}$$

Let $S := M \times [0,T)$ and let the norm $\|\cdot\|_{L^p(M\times[0,T))}$ be shortly denoted by $\|\cdot\|_{L^p(S)}$. Then the previous estimate, using a definition of A_0, can be rewritten as

$$\begin{aligned}\|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} &\le c_n A_0\Big(1+\|H\|^{\frac{2(n+3)}{3}}_{L^{n+3,\frac{2(n+3)}{3}}(M\times[0,T))}\Big)^{\frac{n}{n+2}}\\ &= c_n C_1\Lambda(\beta)\Big(\int_0^T\int_{M_t} v^\beta\Big\{2\eta\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big| + |\nabla\eta|^2\Big\}\mathrm{d}\mu\mathrm{d}t\\ &\quad + 2C_*C_2\|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(S)} + 2C_{**}\|\eta^2 v^\beta\|_{L^1(S)}\Big).\end{aligned} \tag{2.14}$$

Since $1 < \frac{q}{q-1} < \frac{n+2}{n}$, by using the interpolation inequality

$$\|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(S)} \le \varepsilon\|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} + \varepsilon^{-\nu}\|\eta^2 v^\beta\|_{L^1(S)} \tag{2.15}$$

in (2.14), for $\nu = \frac{n+2}{2q-(n+2)} = \frac{\tau(n+2)}{2(n+3)-(n+2)\tau} > 0$, one gets

$$\begin{aligned}&[1-c_n\Lambda(\beta)C_2C_1\varepsilon]\|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)}\\ &\le c_nC_1\Lambda(\beta)\Big[(2C_*C_2+2C_{**})\varepsilon^{-\nu}\|\eta^2v^\beta\|_{L^1(S)} + \Big\|v^\beta\Big(|\nabla\eta|^2+2\eta\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big)\Big\|_{L^1(S)}\Big].\end{aligned}$$

If we choose $\varepsilon = \frac{1}{2\Lambda(\beta)c_nC_2C_1}$, then

$$\begin{aligned}\|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} &\le 2c_nC_1\Lambda(\beta)\Big[((C_*C_2+C_{**})(2\Lambda(\beta)c_nC_2C_1)^\nu)\|\eta^2v^\beta\|_{L^1(S)}\\ &\quad + \Big\|v^\beta\Big(|\nabla\eta|^2+2\eta\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big)\Big\|_{L^1(S)}\Big]\\ &\le C_a(n,C_2,C_1,C_*,C_{**})\Lambda(\beta)^{1+\nu}\Big\|v^\beta\Big(\eta^2+|\nabla\eta|^2+2\eta\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big)\Big\|_{L^1(S)},\end{aligned}$$

where

$$C_a(n,\tau,C_2,C_1,C_*,C_{**}) = (2c_{(n,C_*,C_{**})}(C_2+1)C_1)^{1+\nu}. \tag{2.16}$$

In conclusion, we get a soft reverse Hölder inequality

$$\|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \le C_a(n,\tau,C_2,C_1,C_*,C_{**})\Lambda(\beta)^{1+\nu}\Big\|v^\beta\Big(\eta^2+|\nabla\eta|^2+2\eta\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big|\Big)\Big\|_{L^1(S)}.$$

Next, we show that an L^∞-norm of v over a smaller set can be bounded by an L^β-norm of v on a bigger set, where $\beta \ge 2$. Fix $x_0 \in \mathbb{R}^{n+1}$. As in [7], we consider the following sets in space and time:

$$D = \bigcup_{0\le t\le 1}(B(x_0,1)\cap M_t), \quad D' = \bigcup_{\frac{1}{12}\le t\le 1}\Big(B\Big(x_0,\frac{1}{2}\Big)\cap M_t\Big).$$

Let us denote

$$D_k = \bigcup_{t_k\le t\le 1}(B(x_0,r_k)\cap M_t),$$

where

$$r_k = \frac{1}{2}+\frac{1}{2^{k+1}}, \quad t_k = \frac{1}{12}\Big(1-\frac{1}{4^k}\Big).$$

Then

$$\rho_k := r_{k-1} - r_k = \frac{1}{2^{k+1}}, \quad t_k - t_{k-1} = \rho_k^2.$$

Let us choose a test function $\eta_k = \eta_k(t, x)$, following Ecker [2], of the form

$$\eta_k(t,x) = \varphi_{\rho_k}(t) \times \psi_{\rho_k}(|x - x_0|^2). \tag{2.17}$$

In (2.17), the function φ_{ρ_k} satisfies

$$\varphi_{\rho_k}(t) = \begin{cases} 1, & \text{if } t_k \le t \le 1, \\ \in [0,1], & \text{if } t_{k-1} \le t \le t_k, \\ 0, & \text{if } t \le t_{k-1} \end{cases}$$

and

$$|\varphi'_{\rho_k}|(t) \le \frac{c_n}{\rho_k^2},$$

whereas in (2.17), the function $\psi_{\rho_k}(s)$ satisfies

$$\psi_{\rho_k}(s) = \begin{cases} 0, & \text{if } s \ge r_{k-1}^2, \\ \in [0,1], & \text{if } r_k^2 \le s \le r_{k-1}^2, \\ 1, & \text{if } s \le r_k^2 \end{cases}$$

and

$$|\psi'_{\rho_k}|(s) \le \frac{c_n}{\rho_k^2}.$$

We have

$$0 \le \eta_k \le 1; \quad \eta_k \equiv 1 \text{ in } D_k; \quad \eta_k \equiv 0 \text{ outside } D_{k-1}. \tag{2.18}$$

Use the following identity for the mean curvature flow derived in [1] or [3]:

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)|x - x_0|^2 = -2n, \quad \forall x \in M_t. \tag{2.19}$$

In [7], N. Q. Le and N. Sesum verified the following lemma.

Lemma 2.2 (see [7])

$$\sup_{t\in[0,1]} \sup_{x\in M_t} \Big(\eta_k^2(t,x) + |\nabla \eta_k(t,x)|^2 + 2\eta_k(t,x)\Big|\Big(\frac{\partial}{\partial t} - \Delta\Big)\eta_k(t,x)\Big|\Big) \le \frac{c_n}{\rho_k^2} = c_n 4^{k+1}. \tag{2.20}$$

Then, we have the following Harnack inequality.

Proposition 2.2 *Consider equation* (2.2) *with* $T = 1$. *Let us denote* $\lambda = \frac{n+2}{n}$, *and let* $q = \frac{n+3}{\tau}$ *and* $\beta \ge 2$. *Then, there exists a constant* $C_b = C_b(n, \tau, \beta, C_2, C_1, C_*, C_{**})$, *such that*

$$\|v\|_{L^\infty(D')} \le C_b(n,\tau,\beta,C_2,C_1,C_*,C_{**})\|v\|_{L^\beta(D)}, \tag{2.21}$$

where C_1 *is defined by* (2.3).

In fact, we can choose

$$C_b(n,\tau,\beta,C_2,C_1,C_*,C_{**}) = (4\lambda^{1+\nu} C_z \beta^{1+\nu})^{\frac{n^2}{\beta}}, \tag{2.22}$$

where

$$C_z(n,\tau,C_2,C_1,C_*,C_{**}) := 4^2 \times 100^{1+\nu} c_n C_a(n,\tau,C_2,C_1,C_*,C_{**}). \tag{2.23}$$

The proof of this proposition, by using Proposition 2.1, Lemma 2.2 and Moser iteration, is similar to that of Lemma 5.2 in [7]. We omit it here.

3 Bounding the Mean Curvature

In this section, we prove Proposition 3.2.

First, we establish the following rescaled version of Proposition 3.2.

Proposition 3.1 *There is a universal constant c_0 depending only on n, C_*, C_{**} and τ, such that if*

$$\int_0^1 \int_{M_t} |H|^{n+3} \mathrm{d}\mu \mathrm{d}t \le c_0, \tag{3.1}$$

then

$$\sup_{\frac{1}{12} \le t \le 1} \sup_{x \in M_t} |H(x,t)| \le 1. \tag{3.2}$$

Proof The proposition is now an easy consequence of Proposition 2.2 when $\beta = \frac{n+3}{2}$. In fact, from (2.5), one has

$$C_a(n,\tau,C_2,C_1,C_*,C_{**}) = (2c_{(n,C_*,C_{**})}(C_2+1)C_1)^{1+\nu}.$$

As

$$C_1 = \Big(1 + \|H\|^{\frac{2(n+3)}{3}}_{L^{n+3,\frac{2(n+3)}{3}}(M\times[0,T))}\Big)^{\frac{n}{n+2}}, \quad C_2 = \Big(\int_0^T \int_{M_t} |H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{\tau}{n+3}},$$

so by Hölder inequality

$$C_1 = \Big(1 + \int_0^1 \Big(\int_{M_t} |H|^{n+3}\mathrm{d}\mu\Big)^{\frac{2}{3}}\mathrm{d}t\Big)^{\frac{n}{n+2}} \le \Big(1 + \Big(\int_0^1 \int_{M_t} |H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{2}{3}}\Big)^{\frac{n}{n+2}}.$$

From (2.23), we get

$$\begin{aligned}
C_z(n,C_2,C_1) &= 4^2 \times 100^{1+\nu} c_n (2c_{(n,C_*,C_{**})}(C_2+1)C_1)^{1+\nu} \\
&= c(n,\tau,C_*,C_{**})(C_2+1)^{1+\nu} C_1^{1+\nu} \\
&\le c(n,\tau,C_*,C_{**})\Big(\Big(\int_0^1 \int_{M_t} |H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{2(1+\nu)}{n+3}} + 1\Big) \\
&\quad \times \Big(1 + \Big(\int_0^1 \int_{M_t} |H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{2n(1+\nu)}{3(n+2)}}\Big).
\end{aligned}$$

Then, from (2.22), we have

$$\begin{aligned}
C_b &= (4\lambda^{1+\nu} C_z \beta^{1+\nu})^{\frac{n^2}{\beta}} \\
&= c(n,\tau,C_*,C_{**}) C_z^{\frac{2n^2}{n+3}} \\
&\le c(n,\tau,C_*,C_{**})\Big(\Big(\int_0^1 \int_{M_t} |H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{4n^2(1+\nu)}{(n+3)^2}} + 1\Big) \\
&\quad \times \Big(1 + \Big(\int_0^1 \int_{M_t} |H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{4n^3(1+\nu)}{3(n+2)(n+3)}}\Big).
\end{aligned}$$

Now, by (2.21), one has

$$\begin{aligned}\|v\|_{L^\infty(D')} &\le C_b\|v\|_{L^\beta(D)}\\ &\le c(n,\tau,C_*,C_{**})\Big(\Big(\int_0^1\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{4n^2(1+\nu)}{(n+3)^2}}+1\Big)\\ &\quad\times\Big(1+\Big(\int_0^1\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big)^{\frac{4n^3(1+\nu)}{3(n+2)(n+3)}}\Big)\|v\|_{L^\beta(D)}\\ &\le c(n,\tau,C_*,C_{**})\Big(c_0^{\frac{4n^2(1+\nu)}{(n+3)^2}}+1\Big)\Big(1+c_0^{\frac{4n^3(1+\nu)}{3(n+2)(n+3)}}\Big)c_0^{\frac{2}{n+3}}\\ &\le 1,\end{aligned}$$

if c_0 is small and universal.

Then $\sup\limits_{\frac{1}{12}\le t\le 1}\sup\limits_{x\in M_t}H(x,t)^2\le 1$, so $\sup\limits_{\frac{1}{12}\le t\le 1}\sup\limits_{x\in M_t}|H(x,t)|\le 1$.

Proposition 3.2 *For all $\lambda\in(0,1]$, there is a constant c_λ, such that for all $T\ge\lambda$,*

$$\sup_{x\in M_T}|H(x,T)|\le c_\lambda\Big(1+\int_0^T\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\Big). \tag{3.3}$$

Remark 3.1 As in [6], we can choose $c_\lambda=\frac{1}{\lambda^{\frac{1}{2}}}\big(1+\frac{1}{c_0}\big)$.

Proof of Proposition 3.2 We first consider the special case $\lambda=1$ and $T\ge 1$. There are two cases.

Case 1 This is the case when

$$\int_0^T\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\le c_0.$$

In this case, we consider a new one-parameter family of immersions $\widetilde{F}$ defined by $\widetilde{F}(x,t)=F(x,T-1+t)$. Then

$$\int_0^1\int_{\widetilde{M}_t}|\widetilde{H}|^{n+3}\mathrm{d}\mu\mathrm{d}t=\int_{T-1}^T\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\le\int_0^T\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\le c_0.$$

By Proposition 3.1, one has $\sup\limits_{x\in\widetilde{M}_1}|\widetilde{H}(x,1)|\le 1$. Hence

$$\sup_{x\in M_T}|H(x,T)|\le 1. \tag{3.4}$$

Case 2 This is the case when

$$\int_0^T\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t\ge c_0.$$

In this case, we consider a new one-parameter family of immersions $\widetilde{F}$ defined by $\widetilde{F}(x,t)=QF(x,\frac{t}{Q^2})$. We find that

$$\int_0^{Q^2T}\int_{\widetilde{M}_t}|\widetilde{H}|^{n+3}\mathrm{d}\mu\mathrm{d}t=\frac{1}{Q}\int_0^T\int_{M_t}|H|^{n+3}\mathrm{d}\mu\mathrm{d}t=c_0,$$

if we choose

$$Q = \frac{1}{c_0}\int_0^T \int_{M_t} |H|^{n+3} \mathrm{d}\mu \mathrm{d}t \geq 1.$$

Now, we are back in Case 1 and thus can conclude $\sup\limits_{x\in \widetilde{M}_{Q^2T}} |\widetilde{H}|(x, Q^2T) \leq 1$. This gives

$$\sup_{x\in M_T} |H(x,T)| = Q \sup_{x\in \widetilde{M}_{Q^2T}} |\widetilde{H}|(x, Q^2T) \leq Q = \frac{1}{c_0}\int_0^T \int_{M_t} |H|^{n+3} \mathrm{d}\mu \mathrm{d}t.$$

Combining the above two cases, we find that for $T \geq 1$, one has

$$\sup_{x\in M_T} |H(x,T)| \leq Q = \Big(1 + \frac{1}{c_0}\Big)\Big(1 + \int_0^T \int_{M_t} |H|^{n+3} \mathrm{d}\mu \mathrm{d}t\Big). \tag{3.5}$$

Finally, we consider the general case $\lambda \in (0, 1]$ and $T \geq \lambda$. As usual, let us consider a new one-parameter family of immersions $\widetilde{F}$ defined by $\widetilde{F}(x,t) = QF(x, \frac{t}{Q^2})$ where $Q = \frac{1}{T^{\frac{1}{2}}} \leq \frac{1}{\lambda^{\frac{1}{2}}}$. Then $Q^2T = 1$. Thus, from the estimate (3.5) in the special case, one has

$$\begin{aligned}
\sup_{x\in \widetilde{M}_{Q^2T}} |\widetilde{H}|(x, Q^2T) &\leq \Big(1 + \frac{1}{c_0}\Big)\Big(1 + \int_0^{Q^2T} \int_{M_t} |\widetilde{H}|^{n+3} \mathrm{d}\mu \mathrm{d}t\Big) \\
&= \Big(1 + \frac{1}{c_0}\Big)\Big(1 + \frac{1}{Q}\int_0^T \int_{M_t} |H|^{n+3} \mathrm{d}\mu \mathrm{d}t\Big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sup_{x\in M_T} |H|(x,T) &= Q \sup_{x\in \widetilde{M}_{Q^2T}} |\widetilde{H}|(x, Q^2T) \leq Q\Big(1 + \frac{1}{c_0}\Big)\Big(1 + \frac{1}{Q}\int_0^T \int_{M_t} |H|^{n+3} \mathrm{d}\mu \mathrm{d}t\Big) \\
&\leq \frac{1}{\lambda^{\frac{1}{2}}}\Big(1 + \frac{1}{c_0}\Big)\Big(1 + \int_0^T \int_{M_t} |H|^{n+3} \mathrm{d}\mu \mathrm{d}t\Big).
\end{aligned}$$

4 Proof of Theorem 1.7

Firstly, we bound mean curvature. Fix $\tau_1 < T$ such that $0 < \tau_1 < 1$. Then, by Proposition 3.2, for any $t \geq \tau_1$, there is a universal constant c depending only on τ_1, such that

$$\sup_{x\in M_t} |H(x,t)| \leq c\Big(1 + \int_0^t \int_{M_s} |H|^{n+3} \mathrm{d}\mu \mathrm{d}s\Big). \tag{4.1}$$

Let $f(t) = \sup\limits_{x\in M_t} |H(x,t)|$, $\Psi(s) = s\ln(2+s)$ and $G(s) = \int_{M_s} \frac{|H|^{n+2}}{\ln(2+|H|)}\mathrm{d}\mu$. Then Ψ is an increasing function. Note that (4.1) gives

$$\begin{aligned}
f(t) &\leq c\Big(1 + \int_0^t \int_{M_s} \Psi(|H|)\frac{|H|^{n+2}}{\ln(2+|H|)} \mathrm{d}\mu \mathrm{d}s\Big) \\
&\leq c\Big(1 + \int_0^t \Psi\Big(\sup_{x\in M_s} |H(x,s)|\Big) \int_{M_s} \frac{|H|^{n+2}}{\ln(2+|H|)} \mathrm{d}\mu \mathrm{d}s\Big) = c\Big(1 + \int_0^t \Psi(f(s))G(s) \mathrm{d}s\Big).
\end{aligned}$$

Let $h(t) = c\big(1 + \int_0^t \Psi(f(s))G(s)\mathrm{d}s\big)$. Then for $t \geq \tau_1$, we have $f(t) \leq h(t)$ and $h'(t) = c\Psi(f(t))G(t) \leq c\Psi(h(t))G(t)$. Let $\widetilde{\Psi}(y) = \int_c^y \frac{1}{\Psi(s)}\mathrm{d}s$. Then for $t \geq \tau_1$,

$$\widetilde{\Psi}(h(t)) - \widetilde{\Psi}(h(\tau_1)) \leq c\int_{\tau_1}^t G(s)\mathrm{d}s \leq c\int_0^T G(s)\mathrm{d}s < \infty.$$

Hence, since $h(\tau_1)$ is finite, we get

$$\sup_{\tau_1 \leq t < T} \widetilde{\Psi}(h(t)) \leq \widetilde{\Psi}(h(\tau_1)) + c\int_0^T G(s)\mathrm{d}s < \infty.$$

Since $\int_c^\infty \frac{1}{\Psi(s)}\mathrm{d}s = \infty$, we deduce that $\sup\limits_{\tau_1 \leq t < T} h(t) < \infty$. Hence, $\sup\limits_{\tau_1 \leq t < T} f(t) < \infty$. So, we have bounded the mean curvature.

Secondly, by using condition (1), we have

$$|A|^2(x,t) \leq C_*|H|^\tau(x,t) + C_{**}, \quad (x,t) \in M \times [0,T).$$

Then from Theorem 1.1, the flow can be extended past time T.

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