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Almost Sure Central Limit Theorem for Partial Sums of Markov Chain*

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Abstract The authors prove an almost sure central limit theorem for partial sums based on an irreducible and positive recurrent Markov chain using logarithmic means, which realizes the extension of the almost sure central limit theorem for partial sums from an i.i.d. sequence of random variables to a Markov chain.

Keywords Almost sure central limit theorem, Partial sums, Markov chain, Logarithmic means
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1 Introduction

Let $X_1, X_2, \cdots$ be i.i.d. random variables with $EX_1 = 0$ and $EX_1{}^2 = 1$. If the second moment of X_1 exists, then the central limit theorem holds, i.e.,

$$P\Big\{\frac{S_n^*}{\sqrt{n}} \leqslant x\Big\} \xrightarrow{w} \Phi(x), \quad \text{as } n \to \infty, \tag{1.1}$$

for all $x \in \mathbb{R}$, where $S_n^* = \sum\limits_{i=1}^{n} X_i$, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}} \mathrm{d}t$.

In recent years, many researchers discussed almost sure central limit theorem for partial sums of some sequences of random variables, which was first studied by Brosamler [3] and Schatte [14]. The simplest form of the almost sure central limit theorem states as follows:

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathrm{I}_{\{\frac{S_k^*}{\sqrt{k}} \leqslant x\}} \xrightarrow{\text{a.s.}} \Phi(x), \quad \text{as } n \to \infty \tag{1.2}$$

for all $x \in \mathbb{R}$, where $\mathrm{I}_{\{A\}}$ denotes the indicator function of the event A.

The above result was extended by Lacey and Philipp [7] based on an almost sure invariance principle. Peligrad and Shao [10] and Matula [9] proved the almost sure central limit theorem for strongly mixing and associated sequences, which both satisfy the weak dependence condition

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introduced by Doukhan and Louhichi [4]. For more extensions, we refer to Berkes and Csáki [2] and the reference therein.

Zhang et al. [16] studied an irreducible and positive recurrent Markov chain, and obtained some limit results. To the best of our knowledge, there is no work about the almost sure central limit theorem for partial sums for this type of Markov chain.

In this paper, we will deal with the almost sure central limit theorem for partial sums based on this type of irreducible and positive recurrent Markov chain using logarithmic means. The paper is organized as follows. In Section 2, we present the main results by giving some basic assumptions. In Section 3, we show some lemmas and finish the proofs of the main results.

2 Main Results

Throughout this paper, $\{Y_n, n=1,2,\cdots\}$ is an irreducible and positive recurrent Markov chain, S is its state space, u is the initial distribution and π is the invariant distribution. $\tau_j^{(r)}$ denotes the time that the process $\{Y_n,\ n=1,\ 2,\cdots\}$ reached state j for the r-th time, $r=0,1,2,\cdots$ and $\tau_j^{(0)}=0$. $E_j(\tau_j^{(1)})=E(\tau_j^{(2)}-\tau_j^{(1)})$ denotes the average time that the process reached state j. $T_i^{(r)}=\#\{m\in(\tau_j^{(r)},\tau_j^{(r+1)}]:Y_m=i\}$, $r=0,1,2,\cdots$, denotes the times of the process reaching state i during the time of the r-th and $(r+1)$-th reaching state j. Let N_n be the times of the process reaching state j until time n, $N_{m,n}$ be the times of the process reaching state j from time m to time n.

Now, we state the main results of this paper.

Theorem 2.1 Let $\{Y_n,\ n=1,\ 2,\cdots\}$ be an irreducible and positive recurrent Markov chain. Assume that the real valued function f satisfies

$$E_\pi[|f(Y_1)|]=\sum_{i\in S}\pi_i|f(i)|<+\infty, \tag{2.1}$$

where $\sum\limits_{i\in S}\pi_i=1$. Let $S_k=\sum\limits_{m=1}^{k}[f(Y_m)-\mu]$, $k=1,2,\cdots$, $Z_r=\sum\limits_{k=\tau_j^{(r)}+1}^{\tau_j^{(r+1)}}(f(Y_k)-\mu)$, $r=0,1,2,\cdots$, $j\in S$, $\mu=E_\pi[f(Y_1)]=\sum\limits_{i\in S}\pi_i f(i)$. If

$$E_j(Z_r^2)=\sigma^2<+\infty, \tag{2.2}$$

then for any bounded Lipschitz 1 function g, we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}g\Big(\frac{S_k}{\sigma\sqrt{k\pi_j}}\Big)\xrightarrow{\text{a.s.}}\int_{-\infty}^{+\infty}g(x)\mathrm{d}\Phi(x),\quad \text{as } n\to\infty, \tag{2.3}$$

for any $x\in\mathbb{R}$, where $\sigma>0$, $\pi_j=\lim\limits_{n\to\infty}\frac{N_n}{n}=(E_j(\tau_j^{(1)}))^{-1}$. Specially, if $E_j(Z_r^2)=E_j(\tau_j^{(1)})$, then we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}g\Big(\frac{S_k}{\sqrt{k}}\Big)\xrightarrow{\text{a.s.}}\int_{-\infty}^{+\infty}g(x)\mathrm{d}\Phi(x),\quad \text{as } n\to\infty. \tag{2.4}$$

Corollary 2.1 Under the conditions of Theorem 2.1, for any $x\in\mathbb{R}$, we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathrm{I}_{\{\frac{S_k}{\sigma\sqrt{k\pi_j}}\leqslant x\}}\xrightarrow{\text{a.s.}}\Phi(x),\quad \text{as } n\to\infty. \tag{2.5}$$

If $E_j(Z_r^2)=E_j(\tau_j^{(1)})$, then we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathrm{I}_{\{\frac{S_k}{\sqrt{k}}\leqslant x\}}\xrightarrow{\text{a.s.}}\Phi(x),\quad \text{as } n\to\infty. \tag{2.6}$$

3 Proofs

First, we state and prove several lemmas, which will be used in the proof of the main results.

Lemma 3.1 Let $\{Y_n, n=1,2,\cdots\}$ be an irreducible and positive recurrent Markov chain. Assume that the real valued function f satisfies

$$E_j[|f(Y_1)|+\cdots+|f(Y_{\tau_j^{(1)}})|]<+\infty. \tag{3.1}$$

Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(Y_k)=\frac{E_j[f(Y_1)+\cdots+f(Y_{\tau_j^{(1)}})]}{E_j(\tau_j^{(1)})}\qquad \text{a.s.} \tag{3.2}$$

Proof Let

$$\overline{S}_n=\sum_{k=1}^{n}f(Y_k),\quad n=1,2,\cdots$$

and

$$\overline{Z}_r=\sum_{k=\tau_j^{(r)}+1}^{\tau_j^{(r+1)}}f(Y_k),\quad r=0,1,2,\cdots.$$

According to the strong Markov property, $\overline{Z}_0,\overline{Z}_1,\overline{Z}_2,\cdots$ are i.i.d. random variables. By the strong law of large number, we obtain

$$\lim_{r\to\infty}\frac{1}{r}\sum_{s=1}^{r}\overline{Z}_s=E\overline{Z}_0=E\overline{Z}_1,\quad \text{a.s.} \tag{3.3}$$

Let N_n be the times of the process reaching state j until time n, $\tau_j^{(r)}$ be the time that the process reached state j for the r-th time. We have

$$N_n=\max\{r\geqslant 0,\ \tau_j^{(r)}\leqslant n\}$$

and

$$\overline{S}_n=\sum_{k=1}^{\tau_j^{(1)}}f(Y_k)+\sum_{r=1}^{N_n}\overline{Z}_r-\sum_{k=n+1}^{\tau_j^{(N_n+1)}}f(Y_k). \tag{3.4}$$

By using the condition of (3.1), we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\tau_j^{(1)}}f(Y_k)=0,\quad \text{a.s.}$$

and

$$\Big|\frac{1}{n}\sum_{k=n+1}^{\tau_j^{(N_n+1)}} f(Y_k)\Big| \leqslant \frac{1}{n}\sum_{k=n+1}^{\tau_j^{(N_n+1)}} |f(Y_k)| \xrightarrow{\text{a.s.}} 0, \quad n\to\infty.$$

From (3.4) we get

$$\frac{\overline{S}_n}{n} = \frac{1}{n}\sum_{r=1}^{N_n}\overline{Z}_r + R_n = \frac{N_n}{n}\cdot\frac{1}{N_n}\sum_{r=1}^{N_n}\overline{Z}_r + R_n. \tag{3.5}$$

Note that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^{\tau_j^{(1)}} f(Y_k) - \frac{1}{n}\sum_{k=n+1}^{\tau_j^{(N_n+1)}} f(Y_k)\Big) = 0, \quad \text{a.s.}$$

and

$$\lim_{n\to\infty}\frac{1}{N_n}\sum_{r=1}^{N_n}\overline{Z}_r = E\overline{Z}_0 = E\overline{Z}_1, \quad \text{a.s.} \tag{3.6}$$

Let $\overline{f}\equiv 1$ instead of f in (3.6). We have

$$\lim_{n\to\infty}\frac{\tau_j^{(N_n+1)} - \tau_j^{(1)}}{N_n} = E(\tau_j^{(2)} - \tau_j^{(1)}).$$

Since

$$n - \tau_j^{(1)} \leqslant \tau_j^{(N_n+1)} - \tau_j^{(1)},$$

we have

$$\lim_{n\to\infty}\frac{n}{N_n} = \lim_{n\to\infty}\frac{\tau_j^{(N_n+1)} - \tau_j^{(1)}}{N_n} = E(\tau_j^{(2)} - \tau_j^{(1)}) = E_j(\tau_j^{(1)}). \tag{3.7}$$

Combining (3.5)–(3.7), we obtain the desired result.

Lemma 3.2 Let $\{Y_n,\ n=1,\ 2,\cdots\}$ be an irreducible and positive recurrent Markov chain. We have

$$\pi_j = (E_j(\tau_j^{(1)}))^{-1}, \quad j\in S. \tag{3.8}$$

Proof Let

$$T_i^{(r)} = \#\{m\in(\tau_j^{(r)},\ \tau_j^{(r+1)}] : Y_m = i\}, \quad r=0,1,2,\cdots.$$

Note that $T_i^{(r)}$ are the times of process $\{Y_n,\ n=1,\ 2,\cdots\}$ reaching state i during the time of the r-th and $(r+1)$-th reaching state j. By the strong Markov property, we know that $\{T_i^{(r)}:\ \ r=1,2,\cdots\}$ is an i.i.d. sequence of random variables. Let

$$\theta_j^{(i)} = E_j(T_i^{(r)}). \tag{3.9}$$

Then

$$\sum_{i\in S}\theta_j^{(i)} = \sum_{i\in S}E_j(T_i^{(1)}) = E_j\Big(\sum_{i\in S}(T_i^{(1)})\Big) = E(\tau_j^{(2)} - \tau_j^{(1)}) = \frac{1}{\pi_j}.$$

Let f be the indicator function of state i, and let i instead of j in (3.2). We have

$$\lim_{n\to\infty}\frac{\#\{m\leqslant n: Y_m=i\}}{n}=\pi_i,\quad \text{a.s.}$$

By the strong law of large number and (3.7), (3.9), we obtain

$$\lim_{n\to\infty}\frac{\sum\limits_{r=1}^{N_n}T_i^{(r)}}{n}=\lim_{n\to\infty}\frac{N_n}{n}\Big(\sum_{r=1}^{N_n}\frac{T_i^{(r)}}{N_n}\Big)=\pi_j\theta_j^{(i)}.$$

Thus

$$\pi_i=\pi_j\theta_j^{(i)} \tag{3.10}$$

and

$$\sum_{i\in S}\pi_i=\pi_j\sum_{i\in S}\theta_j^{(i)}=\frac{\pi_j}{\pi_j}=1.$$

By Lebesgue dominated convergence theorem,

$$\sum_{i\in S}\Big|\frac{1}{n}\sum_{m=1}^{n}p_{ji}^{(m)}-\pi_i\Big|\longrightarrow 0,\quad \text{as } n\to\infty.$$

Consequently,

$$\Big|\sum_{i\in S}\pi_i p_{ij}-\sum_{i\in S}\Big(\frac{1}{n}\sum_{m=1}^{n}p_{ji}^{(m)}\Big)p_{ij}\Big|\leqslant\sum_{i\in S}\Big|\pi_i-\frac{1}{n}\sum_{m=1}^{n}p_{ji}^{(m)}\Big|\longrightarrow 0,\quad \text{as } n\to\infty,$$

which implies

$$\begin{aligned}\sum_{i\in S}\pi_i p_{ij}&=\lim_{n\to\infty}\sum_{i\in S}\Big(\frac{1}{n}\sum_{m=1}^{n}p_{ji}^{(m)}\Big)p_{ij}=\lim_{n\to\infty}\frac{1}{n}\sum_{i\in S}\Big(\sum_{m=1}^{n}p_{ji}^{(m)}\Big)p_{ij}\\&=\lim_{n\to\infty}\frac{1}{n}\sum_{i\in S}\sum_{m=1}^{n}p_{ji}^{(m)}p_{ij}=\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}p_{jj}^{(m+1)}=\pi_j.\end{aligned}$$

This completes the proof.

Lemma 3.3 Let $\{Y_n,\ n=1,2,\cdots\}$ be an irreducible and positive recurrent Markov chain. If (2.1) holds, then we get (3.1) and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(Y_k)=E_\pi[f(Y_1)]=\sum_{i\in S}\pi_i f(i)\quad \text{a.s.} \tag{3.11}$$

Proof By the definition of $T_i^{(1)}$, we get

$$\sum_{k=\tau_j^{(1)}+1}^{\tau_j^{(2)}}|f(Y_k)|=\sum_{i\in S}|f(i)|T_i^{(1)}.$$

In view of (3.9) and (3.10), we have

$$E(T_i^{(1)}) = \theta_j^{(i)} = \frac{\pi_i}{\pi_j}.$$

Applying Fubini theorem, we obtain

$$E|\overline{Z}_0| = E|\overline{Z}_1| = E\Big(\sum_{k=\tau_j^{(1)}+1}^{\tau_j^{(2)}} |f(Y_k)|\Big) = E\Big(\sum_{i\in S} |f(i)| T_i^{(1)}\Big) = \sum_{i\in S} |f(i)| \frac{\pi_i}{\pi_j} < \infty. \tag{3.12}$$

Similarly, we get

$$\begin{aligned} E\overline{Z}_0 = E\overline{Z}_1 &= E\Big(\sum_{k=\tau_j^{(1)}+1}^{\tau_j^{(2)}} f(Y_k)\Big) = E\Big(\sum_{i\in S} f(i) T_i^{(1)}\Big) \\ &= \sum_{i\in S} f(i) E(T_i^{(1)}) = \sum_{i\in S} f(i)\frac{\pi_i}{\pi_j} = \frac{1}{\pi_j}\sum_{i\in S}\pi_i f(i). \end{aligned} \tag{3.13}$$

Thus (3.11) follows from Lemma 3.1.

Lemma 3.4 Let $\{X_j,\ j \geqslant 1\}$ be an i.i.d. sequence of random variables with $EX_j = 0$ and $\sigma^2 = EX_j^2 < \infty$. Suppose that $\{v_n,\ n \geqslant 1\}$ is a non-negative and integer valued sequence of random variables satisfying

$$\lim_{n\to\infty} \frac{v_n}{n} = \alpha \tag{3.14}$$

for some constant $\alpha > 0$. Then

$$\lim_{n\to\infty} P\Big\{\frac{\sum\limits_{j=1}^{v_n} X_j}{\sqrt{v_n}} \leqslant x\Big\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t. \tag{3.15}$$

Proof Without loss of generality, we assume that $\sigma = 1$. Let $S_n^* = X_1 + X_2 + \cdots + X_n$. Then $\forall \varepsilon > 0$,

$$\begin{aligned} &P\{|S_{v_n}^* - S_{[n\alpha]}^*| \geqslant \varepsilon[n\alpha]^{\frac{1}{2}}\} \\ &\leqslant P\{|v_n - [n\alpha]| \geqslant \varepsilon^3[n\alpha]\} + P\Big\{\max_{m:|m-[n\alpha]|<\varepsilon^3[n\alpha]} \{|S_m^* - S_{[n\alpha]}^*| \geqslant \varepsilon[n\alpha]^{\frac{1}{2}}\}\Big\}. \end{aligned}$$

By (3.14), we get

$$\lim_{n\to\infty} P\{|v_n - [n\alpha]| \geqslant \varepsilon^3[n\alpha]\} = 0.$$

By Kolmogorov maximal inequality, we have

$$P\Big\{\max_{m:|m-[n\alpha]|<\varepsilon^3[n\alpha]} \{|S_m^* - S_{[n\alpha]}^*| \geqslant \varepsilon[n\alpha]^{\frac{1}{2}}\}\Big\} \leqslant (\varepsilon[n\alpha]^{\frac{1}{2}})^{-2}\varepsilon^3[n\alpha] = \varepsilon,$$

which implies

$$\frac{S_{v_n}^* - S_{[n\alpha]}^*}{\sqrt{[n\alpha]}} \xrightarrow{P} 0, \quad \text{as } n \to \infty. \tag{3.16}$$

According to central limit theorem, for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P\Big\{\frac{S^*_{[n\alpha]}}{\sqrt{[n\alpha]}} \leqslant x\Big\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}}\mathrm{d}t = \Phi(x),$$

which combining with (3.16) implies

$$\lim_{n\to\infty} P\Big\{\frac{S^*_{v_n}}{\sqrt{[n\alpha]}} \leqslant x\Big\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}}\mathrm{d}t = \Phi(x).$$

Applying (3.14), we get the desired result.

Lemma 3.5 Let $\{Y_n,\ n = 1, 2, \cdots\}$ be an irreducible and positive recurrent Markov chain and f be a real function satisfying $E_\pi[|f(Y_1)|] = \sum\limits_{i\in S}\pi_i|f(i)| < +\infty$. Let $S_k = \sum\limits_{m=1}^{k}[f(Y_m) - \mu]$, $k = 1, 2, \cdots$, $Z_r = \sum\limits_{k=\tau_j^{(r)}+1}^{\tau_j^{(r+1)}}(f(Y_k) - \mu)$, $r = 0, 1, 2, \cdots$, $\mu = E_\pi[f(Y_1)] = \sum\limits_{i\in S}\pi_i f(i)$. If $E_j(Z_r - E_jZ_r)^2 = \sigma^2 < +\infty$, then

$$\lim_{n\to\infty} P\Big\{\frac{S_n}{\sigma\sqrt{n\pi_j}} \leqslant x\Big\} = \Phi(x). \tag{3.17}$$

Proof By Lemma 3.3 and (3.13), we have

$$E_j(Z_r) = E_j(\tau_j^{(r)})E_\pi[f(Y_1) - \mu] = 0, \quad r = 0, 1, 2, \cdots.$$

Note that $\{Z_r,\ r = 0, 1, 2, \cdots\}$ is an i.i.d. sequence of random variables with $E_j(Z_r) = 0$ and $E_j(Z_r^2) = \sigma^2 < +\infty$. By (3.7) and (3.8), we have $\pi_j = \lim\limits_{n\to\infty}\frac{N_n}{n} = (E_j(\tau_j^{(1)}))^{-1}$. Using (3.4), (3.5) and Lemma 3.4, we have

$$\lim_{n\to\infty}\frac{S_n}{\sqrt{n}} = \lim_{n\to\infty}\sqrt{\frac{N_n}{n}}\cdot\frac{1}{\sqrt{N_n}}\sum_{r=1}^{N_n} Z_r \sim N(0,\ \pi_j\sigma^2),$$

which implies

$$\lim_{n\to\infty} P\Big\{\frac{S_n}{\sigma\sqrt{n\pi_j}} \leqslant x\Big\} = \Phi(x).$$

This completes the proof of the lemma.

Lemma 3.6 Let $\{\xi_k,\ k = 1, 2, \cdots\}$ be a sequence of bounded random variables, i.e., there exists some $M \in (0,\ \infty)$, such that $|\xi_k| \leqslant M$ a.s. for all $k \in \mathbb{N}$, satisfying $E\xi_k \to \nu$, as $k \to \infty$. Suppose furthermore that $\{p_n,\ n \geqslant 1\}$ is a sequence of non-negative numbers and partial sums $P_n \nearrow \infty$, such that for some $\varepsilon > 0$,

$$\mathrm{Var}\Big(\sum_{k=1}^{n} p_k\xi_k\Big) = O(P_n^{(2-\varepsilon)}), \quad n = 1, 2, \cdots.$$

Then we have

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k\xi_k \xrightarrow{\text{a.s.}} \nu, \quad n \to \infty.$$

Proof See [5, Lemma 1].

Proof of Theorem 2.1 Let $\widetilde{S}_{N_k} = Z_1 + Z_2 + \cdots + Z_{N_k}$, $\widetilde{S}_{N_l} = Z_1 + Z_2 + \cdots + Z_{N_k} + Z_{N_{k+1}} + \cdots + Z_{N_l}$ $(k < l)$ and $\widetilde{S}_{N_{k,l}} = Z_{N_{k+1}} + Z_{N_{k+2}} + \cdots + Z_{N_l}$. Note that $\widetilde{S}_{N_l} - \widetilde{S}_{N_k} = \widetilde{S}_{N_{k,l}}$. According to Lemma 3.1, $\widetilde{S}_{N_k}$ and $\widetilde{S}_{N_{k,l}}$ are independent. By Lemmas 3.1–3.4, we obtain

$$\frac{\widetilde{S}_{N_n}}{\sqrt{n}} = \sqrt{\Big(\frac{N_n}{n}\Big)} \cdot \frac{1}{\sqrt{N_n}} \sum_{r=1}^{N_n} Z_r \sim N(0,\ \pi_j \sigma^2), \quad \text{as } n \to \infty,$$

where $\pi_j = \lim\limits_{n\to\infty} \frac{N_n}{n} = (E_j(\tau_j^{(1)}))^{-1}$, $\sigma^2 = E_j(Z_r - E_j Z_r)^2 = E_j(Z_r^2)$. By Lemma 3.5, we have

$$\lim_{n\to\infty} P\Big\{\frac{S_n}{\sigma\sqrt{n\pi_j}} \leqslant x\Big\} = \lim_{n\to\infty} P\Big\{\frac{\widetilde{S}_{N_n}}{\sigma\sqrt{n\pi_j}} \leqslant x\Big\} = \Phi(x) \tag{3.18}$$

for any $x \in \mathbb{R}$. By the dominated convergence theorem, we have

$$\lim_{k\to\infty} Eg\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big) = \int_{-\infty}^{+\infty} g(x)\mathrm{d}\Phi(x),$$

which implies

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} Eg\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big) = \int_{-\infty}^{+\infty} g(x)\mathrm{d}\Phi(x). \tag{3.19}$$

First, we prove that

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big) \xrightarrow{\text{a.s.}} \int_{-\infty}^{+\infty} g(x)\mathrm{d}\Phi(x), \quad n \to \infty. \tag{3.20}$$

By Lemma 3.6 and (3.19), in order to prove (3.20), we only need to prove that

$$\mathrm{Var}\Big(\sum_{k=1}^{n} \frac{1}{k} g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big)\Big) = O((\log n)^{2-\varepsilon}). \tag{3.21}$$

Note that

$$\begin{aligned}
&\mathrm{Var}\Big(\sum_{k=1}^{n} \frac{1}{k} g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big)\Big) \\
&= \sum_{k=1}^{n} \frac{1}{k^2} \mathrm{Var}\Big(g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big)\Big) + 2 \sum_{1\leqslant k<l\leqslant n} \frac{1}{k} \cdot \frac{1}{l} \mathrm{Cov}\Big(g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big), g\Big(\frac{\widetilde{S}_{N_l}}{\sigma\sqrt{l\pi_j}}\Big)\Big) \\
&=: A_1 + A_2.
\end{aligned}$$

Obviously, for some $M \in (0,\ \infty)$, we have

$$A_1 = \sum_{k=1}^{n} \frac{1}{k^2} \mathrm{Var}\Big(g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big)\Big) \leqslant M \sum_{k=1}^{n} \frac{1}{k} = O(\log n).$$

It is easy to check that

$$\begin{aligned}
&\Big|\mathrm{Cov}\Big(g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big),\ g\Big(\frac{\widetilde{S}_{N_l}}{\sigma\sqrt{l\pi_j}}\Big)\Big)\Big| \\
&= \Big|\mathrm{Cov}\Big(g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big),\ g\Big(\frac{\widetilde{S}_{N_l}}{\sigma\sqrt{l\pi_j}}\Big) - g\Big(\frac{\widetilde{S}_{N_{k,l}}}{\sigma\sqrt{l\pi_j}}\Big) + g\Big(\frac{\widetilde{S}_{N_{k,l}}}{\sigma\sqrt{l\pi_j}}\Big)\Big)\Big| \\
&\leqslant \Big|\mathrm{Cov}\Big(g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big),\ g\Big(\frac{\widetilde{S}_{N_l}}{\sigma\sqrt{l\pi_j}}\Big) - g\Big(\frac{\widetilde{S}_{N_{k,l}}}{\sigma\sqrt{l\pi_j}}\Big)\Big)\Big| + \Big|\mathrm{Cov}\Big(g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big),\ g\Big(\frac{\widetilde{S}_{N_{k,l}}}{\sigma\sqrt{l\pi_j}}\Big)\Big)\Big| \\
&=: B_1 + B_2.
\end{aligned}$$

Since $\widetilde{S}_{N_k}$ and $\widetilde{S}_{N_{k,l}}$ are independent, then $B_2 = 0$. Noting that g is a bounded Lipschitz 1 function, for some $C \in (0, \infty)$, and we have

$$B_1 \leqslant ME\Big|g\Big(\frac{\widetilde{S}_{N_l}}{\sigma\sqrt{l\pi_j}}\Big) - g\Big(\frac{\widetilde{S}_{N_{k,l}}}{\sigma\sqrt{l\pi_j}}\Big)\Big| \leqslant CE\Big|\frac{\widetilde{S}_{N_l} - \widetilde{S}_{N_{k,l}}}{\sqrt{l}}\Big| \leqslant \frac{C}{\sqrt{l}}(E(\widetilde{S}_{N_k})^2)^{\frac{1}{2}}.$$

By Wald equality and (3.7), we have

$$E(\widetilde{S}_{N_k})^2 = \sigma^2 EN_k = O(k).$$

Thus

$$B_1 = O\Big(\sqrt{\frac{k}{l}}\Big).$$

Consequently,

$$A_2 \leqslant M \sum_{1\leqslant k<l\leqslant n} \frac{1}{k}\cdot\frac{1}{l}\cdot\sqrt{\frac{k}{l}} \leqslant M\sum_{l=2}^{n}\frac{1}{l^{\frac{3}{2}}}\sum_{k=1}^{l-1}\frac{1}{\sqrt{k}} \leqslant 2M\sum_{l=2}^{n}\frac{1}{l} = O(\log n).$$

Hence

$$\mathrm{Var}\Big(\sum_{k=1}^{n}\frac{1}{k}g\Big(\frac{\widetilde{S}_{N_k}}{\sigma\sqrt{k\pi_j}}\Big)\Big) = O(\log n)$$

satisfies (3.21) with $\varepsilon = 1$. Thus, (3.20) holds.

From (3.4)–(3.6), we have

$$\frac{S_n}{\sqrt{n}} - \frac{\widetilde{S}_{N_n}}{\sqrt{n}} = o(1), \quad \text{a.s.} \tag{3.22}$$

Thus, for any bounded Lipschitz 1 function g, we have

$$\Big|g\Big(\frac{S_n}{\sigma\sqrt{n\pi_j}}\Big) - g\Big(\frac{\widetilde{S}_{N_n}}{\sigma\sqrt{n\pi_j}}\Big)\Big| \leqslant C\Big|\frac{S_n}{\sqrt{n}} - \frac{\widetilde{S}_{N_n}}{\sqrt{n}}\Big| = o(1), \quad \text{a.s.}$$

By (3.20) and [7, Theorem 1], we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}g\Big(\frac{S_k}{\sigma\sqrt{k\pi_j}}\Big) \xrightarrow{\text{a.s.}} \int_{-\infty}^{+\infty} g(x)\mathrm{d}\Phi(x), \quad n\to\infty.$$

If $E_j(Z_r^2) = E_j(\tau_j^{(1)})$, we have $\pi_j E_j(Z_r^2) = \pi_j \sigma^2 = 1$. Similarly, we get

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}g\Big(\frac{S_k}{\sqrt{k}}\Big) \xrightarrow{\text{a.s.}} \int_{-\infty}^{+\infty} g(x)\mathrm{d}\Phi(x), \quad n \to \infty.$$

The proof is completed now.

Corollary 2.1 is a special case of Theorem 2.1.

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