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On the Wielandt Subgroup in a p-Group of Maximal Class*

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Abstract The Wielandt subgroup of a group G, denoted by $w(G)$, is the intersection of the normalizers of all subnormal subgroups of G. In this paper, the authors show that for a p-group of maximal class G, either $w_i(G) = \zeta_i(G)$ for all integer i or $w_i(G) = \zeta_{i+1}(G)$ for every integer i, and $w(G/K) = \zeta(G/K)$ for every normal subgroup K in G with $K \neq 1$. Meanwhile, a necessary and sufficient condition for a regular p-group of maximal class satisfying $w(G) = \zeta_2(G)$ is given. Finally, the authors prove that the power automorphism group $\mathrm{PAut}(G)$ is an elementary abelian p-group if G is a non-abelian p-group with elementary $\zeta(G) \cap \mho_1(G)$.

Keywords p-Groups of maximal class, Wielandt subgroup, Wielandt series, Upper central series
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1 Introduction

All groups considered in this paper are finite.

The Wielandt subgroup $w(G)$ of a group G is the intersection of the normalizers of all subnormal subgroups of G. Wielandt [14] showed that for a group G, $w(G)$ is a non-trivial characteristic subgroup of G, and defined an ascending normal series terminating at the group. Let $w_0(G) = 1$, and $w_{i+1}(G)/w_i(G) = w(G/w_i(G))$ for $i \geq 0$. The smallest n for which $w_n(G) = G$ is called the Wielandt length of G. A related concept is the norm of a group G, denoted by $N(G)$, which is the intersection of the normalizers of all subgroups of G. This concept was introduced by Baer [1] in 1934. The Wielandt subgroup of a nilpotent group G coincides with the norm of the group. Schenkman [13] showed that the norm is in the second center of a group. Thus for any group G, we have $\zeta(G) \leq N(G) \leq \zeta_2(G)$.

So the interesting question is to investigate the relationship between the Wielandt series and the upper central series of a nilpotent group. Bryce et al. [6] showed that for metabelian p-groups of exponent dividing p^2 and of sufficiently large class, the Wielandt series and the upper central series coincide. Ormerod [11] showed that $w_{r+1}(G) \subseteq \zeta_{n+1}(G)$ for a metabelian p-group G with $w_r(G) \subseteq \zeta_n(G)$, where $r \geq 1$, $n \geq 2$.

In the present paper, we are interested in p-groups of maximal class. We show that for a p-group of maximal class G, either $w_i(G) = \zeta_i(G)$ for all integer i or $w_i(G) = \zeta_{i+1}(G)$ for every integer i and $w(G/K) = \zeta(G/K)$ for every normal subgroup K in G with $K \neq 1$. Meanwhile, we give a necessary and sufficient condition for a regular p-group of maximal class G satisfying

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$w(G) = \zeta_2(G)$. Finally, we prove that the power automorphism group $\mathrm{PAut}(G)$ is an elementary abelian p-group if G is a non-abelian p-group with elementary $\zeta(G) \cap \mho_1(G)$.

2 Preliminaries

In this section we give some basic facts, which will be useful for later use. We recall that an automorphism of a group is a power automorphism if it maps every subgroup onto itself. A power automorphism is said to be universal if every element of the group is mapped to the same power. We denote by $\mathrm{PAut}(G)$ the power automorphisms of G.

Theorem 2.1 (see [5]) *Let G be a p-group of maximal class of order p^n and G_i be the i-th member of the lower center series of G. Then*

(1) $|G : G_2| = p^2$, $G_2 = \Phi(G)$, $d(G) = 2$;

(2) $|G_i/G_{i+1}| = p$, $i = 2, 3, \cdots, n-1$;

(3) *If $i \geq 2$, then G_i is the unique normal subgroup of G of order p^{n-i}*;

(4) *If $N \trianglelefteq G$ and $|G/N| \geq p^2$, then G/N is also of maximal class*;

(5) *If $0 \leq i \leq n-1$, then $\zeta_i(G) = G_{n-i}$, where $1 = \zeta_0(G) < \cdots < \zeta_{n-1}(G) = G$ is the upper central series of G*;

(6) *If $p > 2$ and $n > 3$, then G has no cyclic normal subgroup of order p^2.*

Theorem 2.2 (see [12]) *Let G be a minimal non-abelian p-group. Then G is isomorphic to one of the following p-groups:*

(1) Q_8;

(2) (Metacyclic) $M_p(n, m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1,\ a^b = a^{1+p^{n-1}} \rangle$ with $n \geq 2$;

(3) (Non-metacyclic) $M_p(n, m, 1) = \langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1,\ [a, b] = c,\ [c, a] = [c, b] = 1 \rangle$ with $n \geq m$, and $n + m \geq 3$ when $p = 2$.

Theorem 2.3 (see [9, Theorem 11.9]) *Let G be a 2-group of maximal class of order 2^n. Then G is isomorphic to one of the following 2-groups:*

(1) $D_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ [a, b] = a^{-2} \rangle$, *where* $n \geq 3$;

(2) $SD_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ [a, b] = a^{-2+2^{n-2}} \rangle$, *where* $n \geq 4$;

(3) $Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ [a, b] = a^{-2} \rangle$, *where* $n \geq 3$.

Theorem 2.4 (see [8, Hilfssatz 5]) *Let G be a non-abelian p-group, $\alpha \in \mathrm{PAut}(G)$. Then for any $g \in G$, $g^\alpha = g^i$, where i is an integer and $i \equiv 1 \pmod p$, and* PAut*(G) is an abelian p-group.*

Lemma 2.1 (see [7, Theorem 5.3.1]) *If G is a group whose Sylow subgroups are regular, then every power automorphism of G is universal.*

Since elements of $N(G)$ induce power automorphisms of the group G, we have the following conclusion.

Corollary 2.1 *Let G be a regular p-group with $w \in w(G)$. Then there exists an integer n such that $g^w = g^n$ for all $g \in G$.*

Lemma 2.2 (see [9, Lemma 14.14]) *Let G be a p-group of maximal class of order p^n, $n \leq p + 1$. Then* $\exp(G') = p$.

Lemma 2.3 (see [9, Theorem 14.21]) *Let G be a p-group of maximal class. Then G is regular if and only if $|G| \leq p^p$.*

3 Main Results

In this section, we give our main results. All p-groups of maximal class considered in this section are non-abelian.

For convenience, we denote by $N(G:H)$ and $w(G:H)$ the preimages of $N(G/H)$ and $w(G/H)$, respectively, if H is a normal subgroup of a group G and $\alpha\colon G \to G/H$ is the natural homomorphism.

Lemma 3.1 *Let G be a group and $H \le \zeta(G)$. Then for any $g, h \in G$ and $n \in N(G:H)$, we have*

(1) $[g,n,h],[h,n,g],[g,h,n] \in \zeta(G)$;

(2) $[n,g,g]=1,\ [n,g^{-1}]=[n,g]^{-1}$;

(3) $[g,n,h][h,n,g]=1$.

Proof (1) It follows from [13] that $N(G/H) \le \zeta_2(G/H) \le \zeta_3(G)/H$ and therefore $n \in \zeta_3(G)$. Thus $[g,n,h],[h,n,g],[g,h,n] \in \zeta(G)$.

(2) Let $\overline{G}=G/H$. For any $\overline{n} \in N(\overline{G})$ and $\overline{g} \in \overline{G}$, we have $\overline{g}^{\overline{n}}=\overline{g}^i$, where i is a positive integer. This means that $g^n \equiv g^i \pmod H$ and therefore g^n commutes with g. Thus $1=[g^n,g]=[g[g,n],g]=[g,n,g]=[n,g,g]$ and $[n,g^{-1}]=[n,g]^{-1}$.

(3) By (2) we get that $[gh,n]^{gh}=[gh,n]=[g,n][h,n][g,n,h]$. On the other hand, $[gh,n]^{gh}=([gh,n]^g)^h=([g,n][h,n][h,n,g][g,n,h])^h=[g,n][g,n,h]^2[h,n][h,n,g]$. Thus $[g,n,h][h,n,g]=1$.

Lemma 3.2 *Let G be a group with cyclic $G' \cap \zeta_2(G)$. Then $N(G/\zeta(G))=\zeta(G/\zeta(G))$.*

Proof Suppose that $N(G/\zeta(G)) \ne \zeta(G/\zeta(G))$. Then there exists an $n \in N(G:\zeta(G))$ and $g,h \in G$ such that $[g,n,h] \ne 1$. Noticing that $[g,n],[h,n] \in G' \cap \zeta_2(G)$, we see that either $[g,n]=[h,n]^i$ or $[h,n]=[g,n]^j$. If $[g,n]=[h,n]^i$, then by Lemma 3.1(2), $[g,n,h]=[[h,n]^i,h]=1$. Using the same arguments, we get that $[h,n,g]=1$ if $[h,n]=[g,n]^j$. Thus $[g,n,h]=1$, a contradiction.

Theorem 3.1 *Let G be a p-group of maximal class of order p^n.*

(1) *If $w_i(G) \subseteq \zeta_r(G)$, then $w_{i+1}(G) \subseteq \zeta_{r+1}(G)$, where $i \ge 1$, $r \ge 1$;*

(2) *$w_i(G)=\zeta_i(G)$ for all integer i or $w_i(G)=\zeta_{i+1}(G)$ for every integer i except for $G \cong M_p(2,1)$, where $p>2$;*

(3) *If $1 < K \trianglelefteq G$, then $w(G/K)=\zeta(G/K)$.*

Proof (1) By Theorem 2.1, we get $|\zeta(G)|=p$. Furthermore, $\zeta_2(G) \cong C_p \times C_p$ if $p>2$ and $n>3$. We claim that $w(G/\zeta_i(G))=\zeta(G/\zeta_i(G))$ for any positive integer i.

(i) $p>2$.

It is clear that $w(G/\zeta_i(G))=\zeta(G/\zeta_i(G))$ for $n \le 3$. Now assume that $n \ge 4$. We first consider the case $i=1$. Let $\overline{G}=G/\zeta(G)$. Assume that $w(\overline{G}) \ne \zeta(\overline{G})$. Then there exists an $n \in w(G:\zeta(G))$ and $g,h \in G$ such that $[g,n,h] \ne 1$. Since $[g,n,h] \in \zeta(G)$ and $|\zeta(G)|=p$, $\zeta(G)=\langle [g,n,h] \rangle$. Noticing that $[g,n] \in \zeta_2(G),\ [h,n] \in \zeta_2(G)$ and $\zeta_2(G) \cong C_p \times C_p$, we see that $\zeta_2(G)=\langle [g,n] \rangle \times \langle [h,n] \rangle$. So $[h^j,[g,n]]=([h,[g,n]])^j=[g,n][h,n]^i$, where $p \nmid ij$. This means that $h^{-j}[g,n]^{-1}h^j=[h,n]^i$ and therefore $[g,n]^{-1}=[h,n]^i$, a contradiction.

For $i \ge 2$, if $|G/\zeta_i(G)| \le p^2$, then $w(G/\zeta_i(G))=\zeta(G/\zeta_i(G))$. Assume $|G/\zeta_i(G)| \ge p^3$. Then $|G/\zeta_{i-1}(G)| \ge p^4$ and therefore $G/\zeta_{i-1}(G)$ is a p-group of maximal class by Theorem 2.1(4). By the proof of $i=1$, we have $w(G/\zeta_{i-1}(G)\big/\zeta(G/\zeta_{i-1}(G)))=\zeta(G/\zeta_{i-1}(G)\big/\zeta(G/\zeta_{i-1}(G)))$. Since $G/\zeta_{i-1}(G)\big/\zeta(G/\zeta_{i-1}(G))=G/\zeta_{i-1}(G)\big/\zeta_i(G)/\zeta_{i-1}(G) \cong G/\zeta_i(G)$, we have $w(G/\zeta_i(G))=\zeta(G/\zeta_i(G))$.

(ii) $p=2$.

If G is a 2-group of maximal class, then by Theorem 2.3, we get that G is a metacyclic p-group. By Lemma 3.2, we have $w(G/\zeta(G)) = \zeta(G/\zeta(G))$. For $i \geq 2$, since $(G/\zeta_i(G))'$ is cyclic, we have $w(G/\zeta_{i-1}(G)/\zeta(G/\zeta_{i-1}(G))) = \zeta(G/\zeta_{i-1}(G)/\zeta(G/\zeta_{i-1}(G)))$ by Lemma 3.2. Noticing that $G/\zeta_{i-1}(G)/\zeta(G/\zeta_{i-1}(G)) = G/\zeta_{i-1}(G)/\zeta_i(G)/\zeta_{i-1}(G) \cong G/\zeta_i(G)$, we see that $w(G/\zeta_i(G)) = \zeta(G/\zeta_i(G))$.

Let $w \in w_{i+1}(G)$. Then for each $g \in G$, there is an integer k such that $g^w \equiv g^k \pmod{w_i(G)}$. Hence $g^{-k}g^w \in w_i(G) \leq \zeta_r(G)$. Let $\overline{G} = G/\zeta_r(G)$. Since $g^{-k}g^w \in \zeta_r(G)$, $\overline{g}^{\overline{w}} = \overline{g}^k$. Thus $w_{i+1}(G)\zeta_r(G)/\zeta_r(G) \leq w(G/\zeta_r(G))$. Noticing that $w(G/\zeta_r(G)) = \zeta(G/\zeta_r(G)) = \zeta_{r+1}(G)/\zeta_r(G)$, we see that $w_{i+1}(G) \subseteq \zeta_{r+1}(G)$.

(2) First we claim that either $w(G) = \zeta(G)$ or $w(G) = \zeta_2(G)$ except for $G \cong M_p(2,1)$, where $p > 2$.

If $n > 3$, then by Theorem 2.1 we have $|\zeta(G)| = p$ and $|\zeta_2(G)| = p^2$. Since $\zeta(G) \leq w(G) \leq \zeta_2(G)$, either $w(G) = \zeta(G)$ or $w(G) = \zeta_2(G)$.

Assume that $n = 3$. By Theorem 2.2, G is isomorphic to one of the groups Q_8, $M_p(1,1,1)$ and $M_p(2,1)$.

If $G \cong Q_8$, then $w(G) = \zeta_2(G)$. If $G \cong M_p(1,1,1) = \langle a,b,c \mid a^p = b^p = c^p = 1,\ [a,b] = c\rangle$, then for any $a^ib^jc^k \in w(G)$, we have $a^{a^ib^jc^k} \in \langle a\rangle$. So $p \mid j$. Using the same arguments, we get $p \mid i$. It follows that $w(G) = \zeta(G)$. Assume that $G \cong M_p(2,1) = \langle a,b \mid a^{p^2} = b^p = 1,\ [a,b] = a^p\rangle$, where $p > 2$. We may prove that $w(G) = \langle b, a^p\rangle$. Since $(b^ja^i)^b = b^ja^i[a,b]^i = b^ja^ia^{ip} = (b^ja^i)^{1+p}$ and $a^p \in \zeta(G)$, $\langle b, a^p\rangle \leq w(G)$. Noticing $\langle b, a^p\rangle \lessdot G$, we see that $w(G) = \langle b, a^p\rangle$.

If $w_i(G) = \zeta_r(G)$, then $w_{i+1}(G)/w_i(G) = w(G/w_i(G)) = w(G/\zeta_r(G)) = \zeta(G/\zeta_r(G)) = \zeta_{r+1}(G)/\zeta_r(G)$. It follows that $w_{i+1}(G) = \zeta_{r+1}(G)$. Since $w(G) = \zeta(G)$ or $w(G) = \zeta_2(G)$ except for $G \cong M_p(2,1)$, where $p > 2$, by induction, we get that either $w_i(G) = \zeta_i(G)$ or $w_i(G) = \zeta_{i+1}(G)$ except for $M_p(2,1)$.

(3) If $|K| \geq p^{n-1}$, then it is clear that $w(G/K) = \zeta(G/K)$.

Assume that $|K| = p^{n-i}$ $(2 \leq i \leq n-1)$. Then, since G_i is the unique normal subgroup of G of order p^{n-i}, we have $K = G_i$. Noticing that $G_i = \zeta_{n-i}(G)$, we see that $K = \zeta_{n-i}(G)$. By the proof of (1), we get $w(G/\zeta_{n-i}(G)) = \zeta(G/\zeta_{n-i}(G))$. Hence $w(G/K) = \zeta(G/K)$.

Remark 3.1 The importance of Theorem 3.1 is that we find out a class of p-groups such that $w(G/K) = \zeta(G/K)$ for every group G in this class and every normal subgroup K in G with $K \neq 1$. However, this is not true in general. You may find examples in Section 4.

Corollary 3.1 *Let G be a p-group of order p^n with $w_r(G) \subseteq \zeta_i(G)$, where $n \leq 5$, $r \geq 1$ and $i \geq 1$. Then $w_{r+1}(G) \subseteq \zeta_{i+1}(G)$.*

Proof If $G/\zeta(G) \cong Q_8 = \langle \overline{a}, \overline{b} \mid \overline{a}^4 = 1,\ \overline{a}^2 = \overline{b}^2,\ [\overline{a}, \overline{b}] = \overline{a}^2\rangle$, then $a^2 \equiv b^2 \pmod{\zeta(G)}$ and therefore $a^2 \in \zeta(G)$, a contradiction. Assume that $G/\zeta(G) \cong M_p(2,1) = \langle \overline{a}, \overline{b} \mid \overline{a}^{p^2} = 1, \overline{b}^p = 1, [\overline{a}, \overline{b}] = \overline{a}^p\rangle$. Then $G' = \langle [a,b]\rangle$. By Lemma 3.2, $w(G/\zeta(G)) = \zeta(G/\zeta(G))$. However, by Theorem 3.1(2), $w(G/\zeta(G)) \neq \zeta(G/\zeta(G))$, a contradiction. So $G/\zeta(G)$ is either abelian or isomorphic to $M_p(1,1,1)$ if $n \leq 4$. By Theorem 3.1(2), we get $w(G/\zeta(G)) = \zeta(G/\zeta(G))$ if $n \leq 4$.

Now assume that $n = 5$. If $|\zeta(G)| \geq p^2$, then by the above proof we have $w(G/\zeta(G)) = \zeta(G/\zeta(G))$. If $|\zeta(G)| = p$, then we consider three cases: (i) $c(G) \leq 2$; (ii) $c(G) = 4$; (iii) $c(G) = 3$.

(i) If $c(G) \leq 2$, then it is clear that $w(G/\zeta(G)) = \zeta(G/\zeta(G))$;

(ii) If $c(G) = 4$, then by the proof of Theorem 3.1(1), $w(G/\zeta(G)) = \zeta(G/\zeta(G))$;

(iii) Now assume that $c(G) = 3$. Then $|G'| = p^2$ or $|G'| = p^3$. If $G' \cong C_p \times C_p$, then by using the same arguments as Theorem 3.1(1), we may prove that $w(G/\zeta(G)) = \zeta(G/\zeta(G))$. If $G' \cong C_{p^2}$, then by Lemma 3.2, we get $w(G/\zeta(G)) = \zeta(G/\zeta(G))$. Assume $|G'| = p^3$. Then

$|G/\zeta(G)| = p^4$, $c(G/\zeta(G)) = 2$ and $d(G/\zeta(G)) = 2$. It follows that $\zeta(G/\zeta(G)) = (G/\zeta(G))' = \Phi(G/\zeta(G))$ and therefore $G/\zeta(G)$ is a minimal non-abelian p-group, which is a contradiction to $|(G/\zeta(G))'| = p^2$.

For $i \geq 2$, since $|G/\zeta_{i-1}(G)| \leq p^4$ and $G/\zeta_{i-1}(G)\big/\zeta(G/\zeta_{i-1}(G)) = G/\zeta_{i-1}(G)\big/\zeta_i(G)/\zeta_{i-1}(G) \cong G/\zeta_i(G)$, we have $w(G/\zeta_i(G)) = \zeta(G/\zeta_i(G))$. Using the same arguments as Theorem 3.1(1), we may get the conclusion.

Theorem 3.2 *Let G be a 2-group of maximal class of order 2^n.*

(1) *If $G \cong D_{2^n}$ or SD_{2^n}, then $w(G) = \zeta(G)$;*

(2) *If $G \cong Q_{2^n}$, then $w(G) = \zeta_2(G)$.*

Proof By Theorem 2.3, G is isomorphic to one of groups D_{2^n}, SD_{2^n} and Q_{2^n}.

(1) If $G \cong D_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, [a, b] = a^{-2}\rangle$, we consider two cases: $n = 3$ or $n \geq 4$. If $n = 3$, then $G \cong D_8$. By Theorem 3.1(2), we have $w(G) = \zeta(G)$. If $n \geq 4$, then since $\zeta_2(G) = G_{n-2} = \langle a^{2^{n-3}}\rangle$ and $b^{a^{2^{n-3}}} = ba^{2^{n-2}} \notin \langle b\rangle$, we get $w(G) \neq \zeta_2(G)$ and therefore $w(G) = \zeta(G)$ by Theorem 3.1(2). Using the same arguments, we may prove that $w(SD_{2^n}) = \zeta(SD_{2^n})$.

(2) Now assume that $G \cong Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, [a, b] = a^{-2}\rangle$.

(i) If $n = 3$, then $G \cong Q_8$ and therefore $w(G) = \zeta_2(G)$;

(ii) If $n \geq 4$, then $\zeta_2(G) = G_{n-2} = \langle a^{2^{n-3}}\rangle$. For any $g \in G$, g can be written as ba^i or a^i, where i is a non-negative integer. Clearly, $a^{2^{n-3}} \in N_G(\langle a^i\rangle)$. We assume that ba^i is an element of G, then $(ba^i)^{a^{2^{n-3}}} = b^{-1}a^i = (ba^i)^3 \in \langle ba^i\rangle$. Hence $a^{2^{n-3}} \in w(G)$ and therefore $w(G) = \zeta_2(G)$.

Lemma 3.3 *Let G be a regular p-group with $n \in w(G)$. If $h \in G$ with $o(h) = \exp(G)$ and $h^n = h^i$, then $g^n = g^i$ for all $g \in G$, where i is a positive integer.*

Proof It follows from Corollary 2.1 that there exists an integer m such that $g^n = g^m$ for all $g \in G$. If $o(h) = \exp(G)$ and $h^n = h^i$, then $i \equiv m \pmod{\exp(G)}$. Thus $g^n = g^i$ for all $g \in G$.

Theorem 3.3 *Let $G = \langle g_1, g_2, \cdots, g_t\rangle$ be a regular p-group, where $o(g_i) \leq o(g_1) = p^m$, $\exp\,(\zeta(G)) = p^k$ and $2 \leq i \leq t$.*

(1) *If $\exp(G) = \exp(\zeta(G))$, then $w(G) = \zeta(G)$;*

(2) *If $\exp(G) > \exp(\zeta(G))$ and $o(g_i) \leq p^{m-k}$, then $w(G) = \zeta_2(G)$ if and only if $\zeta_2(G) \leq N_G(\langle g_1\rangle) \cap C_G(g_2) \cap \cdots \cap C_G(g_t)$;*

(3) *If G is a p-group of maximal class of order p^n, then*

(i) $\exp(G) \leq p^2$;

(ii) *If $\exp(G) = p$, then $w(G) = \zeta(G)$;*

(iii) *If $\exp(G) = p^2$, then we may assume that $o(g_1) > o(g_2)$ and $w(G) = \zeta_2(G)$ if and only if $\zeta_2(G) \leq N_G(\langle g_1\rangle) \cap C_G(g_2)$.*

Proof (1) Choose $g \in \zeta(G)$ such that $o(g) = \exp(G)$. Then $g^w = g$ for any $w \in w(G)$. It follows from Lemma 3.3 that $w(G) = \zeta(G)$.

(2) Set $g_1^a = g_1^j$, where $a \in \zeta_2(G)$ and j is an integer. Since $[g_1, a] \in \zeta(G)$ and $\exp(\zeta(G)) = p^k$, $p^{m-k} \mid (j-1)$. So we may assume that $j = 1 + sp^{m-k}$. By Lemma 3.3, we get $g_i^a = g_i$. It follows that $\zeta_2(G) \leq N_G(\langle g_1\rangle) \cap C_G(g_2) \cap \cdots \cap C_G(g_t)$.

Conversely, by the above proof, we may assume that $g_1^a = g_1^{1+sp^{m-k}}$ for any $a \in \zeta_2(G)$. It is clear that $\exp(G') \leq p^{m-k}$ and therefore G is p^{m-k}-abelian. For any $g_1^{i_1} g_2^{i_2} \cdots g_t^{i_t} c \in G$ and $a \in \zeta_2(G)$, where $c \in G'$, we have $(g_1^{i_1} g_2^{i_2} \cdots g_t^{i_t} c)^a = (g_1^{i_1})^{1+sp^{m-k}} g_2^{i_2} \cdots g_t^{i_t} c = g_1^{i_1} g_2^{i_2} \cdots g_t^{i_t} c g_1^{i_1 sp^{m-k}} = (g_1^{i_1} g_2^{i_2} \cdots g_t^{i_t} c)^{1+sp^{m-k}}$. Thus $w(G) = \zeta_2(G)$.

(3) Since G is a regular p-group of maximal class, we may assume that $G = \langle g_1, g_2\rangle$, $o(g_1) \geq o(g_2)$ and $\langle g_1\rangle \cap \langle g_2\rangle = 1$.

(i) By Lemma 2.3, $|G| \leq p^p$. It follows from Lemma 2.2 that $\exp(G') = p$. So $[g_1^p, g_2] = [g_1, g_2]^p = 1$ and therefore $g_1^p \in \zeta(G)$. Since $|\zeta(G)| = p$, $o(g_1) \leq p^2$. Using the same arguments, we get $o(g_2) \leq p^2$. Thus $\exp(G) \leq p^2$.

(ii) By (1), we get the conclusion.

(iii) Suppose that $o(g_1) = o(g_2) = p^2$. By the proof of (i), we get $g_1^p, g_2^p \in \zeta(G)$. Noticing $|\zeta(G)| = p$, we see that $\langle g_1\rangle \cap \langle g_2\rangle \neq 1$, a contradiction. Thus $o(g_1) = p^2$ and $o(g_2) = p$. From (2), we get the conclusion.

However, Theorem 3.3 is not true in general. Examples 3.1 and 3.2 show that the requirement that G is regular in Theorem 3.3 is necessary. Example 3.3 shows that the requirement $o(g_i) \leq p^{m-k}$ is necessary.

Example 3.1 $G = \langle a, b \mid a^9 = b^3 = c^3 = 1,\ [a, b] = c,\ [c, a] = a^3,\ [c, b] = 1\rangle$. It is easy to see that G is a p-group of maximal class of order 3^4 and $\zeta_2(G) = G_2 = \langle c, a^3\rangle$ by Theorem 2.1(5). Since $(a^2b^{-1})^c = a^5b^{-1} \notin \langle a^2b^{-1}\rangle$, $c \notin w(G)$ and therefore $w(G) = \zeta(G)$ by Theorem 3.1(2). However, $\zeta_2(G) \leq N_G(\langle a\rangle) \cap C_G(b)$.

Example 3.2 $G \cong Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, [a, b] = a^{-2}\rangle$, where $n \geq 4$. It is easy to see that G is a 2-group of maximal class of order 2^n. By Theorem 3.2, $w(G) = \zeta_2(G) = \langle a^{2^{n-3}}\rangle$. However, $a^{2^{n-3}} \notin C_G(b)$.

Example 3.3 $G = \langle a, b \mid a^{p^3} = 1,\ b^{p^2} = a^{p^2},\ [a, b] = a^p\rangle$, where $p \geq 5$. It is clear that G is regular. By calculation, we get that $\zeta(G) = \langle a^{p^2}\rangle$ and $\zeta_2(G) = \langle a^p\rangle$. Since $b^{a^p} = b^{1-p^2}$, $\zeta_2(G) \leq N_G(\langle b\rangle) \cap C_G(a)$. However, $(ba^{-1})^{a^p} = b^{1-p^2}a^{-1} \notin \langle ba^{-1}\rangle$. So $w(G) \neq \zeta_2(G)$.

If G is a p-group of maximal class of odd order, then $w(G)$ is an elementary abelian p-group. Next we may prove that $\mathrm{PAut}(G)$ is also an elementary abelian p-group. Furthermore, we can get that $\mathrm{PAut}(G)$ is an elementary abelian p-group if G is a non-abelian p-group with elementary abelian $\zeta(G) \cap \mho_1(G)$.

Theorem 3.4 *Let G be a non-abelian p-group.*

(1) *If $\zeta(G) \cap \mho_1(G)$ is elementary abelian, then*

(i) *for any $\alpha \in \mathrm{PAut}(G)$ and $g \in G$, we have $g^\alpha = g^{1+kp^{n-1}}$, where $o(g) = p^n$, k is an integer and $0 \leq k \leq p-1$;*

(ii) $\mathrm{PAut}(G)$ *is an elementary abelian p-group.*

(2) *If G is a p-group of maximal class, then* $\mathrm{PAut}(G)$ *is an elementary abelian p-group.*

(3) *If $\zeta(G) \cap \mho_1(G) \cap G'$ is elementary abelian, then $w(G)/\zeta(G)$ is elementary abelian.*

Proof (1) (i) Set $g^\alpha = g^i$, where i is an integer, $(i, p) = 1$ and $1 \leq i \leq p^n - 1$. By Lemma 2.4, $i \equiv 1 \pmod p$. Since every power automorphism is central, $g^{i-1} \in \zeta(G) \cap \mho_1(G)$. Noticing that $\exp(\zeta(G) \cap \mho_1(G)) = p$, we see that $o(g^{i-1}) \leq p$ and therefore $p^{n-1} \mid (i-1)$. So we may assume that $i = 1 + kp^{n-1}$, where k is an integer and $0 \leq k \leq p-1$. Thus $g^\alpha = g^{1+kp^{n-1}}$.

(ii) By Lemma 2.4, $\mathrm{PAut}(G)$ is an abelian p-group. So we may assume that $\mathrm{PAut}(G) = \langle\alpha_1\rangle \times \langle\alpha_2\rangle \times \cdots \times \langle\alpha_m\rangle$. Since $g^{\alpha_j} = g^{1+kp^{n-1}}$ by (i), $g^{(\alpha_j)^p} = g^{(1+kp^{n-1})^p} = g$. It follows that $o(\alpha_j) \leq p$.

(2) Since G is a p-group of maximal class, $|\zeta(G)| = p$. By (1), $\mathrm{PAut}(G)$ is elementary abelian.

(3) If $w(G)$ is non-abelian, then $G \cong Q_8 \times C_2^n$. It is clear that $w(G)/\zeta(G) \cong C_2 \times C_2$. Now assume that $w(G)$ is abelian. For any $w \in w(G)$, set $g^w = g^i$, where i is an integer, $(i, p) = 1$. Using the same arguments as (1), we may get that $g^w = g^{1+kp^{m-1}}$, where $o(g) = p^m$. Thus

$g^{w^p} = g^{(1+kp^{m-1})^p} = g$. It follows that $w^p \in \zeta(G)$ and therefore $w(G)/\zeta(G)$ is elementary abelian.

Remark 3.2 We should notice that the converse of Theorem 3.4(1) is not true in general. Let $G = \langle a, b \mid a^8 = b^8 = 1,\ [a,b] = b^4\rangle$. It is clear that $\zeta(G) \cap \mho_1(G) \cong C_4 \times C_4$. By [7, Corollary 6.3.3], we get $\mathrm{PAut}(G) \cong C_2 \times C_2$. In Section 4, we give two examples which show that there exist non-abelian p-groups satisfying Theorem 3.4.

4 Examples

In this section, we give some examples. Example 4.1 and Example 4.2 show that Theorem 3.1 is not true in general. Example 4.3 and Example 4.4 show that there exist non-abelian p-groups satisfying Theorem 3.4. Finally, we give some regular p-groups of maximal class satisfying Theorem 3.3.

Example 4.1 Let $G = \langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1,\ [b,a] = c,\ [c,a] = a^p,\ [c,b] = b^p\rangle$, where $p \geq 5$. It is easy to see that $\zeta(G) = \langle a^p, b^p\rangle$. So $\langle b^p\rangle \trianglelefteq G$. Let $\overline{G} = G/\langle b^p\rangle = \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^2} = \overline{b}^p = \overline{c}^p = \overline{1}, [\overline{b}, \overline{a}] = \overline{c}, [\overline{c}, \overline{a}] = \overline{a}^p\rangle$. For any $\overline{a}^i\overline{b}^j\overline{c}^k \in \overline{G}$, we have $(\overline{a}^i\overline{b}^j\overline{c}^k)^{\overline{c}} = (\overline{a}^i)^{1-p}\overline{b}^j\overline{c}^k = (\overline{a}^i\overline{b}^j\overline{c}^k)^{1-p}$. So $\overline{c} \in w(\overline{G}) \setminus \zeta(\overline{G})$ and therefore $w(\overline{G}) > \zeta(\overline{G})$. So there exists a non-abelian p-group with $w(G/K) > \zeta(G/K)$, where $K \trianglelefteq G$ and $1 < K < G$.

Example 4.2 Let $G = \langle a, b, c, d \mid a^{3^n} = b^{3^n} = d^{3^n} = c^3 = 1, [a,b] = d,\ [b,c] = b^{-3^{n-1}}, [c,a] = a^{3^{n-1}}d^{-3^{n-1}}, [d,c] = d^{3^{n-1}}, [d,a] = [d,b] = 1\rangle$, where $n \geq 2$.

It is easy to see that $\zeta(G) = \langle d^3\rangle$. So $\overline{G} = G/\zeta(G) = \langle \overline{a}, \overline{b}, \overline{c}, \overline{d} \mid \overline{a}^{3^n} = \overline{b}^{3^n} = \overline{c}^3 = \overline{d}^3 = \overline{1}, [\overline{a}, \overline{b}] = \overline{d},\ [\overline{b}, \overline{c}] = \overline{b}^{-3^{n-1}}, [\overline{c}, \overline{a}] = \overline{a}^{3^{n-1}}, [\overline{d}, \overline{c}] = [\overline{d}, \overline{a}] = [\overline{d}, \overline{b}] = \overline{1}\rangle$. It is clear that $c(\overline{G}) = 2$.

Now we may prove that $w(G) = \zeta(G)$ and $w_2(G) > \zeta_2(G)$.

It is clear that $\zeta_2(G) = \langle a^3, b^3, d\rangle$. Since $w(G) \leq \zeta_2(G)$, w can be written as $w = a^{3i}b^{3j}d^l$ for any $w \in w(G)$, where i, j and l are non-negative integers. Now assume that $w = a^{3i}b^{3j}d^l$ is an element of $w(G)$. Then $a^{a^{3i}b^{3j}d^l} = a[a,b]^{3j} = ad^{3j} \in \langle a\rangle$. Noticing that $\langle a\rangle \cap \langle d\rangle = 1$, we see that $3^{n-1} \mid j$. By using the same arguments, we may prove that $3^{n-1} \mid i$ and $3 \mid l$. Hence $w \in \zeta(G)$. It follows that $\zeta(G) = w(G)$.

For any $\overline{g} \in \overline{G}$, $\overline{g}$ can be written as $\overline{a}^i\overline{b}^j\overline{c}^k\overline{d}^l$, where i, j, k, l are non-negative integers. Then $\overline{g}^{\overline{c}} = (\overline{a}^i\overline{b}^j\overline{c}^k\overline{d}^l)^{\overline{c}} = (\overline{a}^{\overline{c}})^i(\overline{b}^{\overline{c}})^j\overline{c}^k\overline{d}^l = (\overline{a}^{1-3^{n-1}})^i(\overline{b}^{1-3^{n-1}})^j\overline{c}^k\overline{d}^l = \overline{a}^i\overline{b}^j\overline{c}^k\overline{d}^l\overline{a}^{-i3^{n-1}}\overline{b}^{-j3^{n-1}} = (\overline{a}^i\overline{b}^j\overline{c}^k\overline{d}^l)^{1-3^{n-1}}$. Hence $\overline{c} \in w(\overline{G}) \setminus \zeta(\overline{G})$ and therefore $\zeta(\overline{G}) < w(\overline{G})$. Since $w_2(G)/\zeta(G) = w_2(G)/w(G) = w(G/w(G)) = w(G/\zeta(G)) > \zeta(G/\zeta(G)) = \zeta_2(G)/\zeta(G)$, we get $w_2(G) > \zeta_2(G)$.

Example 4.3 (see [10, Example (2.3 b)]) Let $A = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle z\rangle$ be an elementary abelian 3-group of order 3^4. Then A has an elementary abelian group of automorphisms $\langle y, x, t\rangle$ of order 3^3, where

$$\begin{array}{llll} a^y = a, & b^y = bz^2, & c^y := cz^2, & z^y := z; \\ a^x = az^2, & b^x = bz^2, & c^x = cz, & z^x = z; \\ a^t = a, & b^t = b, & c^t = cz^2, & z^t = z. \end{array}$$

If we extend A successively by y, x and t, putting $y^3 = x^3 = t^3 = z$, and $[y,x] = a$, $[y,t] = c$, $[x,t] = b^2$, we get an extension G of order 3^7 and class 3, and the power automorphism group $\mathrm{PAut}(G)$ is elementary abelian of order 3^3, generated by the automorphisms induced by the elements a, a^2b and abc^2. It is clear that $\zeta(G) \cap \mho_1(G) = \langle z\rangle$.

Example 4.4 (see [10, Example (2.2)]) Let $A = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{p-1}\rangle$, where a_1 is of order p^{n+1} and a_i is of order p^n, if $2 \le i \le p-1$. Then an endomorphism T of A is defined by

$$a_i^T = a_i a_{i+1} \text{ for } 1 \le i \le p-2, \quad a_{p-1}^T = a_{p-1}a_1^p, \quad a_1^{1+T+T^2+\cdots+T^{p-1}} = 1.$$

Obviously, T is an automorphism of A of order p. The extension G of A by T, where $T^p = a_1^{p^n}$, is a p-group of maximal class, and the power automorphism group $\mathrm{PAut}(G)$ is elementary abelian of rank 2; two linearly independent elements α, $\beta \in \mathrm{PAut}(G)$ are given by $a_1^\alpha = a_1$, $T^\alpha = T^{1+p}$ and $a_1^\beta = a_1^{1+p^n}$, $T^\beta = T$.

Example 4.5 Let $G = \langle a, b, c, d, e \mid a^p = b^p = c^p = d^p = e^p = 1,\ [b, a] = c,\ [c, a] = d,\ [c, b] = [d, a] = e,\ [c, d] = [d, b] = [e, a] = [e, b] = [e, c] = [e, d] = 1\rangle$, where $p \ge 5$. It is clear that $|G| = p^5$ and $c(G) = 4$. So G is a regular p-group of maximal class. Noticing that $\exp(G) = p$, we see that $w(G) = \zeta(G)$ by Theorem 3.3(3).

Example 4.6 Let $G = \langle a, b, c, d \mid a^{p^2} = b^p = c^p = d^p = 1,\ [b, a] = c,\ [c, a] = d,\ [c, b] = a^p,\ [d, a] = a^p,\ [d, b] = 1\rangle$, where $p \ge 5$ and $p \equiv 3 \pmod 4$. It is clear that $|G| = p^5$, $c(G) = 4$ and $\exp(G) = p^2$. So G is a regular p-group of maximal class and therefore $\zeta_2(G) = G_3 = \langle d, a^p\rangle$. It is easy to see that $\zeta_2(G) \le N_G(\langle a\rangle) \cap C_G(b)$. By Theorem 3.3(3), we have $w(G) = \zeta_2(G)$.

Example 4.7 Let $G = \langle a, b, c, d \mid a^{p^2} = b^p = c^p = d^p = 1,\ [b, a] = c,\ [c, a] = a^p,\ [c, b] = d,\ [d, a] = 1,\ [d, b] = a^p\rangle$, where $p \ge 5$ and $p \equiv 2 \pmod 3$. It is clear that $|G| = p^5$, $c(G) = 4$ and $\exp(G) = p^2$. So G is a regular p-group of maximal class and therefore $\zeta_2(G) = G_3 = \langle d, a^p\rangle$. Since $d \notin C_G(b)$, we have $w(G) = \zeta(G)$ by Theorem 3.3(3).

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