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Sharp Observability Inequalities for the 1-D Plate Equation with a Potential*

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Abstract This paper deals with the problem of sharp observability inequality for the 1-D plate equation $w_{tt} + w_{xxxx} + q(t,x)w = 0$ with two types of boundary conditions $w = w_{xx} = 0$ or $w = w_x = 0$, and $q(t,x)$ being a suitable potential. The author shows that the sharp observability constant is of order $\exp(C\|q\|_\infty^{\frac{2}{7}})$ for $\|q\|_\infty \geq 1$. The main tools to derive the desired observability inequalities are the global Carleman inequalities, based on a new point wise inequality for the fourth order plate operator.

Keywords Observability inequality, Plate equation, Point-wise estimate, Carleman estimate
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1 Introduction

Let $T > 0$ be given, Ω be a bounded domain (i.e., a bounded open interval) in $\mathbb{R}$ with boundary Γ (i.e., the endpoints of the interval). Put $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$.

We are interested in the following plate equation with a potential:

$$\begin{cases} w_{tt} + w_{xxxx} + q_1 w = 0, & \text{in } Q, \\ w = w_{xx} = 0, & \text{on } \Sigma, \\ w(0) = w^0, \quad w_t(0) = w^1, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $q_1 \in L^\infty(0,T; W^{1,\infty}(\Omega))$. Also, we shall consider the same plate equation but with different boundary conditions

$$\begin{cases} w_{tt} + w_{xxxx} + q_2 w = 0, & \text{in } Q, \\ w = w_x = 0, & \text{on } \Sigma, \\ w(0) = w^0, \quad w_t(0) = w^1, & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $q_2 \in L^\infty(0,T; W^{1,\infty}(\Omega))$. Set

$$W_1 \stackrel{\triangle}{=} \{w \in H^3(\Omega) \mid w|_\Gamma = w_{xx}|_\Gamma = 0\}, \quad W_2 \stackrel{\triangle}{=} \{w \in H^3(\Omega) \mid w|_\Gamma = w_x|_\Gamma = 0\}.$$

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Based on the semigroup theory (see [12, Theorem 1.4, p. 185]), it is easy to show that system (1.1) admits a unique weak solution $w \in C([0,T];W_1) \cap C^1([0,T];H_0^1(\Omega))$, while system (1.2) admits one and only one weak solution $w \in C([0,T];W_2) \cap C^1([0,T];H_0^1(\Omega))$.

In what follows, we shall denote by $\|\cdot\|_\infty$ and $\|\cdot\|_{1,\infty}$ the (usual) norms on $L^\infty(Q)$ and $L^\infty(0,T;W^{1,\infty}(\Omega))$, respectively. Denote the energy of (1.1) by

$$E_1(t) = \frac{1}{2}[\|w_{tx}(t,\cdot)\|^2_{L^2(\Omega)} + \|w_{xxx}(t,\cdot)\|^2_{L^2(\Omega)} + \|q_1\|_\infty \|w_x(t,\cdot)\|^2_{L^2(\Omega)}] \tag{1.3}$$

for the solution w to system (1.1). Next, we define an unbounded operator $\mathcal{A}$ in $L^2(\Omega)$ as follows:

$$\mathcal{A} \triangleq \partial_{xxxx}, \quad D(\mathcal{A}) \triangleq H^4(\Omega) \cap H_0^2(\Omega). \tag{1.4}$$

It is easy to see that $\mathcal{A}$ is a self-adjoint operator. Assume that $\{\mu_j\}_{j=1}^\infty$ are the eigenvalues of $\mathcal{A}$ and the corresponding eigenvectors $\{e_j\}_{j=1}^\infty$ consist of the complete orthogonal basis in $L^2(\Omega)$. Then, we define $\mathcal{A}^{\frac{1}{4}}$, $\mathcal{A}^{\frac{3}{4}}$ as follows:

$$\begin{cases} D(\mathcal{A}^{\frac{1}{4}}) \triangleq \Big\{ w = \displaystyle\sum_{j=1}^\infty a_j e_j \,\Big|\, a_j \in \mathbb{R},\ \sum_{j=1}^\infty a_j^2 \mu_j^{\frac{1}{2}} < \infty \Big\}, \quad \mathcal{A}^{\frac{1}{4}} w \triangleq \sum_{j=1}^\infty a_j \mu_j^{\frac{1}{4}} e_j \in L^2(\Omega), \\ D(\mathcal{A}^{\frac{3}{4}}) \triangleq \Big\{ w = \displaystyle\sum_{j=1}^\infty a_j e_j \,\Big|\, a_j \in \mathbb{R},\ \sum_{j=1}^\infty a_j^2 \mu_j^{\frac{3}{2}} < \infty \Big\}, \quad \mathcal{A}^{\frac{3}{4}} w \triangleq \sum_{j=1}^\infty a_j \mu_j^{\frac{3}{2}} e_j \in L^2(\Omega). \end{cases} \tag{1.5}$$

We claim that $\mathcal{A}^{\frac{1}{4}}$ is also a self-adjoint operator on the domain $D(\mathcal{A}^{\frac{1}{4}})$. More precisely, for any $w, \widehat{w} \in D(\mathcal{A}^{\frac{1}{4}})$, there exist $\{a_j\}_{j=1}^\infty$, $\{b_k\}_{k=1}^\infty \subset \mathbb{R}$ such that

$$w = \sum_{j=1}^\infty a_j e_j,\ \ \widehat{w} = \sum_{k=1}^\infty b_k e_k.$$

Then, it is easy to see that (remember that $\{e_j\}_{j=1}^\infty$ consist of the complete orthogonal basis)

$$(\mathcal{A}^{\frac{1}{4}} w, \widehat{w})_{L^2(\Omega)} = \Big(\sum_{j=1}^\infty a_j \mu_j^{\frac{1}{4}} e_j, \sum_{k=1}^\infty b_k e_k \Big)_{L^2(\Omega)} = \Big(\sum_{j=1}^\infty a_j e_j, \sum_{k=1}^\infty b_k \mu_k^{\frac{1}{4}} e_k \Big)_{L^2(\Omega)} = (w, \mathcal{A}^{\frac{1}{4}} \widehat{w})_{L^2(\Omega)}.$$

Now, for the solution w to (1.2), we define the energy as follows:

$$E_2(t) = \frac{1}{2}[\|\mathcal{A}^{\frac{1}{4}} w_t(t,\cdot)\|^2_{L^2(\Omega)} + \|\mathcal{A}^{\frac{3}{4}} w(t,\cdot)\|^2_{L^2(\Omega)} + \|q_2\|_\infty \|\mathcal{A}^{\frac{1}{4}} w(t,\cdot)\|^2_{L^2(\Omega)}]. \tag{1.6}$$

The main purpose of this paper is to study the observability constant $P(T;q_1)$ of system (1.1), defined as the smallest (possibly infinite) constant such that the following observability estimate for system (1.1) holds:

$$E_1(0) \le P(T,q_1) \int_0^T \int_\omega (w_x^2 + w_{tx}^2 + w_{xxx}^2) \mathrm{d}x\mathrm{d}t, \quad \forall\ (w^0, w^1) \in W_1 \times H_0^1(\Omega). \tag{1.7}$$

Also, we shall study the observability constant $P(T;q_2)$ of system (1.2) such that the following observability estimate for system (1.2) holds:

$$E_2(0) \le P(T,q_2) \int_0^T \int_\omega (w_x^2 + w_{tx}^2 + w_{xxx}^2) \mathrm{d}x\mathrm{d}t, \quad \forall\ (w^0, w^1) \in W_2 \times H_0^1(\Omega). \tag{1.8}$$

In (1.7)–(1.8), ω is a nonempty open subset of Ω. These inequalities, the observability inequalities, allow us to estimate the total energy of solutions in terms of the energy localized in the observation subdomain ω.

It is well-known that, observability inequalities are relevant to the control problems (see [13]). Several authors have already studied the problem of controllability for the plate equation. We can mention among them [9, 13]. In 1988, J.-L. Lions introduced the so-called Hilbert Uniqueness Method (HUM), which reduces the exact controllability problem for a large class of partial differential equations to the obtention of suitable observability estimates for their dual systems. After that, great progress on control problem of PDEs has been made (see [6, 16] and the rich references therein). In this paper, we are interested in the problem of sharp observability inequality for the plate equation with a potential. Similar observability problems have been considered for the parabolic and hyperbolic equations in [2], and for the Schrödinger equations in [11]. We refer to [5, 13] for observability of Euler-Bernoulli plate equation, and [14] for observability of Kirchhoff plate systems. However, to the author's best knowledge, there are no references considering the sharp observability of system (1.2). Moreover, the observability constant derived in [5, 14] is of order $\exp(C\|q\|_\infty^{\frac{2}{6}})$, q being the potential involved in the system. We shall see later that the observability constant for one dimensional plate equation can be improved to $\exp(C\|q\|_\infty^{\frac{2}{7}})$ for $\|q\|_\infty \geq 1$. We refer to [8] for a related result for the observability inequality of a one-dimensional fourth order parabolic equation with potential.

The rest of this paper is organized as follows. The main results are stated in Section 2. In Section 3, we shall collect some preliminary results for the plate equation. In Section 4, we establish a point wise inequality for the fourth order plate operator, via which in Section 5, we show Carleman estimates for the plate equation with two types of boundary conditions. Section 6 is devoted to the proof of our main results.

2 Statement of the Main Results

For any fixed $x_0 \in \mathbb{R}$ and $\delta > 0$, we define

$$\begin{cases} \omega = \mathcal{O}_\delta(\Gamma_0) \cap \Omega, \\ \Gamma_0 \overset{\triangle}{=} \{x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0\}, \end{cases} \tag{2.1}$$

where $\nu = \nu(x)$ is the unit outward normal vector.

Throughout this paper, we shall use $C = C(\Omega, \omega)$ to denote generic positive constants which may vary from line to line. Our main results can be stated as the following theorem.

Theorem 2.1 *Let Γ_0 and ω be given by* (2.1)*. Then there is a constant $C > 0$ such that for any $T > 0$ and any $q_1 \in L^\infty(0, T; W^{1,\infty}(\Omega))$, the weak solution w to system* (1.1) *satisfies estimate* (1.7) *with the same observability constant $P(T, q_1) > 0$ verifying*

$$P(T, q_1) \leq P_1(T,\, q_1) P_2(T,\, q_1) P_3(T,\, q_1), \tag{2.2}$$

where

$$\begin{aligned} &P_1(T,\, q_1) = \exp\Big[C\Big(1 + \frac{1}{T}\Big)\Big], \quad P_2(T,\, q_1) = \exp(CT\|q_1\|_{1,\infty}), \\ &P_3(T,\, q_1) = \exp(C\|q_1\|_\infty^{\frac{2}{7}} + C\|q_1\|_\infty^{-1}). \end{aligned} \tag{2.3}$$

Also, we have the following observability estimate for system (1.2).

Theorem 2.2 *Let Γ_0 and ω be given by* (2.1)*. Then there is a constant $C > 0$ such that for any $T > 0$ and any $q_2 \in L^\infty(0, T; W^{1,\infty}(\Omega))$, the weak solution w to system* (1.2) *satisfies estimate* (1.8) *with observability constant $P(T, q_2) > 0$ verifying*

$$P(T, q_2) \le P_1(T, q_2)P_2(T, q_2)P_3(T, q_2), \tag{2.4}$$

where

$$\begin{aligned} &P_1(T, q_2) = \exp\Big[C\Big(1 + \frac{1}{T}\Big)\Big], \quad P_2(T, q_2) = \exp(CT\|q_2\|_{1,\infty}), \\ &P_3(T, q_2) = \exp(C\|q_2\|_\infty^{\frac{2}{7}} + C\|q_2\|_\infty^{-1}). \end{aligned} \tag{2.5}$$

Remark 2.1 For the multidimensional plate equations with zero Dirichlet boundary condition $w = \Delta w = 0$, it was shown in [5] that the optimal observability constant P_3 should be $\exp(C\|q\|_\infty^{\frac{2}{6}})$ $(q = q_1, q_2)$ in even dimensional space. For one space-dimensional case, Theorems 2.1–2.2 show that the observability constant $P_3(T, q)$ can be improved to be $\exp(C\|q\|_\infty^{\frac{2}{7}})$ for $\|q\|_\infty \ge 1$. This phenomenon is similar to the observability estimate for the wave equation with a potential q. Indeed, in even dimensional space, the corresponding optimal observability constant $P_2(T, q)$ should be $\exp(C\|q\|_\infty^{\frac{2}{3}})$ (see [2]), while for the case of one-dimensional space, the observability constant can be improved to $\exp(C\|q\|_\infty^{\frac{2}{4}})$ (see [15]).

Remark 2.2 It should be pointed out that the technique developed here to prove Theorem 2.1 and Theorem 2.2 cannot be applied to multidimensional plate equations. Moreover, according to [5], we know that the optimal observability constant $P_3(T, q)$ cannot arrive at this level $\exp(C\|q\|_\infty^{\frac{2}{7}})$ for $\|q\|_\infty \ge 1$.

3 Preliminaries

In this section, we shall collect some preliminary results that we need. First, by using the usual energy estimate, one can easily obtain the following result.

Lemma 3.1 *Let $T > 0$ and $q_1(\cdot) \in L^\infty(0, T; W^{1,\infty}(\Omega))$. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$E_1(t) \le CE_1(s)\mathrm{e}^{CT(1+\|q_1\|_{1,\infty})}, \quad \forall t, s \in [0, T]. \tag{3.1}$$

Proof Multiplying both sides of the first equation in (1.1) by w_{txx}, integrating it on Ω, using integration by parts, by Hölder inequality and Poincaré inequality, we obtain

$$\begin{aligned} \frac{\mathrm{d}E_1(t)}{\mathrm{d}t} &= -\int_\Omega (q_1)_x w w_{tx}\mathrm{d}x - \int_\Omega q_1 w_x w_{tx}\mathrm{d}x + \|q_1\|_\infty \int_\Omega w_x w_{tx}\mathrm{d}x \\ &\le C(1 + \|q_1\|_{1,\infty})E_1(t). \end{aligned} \tag{3.2}$$

Now, by (3.2) and noting the time reversibility of (1.1), one gets (3.1) immediately.

In a way similar to Lemma 3.1, we have the following energy estimate for system (1.2).

Lemma 3.2 *Let $T > 0$ and $q_2(\cdot) \in L^\infty(0, T; W^{1,\infty}(\Omega))$. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$E_2(t) \le CE_2(s)\mathrm{e}^{CT(1+\|q_2\|_{1,\infty})}, \quad \forall t, s \in [0, T]. \tag{3.3}$$

Proof By the elementary calculus, and using Hölder inequality and Poincaré inequality, we obtain

$$\begin{aligned}\frac{\mathrm{d}E_2(t)}{\mathrm{d}t} &= (\mathcal{A}^{\frac14}w_t, \mathcal{A}^{\frac14}w_{tt})_{L^2(\Omega)} + (\mathcal{A}^{\frac34}w, \mathcal{A}^{\frac34}w_t)_{L^2(\Omega)} + \|q_2\|_\infty(\mathcal{A}^{\frac14}w, \mathcal{A}^{\frac14}w_t)_{L^2(\Omega)}\\ &= -(\mathcal{A}^{\frac12}w_t, q_2w)_{L^2(\Omega)} + \|q_2\|_\infty(\mathcal{A}^{\frac14}w, \mathcal{A}^{\frac14}w_t)_{L^2(\Omega)}\\ &= -(\mathcal{A}^{\frac14}(q_2w), \mathcal{A}^{\frac14}w_t)_{L^2(\Omega)} + \|q_2\|_\infty(\mathcal{A}^{\frac14}w, \mathcal{A}^{\frac14}w_t)_{L^2(\Omega)}\\ &\le C(1+\|q_2\|_{1,\infty})E_2(t).\end{aligned} \tag{3.4}$$

Now, by (3.4) and applying Gronwall's inequality, we conclude the desired result.

Further, we recall the following boundary trace estimates for the plate equation (1.1).

Lemma 3.3 *Let Γ_0 and ω satisfy* (2.1). *Let $T > 0$, $0 \le s_1 < s_0 < s_0' < s_1' \le T$ and $q_1 \in L^\infty(Q)$. Suppose that $w(\cdot)$ satisfies* (1.1). *Then*

$$\begin{aligned}&\int_{s_0}^{s_0'}\int_{\Gamma_0}(w_x^2 + w_{tx}^2 + w_{xxx}^2)\mathrm{d}x\mathrm{d}t\\ &\le \frac{CT^2(1+T^2)(1+\|q_1\|_\infty)}{(s_0-s_1)^2(s_1'-s_0')^2}\int_{s_1}^{s_1'}\int_\omega (w_x^2 + w_{tx}^2 + w_{xxx}^2)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.5}$$

Proof The proof is very close to that of [13, Lemma 4.4]. However, for the reader's convenience, we give some details here. We divide the proof into several steps.

Step 1 Fix δ_1 such that $0 < \delta_1 < \delta$, where δ is given in assumption (2.1). Denote

$$\omega_1 \stackrel{\triangle}{=} \mathcal{O}_{\delta_1}(\Gamma_0) \cap \Omega. \tag{3.6}$$

Then it is easy to see that

$$\omega_1 \subset \omega. \tag{3.7}$$

Choose $h_1 = h_1(x) \in C^3(\overline{\Omega}; \mathbb{R})$ such that $h_1 = \nu$ on Γ, and choose $h_2 = h_2(x) \in C^\infty(\overline{\Omega}; [0,1])$ such that

$$\begin{cases} 0 \le h_2(x) \le 1, & x \in \overline{\Omega},\\ h_2(x) \equiv 1, & x \in \omega_1,\\ h_2(x) \equiv 0, & x \in \Omega\setminus\omega.\end{cases} \tag{3.8}$$

Put

$$h = h(t,x) \stackrel{\triangle}{=} (t-s_1)^2(s_1'-t)^2h_1(x)h_2(x). \tag{3.9}$$

By [13, Lemma 4.1], it is easy to see that

$$\begin{aligned}&[hw_{xxx}^2 + hw_{tx}^2 - 2hw_tw_{xxt} + 2h_xw_tw_{tx} - h_{xx}w_t^2]_x\\ &= 2(w_{tt}+w_{xxxx})hw_{xxx} - 2(w_thw_{xxx})_t + h_x(3w_{tx}^2 + w_{xxx}^2) + 2w_th_tw_{xxx} - h_{xxx}w_t^2.\end{aligned} \tag{3.10}$$

Next, integrating (3.10) on $(s_1, s_1') \times \Omega$, using integration by parts, by (1.1) and Poincaré inequality, we have

$$\begin{aligned}
&\int_{s_1}^{s_1'} \int_\Gamma h \cdot \nu (w_{xxx}^2 + w_{tx}^2) \mathrm{d}x\mathrm{d}t \\
=& \int_{s_1}^{s_1'} \int_\Omega [-2q_1 h w w_{xxx} + h_x(3w_{tx}^2 + w_{xxx}^2) + 2w_t h_t w_{xxx} - h_{xxx} w_t^2] \mathrm{d}x\mathrm{d}t \\
\le& CT^2(1+T^2)(1+\|q_1\|_\infty) \int_{s_1}^{s_1'} \int_\omega (w_x^2 + w_{tx}^2 + w_{xxx}^2) \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.11}$$

Step 2 Let us estimate the term $\int_{s_0}^{s_0'} \int_{\Gamma_0} w_x^2 \mathrm{d}x\mathrm{d}t$.
Set

$$z = -\mathrm{i}w_t + w_{xx}.$$

By (1.1), it is easy to see that

$$\begin{cases} -\mathrm{i}w_t + w_{xx} = z, & \text{in } Q, \\ w = 0, & \text{on } \Sigma, \\ w(0) = w^0, & \text{in } \Omega. \end{cases} \tag{3.12}$$

Put

$$g = g(t,x) \stackrel{\triangle}{=} (t - s_1)(s_1' - t) h_1(x) h_2^2(x). \tag{3.13}$$

Using multiplier $g w_x$ as above (see [10]), for any $x \in \omega$, we choose h_2 such that $|\partial_x h_2| \le \frac{1}{2} h_2$. By using Poincaré inequality, one can obtain

$$\begin{aligned}
\int_{s_0}^{s_0'} \int_\Gamma g \cdot \nu w_x^2 \mathrm{d}x\mathrm{d}t &\le CT^2(1+\|q_1\|_\infty) \int_{s_1}^{s_1'} \int_\omega h_2^2(|w_t|^2 + |w_{xx}|^2 + w_x^2) \mathrm{d}x\mathrm{d}t \\
&\le CT^2(1+\|q_1\|_\infty) \int_{s_1}^{s_1'} \int_\omega [(h_2 w_t)_x^2 + (h_2 w_{xx})_x^2 + w_x^2] \mathrm{d}x\mathrm{d}t \\
&\le CT^2(1+\|q_1\|_\infty) \int_{s_1}^{s_1'} \int_\omega (w_{tx}^2 + w_{xxx}^2 + w_x^2) \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{3.14}$$

where the following fact is used:

$$\begin{aligned}
&\int_{s_1}^{s_1'} \int_\omega [(h_2 w_t)_x^2 + (h_2 w_{xx})_x^2] \mathrm{d}x\mathrm{d}t \\
\le& 2 \int_{s_1}^{s_1'} \int_\omega (\partial_x h_2)^2 (w_t^2 + w_{xx}^2) \mathrm{d}x\mathrm{d}t + 2 \int_{s_1}^{s_1'} \int_\omega h_2^2 (w_{tx}^2 + w_{xxx}^2) \mathrm{d}x\mathrm{d}t \\
\le& \frac{1}{2} \int_{s_1}^{s_1'} \int_\omega h_2^2 (w_t^2 + w_{xx}^2) \mathrm{d}x\mathrm{d}t + 2 \int_{s_1}^{s_1'} \int_\omega h_2^2 (w_{tx}^2 + w_{xxx}^2) \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Combining (3.11) and (3.14), we get the desired result.

Finally, we need the following boundary trace estimates for the plate equation (1.2).

Lemma 3.4 *Let Γ_0 and ω satisfy* (2.1). *Let $T > 0$, $0 \le s_1 < s_0 < s_0' < s_1' \le T$ and $q_2 \in L^\infty(Q)$. Suppose that $w(\cdot)$ satisfies* (1.2). *Then*

$$\begin{aligned}&\int_{s_0}^{s_0'}\int_{\Gamma_0}(w_{xx}^2+w_{xxx}^2)\mathrm{d}x\mathrm{d}t\\ \le\ &\frac{CT^2(1+T^2)(1+\|q_2\|_\infty)}{(s_0-s_1)^2(s_1'-s_0')^2}\int_{s_1}^{s_1'}\int_\omega(w_x^2+w_{tx}^2+w_{xxx}^2)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.15}$$

Proof Integrating (3.10) on $(s_1, s_1')\times\Omega$, using integration by parts, by (1.2) and Poincaré inequality, we have

$$\int_{s_1}^{s_1'}\int_\Gamma h\cdot\nu w_{xxx}^2\mathrm{d}x\mathrm{d}t \le CT^2(1+T^2)(1+\|q_2\|_\infty)\int_{s_1}^{s_1'}\int_\omega(w_{xxx}^2+w_{tx}^2)\mathrm{d}x\mathrm{d}t.\tag{3.16}$$

On the other hand, by elementary calculus, we have

$$\begin{aligned}2hw_x(w_{tt}+w_{xxxx}) =\ &(2hw_xw_t)_t-(hw_t^2)_x+h_xw_t^2-2h_tw_xw_t+(2hw_xw_{xxx}-hw_{xx}^2)_x\\ &-2h_xw_xw_{xxx}+h_xw_{xx}^2w_{xxx}^2+h_{xx}w_{xx}^2.\end{aligned}\tag{3.17}$$

Integrating (3.17) on $(s_1, s_1')\times\Omega$, using integration by parts, by (1.2) and Poincaré inequality, we have

$$\int_{s_0}^{s_0'}\int_\Gamma h\cdot\nu w_{xx}^2\mathrm{d}x\mathrm{d}t \le CT^2(1+T^2)(1+\|q_2\|_\infty)\int_{s_1}^{s_1'}\int_\omega(w_x^2+w_{tx}^2+w_{xxx}^2)\mathrm{d}x\mathrm{d}t.\tag{3.18}$$

Combining (3.16) and (3.18), we get the desired result.

4 A Point-Wise Identity for the Plate Operator

In this section, we shall establish a point wise weighted identity for 1-D plate operator, which will play an important role in the sequel. We have the following point wise inequality.

Theorem 4.1 *Let $w \in C^4(\mathbb{R}^2)$, $\ell \in C^2(\mathbb{R}^2)$ and $\Psi \in C^2(\mathbb{R})$. For any fixed $x_0 \in \mathbb{R}$ and $\lambda, c > 0$, set*

$$\begin{cases}\ell(t,x)=\dfrac{1}{2}\Big[(x-x_0)^2-c\Big(t-\dfrac{T}{2}\Big)^2\Big],\\ \theta(t,x)=\mathrm{e}^{\ell(t,x)},\quad z(t,x)=\theta(t,x)w(t,x).\end{cases}\tag{4.1}$$

Then

$$\begin{aligned}&\theta(w_{tt}+w_{xxxx})I_1+M_t+V_x\\ =\ &|I_1|^2+Bz^2+Ez_x^2+6\ell_{xx}z_{tx}^2+2\ell_{xx}z_{xxx}^2+24\ell_t\ell_xz_tz_x+\Psi_xz_xz_{xx}-\Psi_xzz_{xxx}\\ &+[\ell_{tt}+6\ell_{xx}^2+\Psi+18\ell_{xx}\ell_x^2]z_{xx}^2+[\ell_{tt}+6\ell_{xx}^2-\Psi-6\ell_{xx}\ell_x^2]z_t^2,\end{aligned}\tag{4.2}$$

where

$$\begin{cases}I_1\triangleq z_{tt}+z_{xxxx}+6(\ell_x^2-\ell_{xx})z_{xx}+Az,\\ M\triangleq \ell_t[z_t^2+Az^2+z_{xx}^2-6(\ell_x^2-\ell_{xx})z_x^2]+4\ell_xz_{xxx}z_t+4(\ell_x^2-3\ell_{xx})\ell_xz_xz_t-\Psi zz_t,\\ V\triangleq \widetilde{V}+2(5\ell_x^2-3\ell_{xx})\ell_xz_{xx}^2+2\ell_xz_{xxx}^2+4\ell_x(\ell_x^2-3\ell_{xx})z_{xxx}z_x+2\ell_xz_{tx}^2\\ \qquad +2\ell_x[6(\ell_x^2-3\ell_{xx})(\ell_x^2-\ell_{xx})+6\ell_{xx}^2-A]z_x^2\\ \qquad +2\ell_tz_{tx}z_{xx}+\Psi z_xz_{xx}-12\ell_{xx}(\ell_x^2-\ell_{xx})z_{xx}z_x\end{cases}\tag{4.3}$$

and

$$\begin{aligned}\widetilde{V} \stackrel{\triangle}{=} & 2[\ell_t z_{xxx} + 6(\ell_x^2 - \ell_{xx})\ell_t z_x - 2\ell_x z_{xxt} + 2\ell_{xx} z_{xt} - \ell_x(\ell_x^2 - 3\ell_{xx})z_t]z_t \\ & + [4A\ell_x z_{xx} - 4(A\ell_x)_x z_x + 2(A\ell_x)_{xx} + 2A\ell_x(\ell_x^2 - 3\ell_{xx})z \\ & + 3(\Psi\ell_x^2 - \Psi\ell_{xx})_x z - 6\Psi(\ell_x^2 - \ell_{xx})z_x - \Psi z_{xxx}]z, \end{aligned} \tag{4.4}$$

and moreover,

$$\begin{cases} A \stackrel{\triangle}{=} \ell_t^2 - \ell_{tt} + (\ell_x^2 - \ell_{xx})^2 - 4\ell_{xx}\ell_x^2 + 2\ell_{xx}^2 - \Psi, \\ B \stackrel{\triangle}{=} (A\ell_t)_t + 2(A\ell_x)_{xxx} + 2[A\ell_x(\ell_x^2 - 3\ell_{xx})]_x + A\Psi + 3(\Psi\ell_x^2 - \Psi\ell_{xx})_{xx}, \\ E \stackrel{\triangle}{=} -6\ell_{tt}(\ell_x^2 - \ell_{xx}) - 6(A\ell_x)_x + 12\ell_{xx}^3 - 6\Psi(\ell_x^2 - \ell_{xx}) \\ \qquad +12[\ell_x(\ell_x^2 - 3\ell_{xx})(\ell_x^2 - \ell_{xx})]_x. \end{cases} \tag{4.5}$$

Remark 4.1 If we assume that $w = w_{xx} = 0$ or $w = w_x = 0$ on the boundary, noting that $z = \theta w$, then we have $\int_\Gamma \widetilde{V}_x \mathrm{d}x\mathrm{d}t = \int_\Gamma \widetilde{V} \cdot \nu \mathrm{d}x\mathrm{d}t = 0$.

Remark 4.2 Unlike [13], we do not divide the plate operator into two Schrödinger operators. Here, we establish the point wise estimate for the fourth order plate operator directly.

Remark 4.3 Equation (4.2) can be regarded as a weighted identity. The main idea of (4.2) is to establish a point wise identity (and/or estimate) on the principal operator $w_{tt} + w_{xxxx}$ in terms of the sum of "divergence" terms $M_t + V_x$, "energy" terms $z^2(\cdot) + z_x^2(\cdot) + z_{tx}^2(\cdot) + z_{xxx}^2(\cdot) + z_{xx}^2(\cdot) + z_t^2(\cdot)$ and lower order terms.

Proof The proof is long, so we divided it into several steps.

Step 1 Note that $\theta = \mathrm{e}^\ell$, $z = \theta w$, it is easy to check that

$$\begin{cases} \theta w_t = z_t - \ell_t z, \quad \theta w_x = z_x - \ell_x z, \\ \theta w_{tt} = z_{tt} - 2\ell_t z_t + (\ell_t^2 - \ell_{tt})z, \\ \theta w_{xx} = z_{xx} - 2\ell_x z_x + (\ell_x^2 - \ell_{xx})z. \end{cases} \tag{4.6}$$

Next, recalling (4.1) for the definition of $\ell(t,x)$ and noting that Ψ only depends on x, it is easy to see that

$$\ell_{xxx} = \ell_{xxxx} = \ell_{tx} = \Psi_t = 0. \tag{4.7}$$

By (4.6)–(4.7) and recalling the definition of A in (4.5), we have

$$\begin{aligned}\theta w_{xxxx} &= [z_{xx} - 2\ell_x z_x + (\ell_x^2 - \ell_{xx})z]_{xx} - 2\ell_x[z_{xx} - 2\ell_x z_x + (\ell_x^2 - \ell_{xx})z]_x \\ &\quad + (\ell_x^2 - \ell_{xx})[z_{xx} - 2\ell_x z_x + (\ell_x^2 - \ell_{xx})z] \\ &= z_{xxxx} - 4\ell_x z_{xxx} + 6(\ell_x^2 - \ell_{xx})z_{xx} - 4\ell_x(\ell_x^2 - 3\ell_{xx})z_x \\ &\quad + [(\ell_x^2 - \ell_{xx})^2 - 4\ell_{xx}\ell_x^2 + 2\ell_{xx}^2]z. \end{aligned} \tag{4.8}$$

Combining (4.6) and (4.8), recalling the definition of I_1 in (4.3), we get

$$\theta(w_{tt} + w_{xxxx}) = I_1 + I_2 + I_3, \tag{4.9}$$

where

$$I_2 \stackrel{\triangle}{=} -2\ell_t z_t - 4\ell_x z_{xxx}, \quad I_3 \stackrel{\triangle}{=} -4\ell_x(\ell_x^2 - 3\ell_{xx})z_x + \Psi z. \tag{4.10}$$

By (4.9), it is easy to see that

$$\theta(w_{tt} + w_{xxxx})I_1 = |I_1|^2 + I_1 I_2 + I_1 I_3. \tag{4.11}$$

Step 2 Let us compute $I_1 I_2$. First, by (4.10), we have

$$\begin{aligned} I_1 I_2 = & -2\ell_t z_t (z_{tt} + Az) - 2\ell_t z_t z_{xxxx} - 12\ell_t(\ell_x^2 - \ell_{xx}) z_{xx} z_t \\ & - 4\ell_x z_{xxx}(z_{tt} + Az) - 4\ell_x z_{xxx} z_{xxxx} - 24\ell_x(\ell_x^2 - \ell_{xx}) z_{xx} z_{xxx}. \end{aligned} \tag{4.12}$$

However,

$$-2\ell_t z_t (z_{tt} + Az) = -(\ell_t z_t^2)_t + \ell_{tt} z_t^2 - (A\ell_t z^2)_t + (A\ell_t)_t z^2. \tag{4.13}$$

Next, by (4.7), we have

$$-2\ell_t z_t z_{xxxx} = -2(\ell_t z_t z_{xxx} - \ell_t z_{tx} z_{xx})_x - (\ell_t z_{xx}^2)_t + \ell_{tt} z_{xx}^2 \tag{4.14}$$

and

$$\begin{aligned} -12\ell_t(\ell_x^2 - \ell_{xx}) z_{xx} z_t = & -12[\ell_t(\ell_x^2 - \ell_{xx}) z_t z_x]_x + 24\ell_t \ell_x z_t z_x \\ & + 6[\ell_t(\ell_x^2 - \ell_{xx}) z_x^2]_t - 6\ell_{tt}(\ell_x^2 - \ell_{xx}) z_x^2. \end{aligned} \tag{4.15}$$

On the other hand, by (4.7), we have

$$\begin{aligned} & - 4\ell_x z_{xxx}(z_{tt} + Az) \\ = & -4(\ell_x z_{xxx} z_t)_t + 4(\ell_x z_{xxt} z_t)_x - 4(\ell_{xx} z_{xt} z_t)_x - 2(\ell_x z_{tx}^2)_x + 6\ell_{xx} z_{tx}^2 \\ & - 4(A\ell_x z_{xx} z)_x + 4[(A\ell_x)_x z z_x]_x - 2[(A\ell_x)_{xx} z^2]_x \\ & + 2(A\ell_x z_x^2)_x - 6(A\ell_x)_x z_x^2 + 2(A\ell_x)_{xxx} z^2. \end{aligned} \tag{4.16}$$

Further, it is easy to see that

$$\begin{aligned} & - 4\ell_x z_{xxx} z_{xxxx} - 24\ell_x(\ell_x^2 - \ell_{xx}) z_{xx} z_{xxx} \\ = & -2(\ell_x z_{xxx}^2)_x + 2\ell_{xx} z_{xxx}^2 - 12[\ell_x(\ell_x^2 - \ell_{xx}) z_{xx}^2]_x + 12\ell_{xx}(3\ell_x^2 - \ell_{xx}) z_{xx}^2. \end{aligned} \tag{4.17}$$

Step 3 Let us compute $I_1 I_3$. By (4.10), we have

$$\begin{aligned} I_1 I_3 = & -4\ell_x(\ell_x^2 - 3\ell_{xx}) z_x (z_{tt} + Az) - 24\ell_x(\ell_x^2 - 3\ell_{xx})(\ell_x^2 - \ell_{xx}) z_x z_{xx} \\ & - 4\ell_x(\ell_x^2 - 3\ell_{xx}) z_x z_{xxxx} + \Psi z(z_{tt} + Az) + 6\Psi(\ell_x^2 - \ell_{xx}) z z_{xx} + \Psi z z_{xxxx}. \end{aligned} \tag{4.18}$$

First, by using (4.7) again, we have

$$\begin{aligned} & - 4\ell_x(\ell_x^2 - 3\ell_{xx}) z_x (z_{tt} + Az) \\ = & -4[\ell_x(\ell_x^2 - 3\ell_{xx}) z_x z_t]_t + 2[\ell_x(\ell_x^2 - 3\ell_{xx}) z_t^2]_x - 6\ell_{xx}(\ell_x^2 - \ell_{xx}) z_t^2 \\ & - 2[A\ell_x(\ell_x^2 - 3\ell_{xx}) z^2]_x + 2[A\ell_x(\ell_x^2 - 3\ell_{xx})]_x z^2. \end{aligned} \tag{4.19}$$

Next, it is easy to see that

$$\begin{aligned} & - 24\ell_x(\ell_x^2 - 3\ell_{xx})(\ell_x^2 - \ell_{xx}) z_x z_{xx} \\ = & -12[\ell_x(\ell_x^2 - 3\ell_{xx})(\ell_x^2 - \ell_{xx}) z_x^2]_x + 12[\ell_x(\ell_x^2 - 3\ell_{xx})(\ell_x^2 - \ell_{xx})]_x z_x^2. \end{aligned} \tag{4.20}$$

Furthermore, by (4.7), we have

$$\begin{aligned}&-4\ell_x(\ell_x^2-3\ell_{xx})z_xz_{xxxx}\\ &=-4[\ell_x(\ell_x^2-3\ell_{xx})z_xz_{xxx}]_x+12[\ell_{xx}(\ell_x^2-\ell_{xx})z_xz_{xx}]_x-12(\ell_x\ell_{xx}^2z_x^2)_x\\ &\quad+12\ell_{xx}^3z_x^2+2[\ell_x(\ell_x^2-3\ell_{xx})z_{xx}^2]_x-18\ell_{xx}(\ell_x^2-\ell_{xx})z_{xx}^2.\end{aligned}\tag{4.21}$$

On the other hand, noting $\Psi_t=0$, we have

$$\Psi z(z_{tt}+Az)=(\Psi zz_t)_t-\Psi z_t^2+A\Psi z.\tag{4.22}$$

Next,

$$\begin{aligned}6\Psi(\ell_x^2-\ell_{xx})zz_{xx}&=[6\Psi(\ell_x^2-\ell_{xx})zz_x]_x-3[(\Psi\ell_x^2-\Psi\ell_{xx})_xz^2]_x\\ &\quad+3(\Psi\ell_x^2-\Psi\ell_{xx})_{xx}z^2-6\Psi(\ell_x^2-\ell_{xx})z_x^2.\end{aligned}\tag{4.23}$$

Finally,

$$\Psi zz_{xxxx}=(\Psi zz_{xxx})_x-(\Psi z_xz_{xx})_x+\Psi_xz_xz_{xx}+\Psi z_{xx}^2-\Psi_xzz_{xxx}.\tag{4.24}$$

Step 4 By (4.12)–(4.24), combining all "$\frac{\partial}{\partial t}$-terms", all "$\frac{\partial}{\partial x}$-terms" and (4.11), we arrive at the desired identity (4.2).

5 Carleman Estimate for the Plate Equation

Based on the point wise estimate (4.2) in Theorem 4.1, in this section, we shall establish the global Carleman estimate for the 1-D plate equation. For given $0\le T_1<T_1'\le T$, denote $Q_1=(T_1,T_1')\times\Omega$. We have the following result.

Theorem 5.1 *Let $T>0$ and $q_1\in L^\infty(0,T;W^{1,\infty}(\Omega))$, and let Γ_0 be given by* (2.1). *Then there is a constant $\lambda_0=\lambda_0(\frac{1}{T},\ \|q_1\|_\infty^{\frac{2}{7}})>1$ such that for any $\lambda>\lambda_0$ and for all weak solutions to* (1.1)*, it holds that*

$$\begin{aligned}&\lambda\int_Q\theta^2(\lambda^6w^2+\lambda^4w_x^2+\lambda^2w_{xx}^2+\lambda^2w_t^2+w_{xxx}^2+w_{tx}^2)\mathrm{d}t\mathrm{d}x\\ &\le C\Big[\lambda^7T(1+\|q_1\|_\infty^{-1})\mathrm{e}^{CT(1+\|q_1\|_{1,\infty})}E_1(0)+\lambda\mathrm{e}^{C\lambda}\int_{T_1}^{T_1'}\int_{\Gamma_0}(w_{xxx}^2+w_{tx}^2+\lambda^4w_x^2)\mathrm{d}x\mathrm{d}t\Big].\end{aligned}\tag{5.1}$$

Also, we have the following global Carleman estimate for system (1.2).

Theorem 5.2 *Let $T>0$ and $q_2\in L^\infty(0,T;W^{1,\infty}(\Omega))$, and let Γ_0 be given by* (2.1). *Then there exists a constant $\lambda_0=\lambda_0(\frac{1}{T},\|q_2\|_\infty^{\frac{2}{7}})>1$ such that for any $\lambda>\lambda_0$ and for all weak solutions to* (1.2)*, it holds that*

$$\begin{aligned}&\lambda\int_Q\theta^2(\lambda^6w^2+\lambda^4w_x^2+\lambda^2w_{xx}^2+\lambda^2w_t^2+w_{xxx}^2+w_{tx}^2)\mathrm{d}t\mathrm{d}x\\ &\le C\Big[\lambda^7T(1+\|q_1\|_\infty^{-1})\mathrm{e}^{CT(1+\|q_2\|_{1,\infty})}E_2(0)+\lambda\mathrm{e}^{C\lambda}\int_{T_1}^{T_1'}\int_{\Gamma_0}(\lambda^2w_{xx}^2+w_{xxx}^2)\mathrm{d}x\mathrm{d}t\Big].\end{aligned}\tag{5.2}$$

Remark 5.1 If we divide the plate operator into two Schrödinger operators, we can just get the coefficient of order λ^6 before w^2 (see inequality (3.7) in [5]).

Proof of Theorem 5.1 We borrow some ideas from [13]. The proof is long, so we divide it into several steps.

Step 1 (Choice of weight functions) For simplicity, we assume $x_0 \in \mathbb{R} \setminus \overline{\Omega}$. For the case $x_0 \in \overline{\Omega}$, using the argument in [7], one can prove Theorem 5.1 similarly. Denote

$$R_1 \stackrel{\triangle}{=} \max_{x\in\overline{\Omega}} |x - x_0| > R_0 \stackrel{\triangle}{=} \min_{x\in\overline{\Omega}} |x - x_0| (> 0). \tag{5.3}$$

Next, we choose the constant c in ℓ defined by (4.1) as

$$c = \frac{9R_1^2}{2T^2}. \tag{5.4}$$

From (4.1) and (5.4), for any $\lambda \geq 1$, it is easy to see that

$$\ell(0, x) = \ell(T, x) \leq \frac{\lambda}{2}\Big(R_1^2 - \frac{cT^2}{4}\Big) = -\frac{\lambda R_1^2}{16} < 0, \quad \forall\, x \in \Omega. \tag{5.5}$$

Therefore, there exists an $\varepsilon_1 \in (0, \frac{1}{2})$ independent of T, close to $\frac{1}{2}$, and a constant $r_0 = \frac{\lambda R_1^2}{32} (> 0)$ such that

$$\ell(t, x) \leq -r_0 < 0, \quad \forall\, (t, x) \in ((0, T_1) \cup (T_1', T)) \times \Omega, \tag{5.6}$$

where

$$T_1 = \frac{T}{2} - \varepsilon_1 T, \quad T_1' = \frac{T}{2} + \varepsilon_1 T.$$

Step 2 (Estimate of the "energy" terms) First, take

$$\Psi(t, x) = 9\lambda^3 |x - x_0|^2. \tag{5.7}$$

Recalling (4.5) for the definition of A and by using (4.1), it is easy to check that

$$A = \lambda^4 |x - x_0|^4 + O(\lambda^3). \tag{5.8}$$

Then, by (4.1), (5.7)–(5.8) and (4.5), we have

$$B = 14\lambda^7 |x - x_0|^6 + O(\lambda^6), \quad E = 60\lambda^5 |x - x_0|^4 + O(\lambda^4). \tag{5.9}$$

Similarly, by (4.1) and (5.7), it is easy to check that

$$\begin{aligned}&[\ell_{tt} + 6\ell_{xx}^2 + \Psi + 18\ell_{xx}\ell_x^2] z_{xx}^2 + [\ell_{tt} + 6\ell_{xx}^2 - \Psi - 6\ell_{xx}\ell_x^2] z_t^2 \\ &= [9\lambda^3 |x - x_0|^2 + O(\lambda^2)] z_{xx}^2 + [3\lambda^3 |x - x_0|^2 + O(\lambda^2)] z_t^2.\end{aligned} \tag{5.10}$$

Next, by using (4.1) again, there exists a constant $c_0 > 0$, such that

$$\begin{aligned}&24\ell_t \ell_x z_t z_x + \Psi_x z_x z_{xx} - \Psi_x z z_{xxx} \\ &\geq -c_0 \lambda^2 T R_1 (z_t^2 + z_x^2) - c_0 \lambda^4 z_x^2 - c_0 \lambda^2 z_{xx}^2 - c_0 \lambda^6 z^2 - c_0 z_{xxx}^2.\end{aligned} \tag{5.11}$$

Noting that the lower order terms with respect to the power of λ in (5.9)–(5.11) depend on c, therefore, one can find a $\lambda_1(\frac{1}{T}) > 1$ such that for any $\lambda \geq \lambda_1$, there exists a constant $c_1 > 0$, such that

$$\text{Left-hand side of (4.2)} \geq c_1 [\lambda^7 z^2 + \lambda^5 z_x^3 + \lambda^3 z_{xx}^2 + \lambda^3 z_t^2 + \lambda z_{xxx}^2 + \lambda z_{xt}^2]. \tag{5.12}$$

Step 3 (Estimate of the boundary terms) Integrating (4.2) on Q_1, using integration by parts, by (5.9)–(5.10), it holds

$$\begin{aligned}&\lambda\int_{Q_1}(\lambda^6 z^2+\lambda^4 z_x^2+\lambda^2 z_{xx}^2+\lambda^2 z_t^2+z_{xxx}^2+z_{tx}^2)\mathrm{d}t\mathrm{d}x\\ &\le C\Big[\|\theta(w_{tt}+w_{xxxx})\|_{L^2(Q)}^2+\int_\Omega M(t,x)\mathrm{d}x\Big|_{T_1}^{T_1'}+\int_{T_1}^{T_1'}\int_\Omega V_x\mathrm{d}x\mathrm{d}t\Big].\end{aligned}\tag{5.13}$$

Next, recalling (4.3) for the definition of M, noting $z=\theta w$, by (5.6), (5.8) and Poincaré inequality, we have

$$\begin{aligned}\int_\Omega M(t,x)\mathrm{d}x\Big|_{T_1}^{T_1'}&\le C\int_\Omega(z_t^2+z_{xx}^2+\lambda^4 z^2+\lambda^5 z_x^2+\lambda z_{xxx}^2)\mathrm{d}x\Big|_{T_1}^{T_1'}\\ &\le C\lambda^7\int_\Omega\theta^2(w_t^2+w_{xx}^2+w^2+w_x^2+w_{xxx}^2)\mathrm{d}x\Big|_{T_1}^{T_1'}\\ &\le C\lambda^7\mathrm{e}^{-2\lambda r_0}\int_\Omega(w_x^2+w_{tx}^2+w_{xxx}^2)\mathrm{d}x\Big|_{T_1}^{T_1'}\\ &\le C\lambda^7(1+\|q_1\|_\infty^{-1})[E_1(T_1)+E_1(T_1')].\end{aligned}\tag{5.14}$$

Recalling (4.3) for the definition of V, by Remark 4.1, we arrive at

$$\begin{aligned}&\int_{T_1}^{T_1'}\int_\Omega V_x\mathrm{d}x\mathrm{d}t\\ &=\int_{T_1}^{T_1'}\int_\Gamma \ell_x\cdot\nu[2(5\ell_x^2-3\ell_{xx})z_{xx}^2+2z_{xxx}^2+4(\ell_x^2-3\ell_{xx})z_{xxx}z_x+2z_{tx}^2]\mathrm{d}x\mathrm{d}t\\ &\quad+2\int_{T_1}^{T_1'}\int_\Gamma \ell_x\cdot\nu[6(\ell_x^2-3\ell_{xx})(\ell_x^2-\ell_{xx})+6\ell_{xx}^2-A]z_x^2\mathrm{d}x\mathrm{d}t\\ &\quad+\int_{T_1}^{T_1'}\int_\Gamma[2\ell_t z_{tx}z_{xx}+\Psi z_x z_{xx}-12\ell_{xx}(\ell_x^2-\ell_{xx})z_{xx}z_x]\cdot\nu\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{5.15}$$

However, by (4.1) and (5.8), we have

$$\begin{aligned}&4(\ell_x^2-3\ell_{xx})z_{xxx}z_x+2[6(\ell_x^2-3\ell_{xx})(\ell_x^2-\ell_{xx})+6\ell_{xx}^2-A]z_x^2\\ &=|z_{xxx}-2(\ell_x^2-3\ell_{xx})z_x|^2-z_{xxx}^2+[6\lambda^4|x-x_0|^4+O(\lambda^3)]z_x^2.\end{aligned}\tag{5.16}$$

Further, noting that $w=w_{xx}=0$ on boundary and $z=\theta w=\mathrm{e}^\ell w$, it is easy to see that

$$z=z_t=0,\quad z_x=\theta w_x,\quad z_{xx}=2\theta\ell_x w_x,\quad \text{on boundary.}\tag{5.17}$$

Thus, by (4.1) and (5.7), we have

$$\begin{aligned}&\int_{T_1}^{T_1'}\int_\Gamma[2\ell_t z_{tx}z_{xx}+\Psi z_x z_{xx}-12\ell_{xx}(\ell_x^2-\ell_{xx})z_{xx}z_x]\cdot\nu\mathrm{d}x\mathrm{d}t\\ &=\int_{T_1}^{T_1'}\int_\Gamma \ell_x\cdot\nu[4\theta\ell_t w_x z_{tx}+2\Psi\theta^2 w_x^2-24\ell_{xx}(\ell_x^2-\ell_{xx})\theta^2 w_x^2]\mathrm{d}x\mathrm{d}t\\ &=\int_{T_1}^{T_1'}\int_\Gamma \ell_x\cdot\nu[(z_{tx}+2\ell_t z_x)^2-z_{tx}^2+2(-2\ell_t^2+\Psi-12\ell_{xx}\ell_x^2+12\ell_{xx}^2)z_x^2]\mathrm{d}x\mathrm{d}t\\ &=\int_{T_1}^{T_1'}\int_\Gamma \ell_x\cdot\nu[(z_{tx}+2\ell_t z_x)^2-z_{tx}^2-6\lambda^3|x-x_0|^2z_x^2+O(\lambda^2)z_x^2]\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{5.18}$$

Now, combining (5.15)–(5.18), one can find a $\lambda_2(\frac{1}{T}) > 0$ such that for all $\lambda \geq \max\{\lambda_1, \lambda_2\}$, and by (2.1), we have

$$\int_{T_1}^{T_1'} \int_\Omega V_x \mathrm{d}x\mathrm{d}t \leq C \int_{T_1}^{T_1'} \int_{\Gamma_0} (\lambda z_{xxx}^2 + \lambda z_{tx}^2 + \lambda^5 z_x^2)\mathrm{d}t\mathrm{d}x. \tag{5.19}$$

Combining (5.13)–(5.15) and (1.1), noting $z = \theta w$, we end up with

$$\begin{aligned}
&\lambda \int_{T_1}^{T_1'} \int_\Omega \theta^2 (\lambda^6 w^2 + \lambda^4 w_x^2 + \lambda^2 w_{xx}^2 + \lambda^2 w_t^2 + w_{xxx}^2 + w_{tx}^2)\mathrm{d}t\mathrm{d}x \\
&\leq C\Big\{ \|\theta q_1 w\|_{L^2(Q)}^2 + \lambda^7 (1 + \|q_1\|_\infty^{-1})[E_1(T_1) + E_1(T_1')] \\
&\quad + \lambda \int_{T_1}^{T_1'} \int_{\Gamma_0} \theta^2 (w_{xxx}^2 + w_{xt}^2 + \lambda^4 w_x^2)\mathrm{d}t\mathrm{d}x \Big\}.
\end{aligned} \tag{5.20}$$

Step 4 (End of the proof) By (5.6) and noting that $\theta = \mathrm{e}^\ell$, we get

$$\begin{aligned}
&\lambda \int_{T_1}^{T_1'} \int_\Omega \theta^2 (\lambda^6 w^2 + \lambda^4 w_x^2 + \lambda^2 w_{xx}^2 + \lambda^2 w_t^2 + w_{xxx}^2 + w_{tx}^2)\mathrm{d}t\mathrm{d}x \\
&\geq \lambda \int_Q \theta^2 (\lambda^6 w^2 + \lambda^4 w_x^2 + \lambda^2 w_{xx}^2 + \lambda^2 w_t^2 + w_{xxx}^2 + w_{tx}^2)\mathrm{d}t\mathrm{d}x \\
&\quad - \lambda \Big[\int_0^{T_1} + \int_{T_1'}^T \Big] \int_\Omega (\lambda^6 w^2 + \lambda^4 w_x^2 + \lambda^2 w_{xx}^2 + \lambda^2 w_t^2 + w_{xxx}^2 + w_{tx}^2)\mathrm{d}t\mathrm{d}x.
\end{aligned} \tag{5.21}$$

Combining (5.20)–(5.21), for all $\lambda \geq \max\{\lambda_1, \lambda_2\}$, by Lemma 3.1, we get for some $C_1 > 0$ that

$$\begin{aligned}
&\lambda \int_Q \theta^2 (\lambda^6 w^2 + \lambda^4 w_x^2 + \lambda^2 w_{xx}^2 + \lambda^2 w_t^2 + w_{xxx}^2 + w_{tx}^2)\mathrm{d}t\mathrm{d}x \\
&\leq C_1 \Big\{ \|q_1\|_\infty^2 \|\theta w\|_{L^2(Q)}^2 + \lambda^7 (1 + \|q_1\|_\infty^{-1}) T \mathrm{e}^{CT(1+\|q_1\|_{1,\infty})} E_1(0) \\
&\quad + \lambda \mathrm{e}^{C\lambda} \int_{T_1}^{T_1'} \int_{\Gamma_0} (w_{xxx}^2 + w_{tx}^2 + \lambda^4 w_x^2)\mathrm{d}t\mathrm{d}x \Big\}.
\end{aligned} \tag{5.22}$$

Define

$$\lambda_3 \triangleq 1 + C_1^{\frac{1}{7}} \|q_1\|_\infty^{\frac{2}{7}}, \quad \lambda_0 \triangleq \max\{\lambda_1, \lambda_2, \lambda_3\} = \lambda_0 \Big(\frac{1}{T}, \|q_1\|_\infty^{\frac{2}{7}} \Big). \tag{5.23}$$

Then, for any $\lambda \geq \lambda_0$, we obtain

$$\begin{aligned}
&\lambda \int_Q \theta^2 (\lambda^6 w^2 + \lambda^4 w_x^2 + \lambda^2 w_{xx}^2 + \lambda^2 w_t^2 + w_{xxx}^2 + w_{tx}^2)\mathrm{d}t\mathrm{d}x \\
&\leq C \Big\{ \lambda^7 T (1 + \|q_1\|_\infty^{-1}) \mathrm{e}^{CT(1+\|q_1\|_{1,\infty})} E_1(0) \\
&\quad + \lambda \mathrm{e}^{C\lambda} \int_{T_1}^{T_1'} \int_{\Gamma_0} (w_{xxx}^2 + w_{tx}^2 + \lambda^4 w_x^2)\mathrm{d}t\mathrm{d}x \Big\}.
\end{aligned} \tag{5.24}$$

This gives the proof of Theorem 5.1.

Next, we shall give a brief proof of Theorem 5.2.

Proof of Theorem 5.2 In a way similar to Theorem 5.1, we choose the same weight functions, where the only different terms are the boundary terms. Noting that $w = w_x = 0$ on boundary, by recalling (4.3) for the definition of V, combining Remark 4.1 and noting $z = \theta w$, we arrive at

$$\int_{T_1}^{T_1'} \int_\Omega V_x \mathrm{d}x\mathrm{d}t \le C\lambda \int_{T_1}^{T_1'} \int_{\Gamma_0} (\lambda^2 z_{xx}^2 + z_{xxx}^2)\mathrm{d}t\mathrm{d}x. \tag{5.25}$$

Proceeding exactly as in Theorem 5.1, we obtain the desired result immediately.

6 Proof of Theorems 2.1 and 2.2

In this section, we shall prove the sharp observability inequalities for the 1-D plate equation.

Proof of Theorem 2.1 Recalling (4.1) and (5.3) for the definition of ℓ and R_0, respectively, we see that

$$\ell\Big(\frac{T}{2}, x\Big) \ge \frac{\lambda}{2} R_0^2, \quad \forall\, x \in \Omega.$$

Then, one can find an $\varepsilon_0 \in (0, \frac{1}{2})$ independent of T, close to 0, such that

$$\ell(t,x) \ge \frac{\lambda R_0^2}{4}, \quad \forall\, (t,x) \in Q_0 = (T_0, T_0') \times \Omega, \tag{6.1}$$

where

$$T_0 \triangleq \frac{T}{2} - \varepsilon_0 T, \quad T_0' \triangleq \frac{T}{2} + \varepsilon_0 T. \tag{6.2}$$

By (6.1)–(6.2) and Lemma 3.1, taking $C_1 > 1$ in (5.23), for any $\lambda \ge \lambda_0 \ge \lambda_3$, we have

$$\begin{aligned}
&\int_Q \theta^2(\lambda^6 w^2 + \lambda^4 w_x^2 + \lambda^2 w_{xx}^2 + \lambda^2 w_t^2 + w_{xxx}^2 + w_{tx}^2)\mathrm{d}t\mathrm{d}x \\
\ge{}& \mathrm{e}^{\frac{R_0^2}{2}\lambda} \int_{Q_0} (\lambda^6 w^2 + \lambda^4 w_x^2 + \lambda^2 w_{xx}^2 + \lambda^2 w_t^2 + w_{xxx}^2 + w_{tx}^2)\mathrm{d}t\mathrm{d}x \\
\ge{}& \mathrm{e}^{\frac{R_0^2}{2}\lambda} \int_{Q_0} (\lambda_0^4 w_x^2 + w_{tx}^2 + w_{xxx}^2)\mathrm{d}t\mathrm{d}x \\
\ge{}& \mathrm{e}^{\frac{R_0^2}{2}\lambda} \int_{Q_0} [(1 + \|q_1\|_\infty^{\frac{2}{7}})^4 w_x^2 + w_{tx}^2 + w_{xxx}^2]\mathrm{d}t\mathrm{d}x \\
\ge{}& \mathrm{e}^{\frac{R_0^2}{2}\lambda} \int_{T_0}^{T_0'} E_1(t)\mathrm{d}t \\
\ge{}& \mathrm{e}^{\frac{R_0^2}{2}\lambda} \int_{T_0}^{T_0'} E_1(0)\mathrm{e}^{\frac{-CT(1+\|q_1\|_{1,\infty})}{C}}\mathrm{d}t \\
={}& \frac{2\varepsilon_0 T}{C} \mathrm{e}^{\frac{R_0^2}{2}\lambda - CT(1+\|q_1\|_{1,\infty})} E_1(0).
\end{aligned} \tag{6.3}$$

By (5.1) and (6.3), for any $\lambda \ge \lambda_0$, we get for some $C_2 > 0$ that

$$\begin{aligned}
&\mathrm{e}^{C_2T(1+\|q_1\|_{1,\infty})} \Big[\mathrm{e}^{\frac{R_0^2}{2}\lambda - 2C_2T(1+\|q_1\|_{1,\infty})} - (1 + \|q_1\|_\infty^{-1}) \frac{C_2\lambda^6}{2\varepsilon_0}\Big] E_1(0) \\
\le{}& 2C_2\varepsilon_0^{-1}T^{-1}\mathrm{e}^{C_2\lambda} \int_{T_1}^{T_1'} \int_{\Gamma_0} (w_{xxx}^2 + w_{tx}^2 + w_x^2)\mathrm{d}t\mathrm{d}x.
\end{aligned} \tag{6.4}$$

Now, let λ_4 be such that

$$\frac{R_0^2\lambda_4}{4} \geq 2C_2T(1+\|q_1\|_{1,\infty}). \tag{6.5}$$

On the other hand, it is easy to see that

$$\mathrm{e}^{\frac{R_0^2}{4}\lambda} - (1+\|q_1\|_\infty^{-1})\frac{C_2\lambda^6}{2\varepsilon_0} \geq \frac{(\frac{R_0^2}{4}\lambda)^7}{7!} - 2(1+\|q_1\|_\infty^{-1})\frac{C_2\lambda^6}{2\varepsilon_0}.$$

Put

$$\lambda_5 = C\Big(1+\frac{1}{T}+\|q_1\|_\infty^{\frac{2}{7}}+T\|q_1\|_{1,\infty}+\|q_1\|_\infty^{-1}\Big). \tag{6.6}$$

By (6.4), for any $\lambda \geq \lambda_5$, we obtain

$$E_1(0) \leq P(T,q_1)\int_{T_1}^{T_1'}\int_{\Gamma_0}(w_x^2+w_{tx}^2+w_{xxx}^2)\mathrm{d}t\mathrm{d}x. \tag{6.7}$$

Finally, combining (6.7) and Lemma 3.3, we get the desired result.

Proof of Theorem 2.2 In a way similar to (6.3), by (6.1)–(6.2), we have

$$\begin{aligned}&\int_Q \theta^2(\lambda^6w^2+\lambda^4w_x^2+\lambda^2w_{xx}^2+\lambda^2w_t^2+w_{xxx}^2+w_{tx}^2)\mathrm{d}t\mathrm{d}x\\ \geq\ &\frac{2\varepsilon_0T}{C}\mathrm{e}^{\frac{R_0^2}{2}\lambda-CT(1+\|q_2\|_{1,\infty})}E_2(0).\end{aligned} \tag{6.8}$$

Now, by (5.2) and (6.8), for any $\lambda \geq \max\{\lambda_0,\lambda_1,\lambda_2\}$, we get

$$\begin{aligned}&\mathrm{e}^{C_2T(1+\|q_2\|_{1,\infty})}\Big[\mathrm{e}^{\frac{R_0^2}{2}\lambda-2C_2T(1+\|q_2\|_{1,\infty})} - (1+\|q_1\|_\infty^{-1})\frac{C_2\lambda^6}{2\varepsilon_0}\Big]E_2(0)\\ \leq\ &2C_2\varepsilon_0^{-1}T^{-1}\mathrm{e}^{C_2\lambda}\int_{T_1}^{T'}\int_{\Gamma_0}(w_{xxx}^2+w_{xx}^2)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{6.9}$$

Now, one can find a λ_4 such that for any $\lambda \geq \lambda_4$,

$$\frac{R_0^2\lambda}{4} \geq 2C_2T(1+\|q_2\|_{1,\infty}) \quad \text{and} \quad \frac{(\frac{R_0^2}{4}\lambda)^7}{7!} \geq 2(1+\|q_1\|_\infty^{-1})\frac{C_2\lambda^6}{2\varepsilon_0}. \tag{6.10}$$

Put

$$\lambda_5 \stackrel{\triangle}{=} \max\{\lambda_0,\lambda_4\} = C\Big(1+\frac{1}{T}+\|q_2\|_\infty^{\frac{2}{7}}+T\|q_2\|_{1,\infty}+\|q_1\|_\infty^{-1}\Big). \tag{6.11}$$

By (6.4), for any $\lambda \geq \lambda_5$, we obtain

$$E_2(0) \leq P(T,q_2)\int_{T_1}^{T_1'}\int_{\Gamma_0}(w_{xx}^2+w_{xxx}^2)\mathrm{d}t\mathrm{d}x. \tag{6.12}$$

Finally, combining (6.12) and Lemma 3.4, we get the desired result.

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