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On Trees with Double Domination Number Equal to the 2-Outer-Independent Domination Number Plus One

Marcin KRZYWKOWSKI[1]

Abstract A vertex of a graph is said to dominate itself and all of its neighbors. A double dominating set of a graph G is a set D of vertices of G, such that every vertex of G is dominated by at least two vertices of D. The double domination number of a graph G is the minimum cardinality of a double dominating set of G. For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a 2-dominating set if every vertex of $V(G) \setminus D$ has at least two neighbors in D, while it is a 2-outer-independent dominating set of G if additionally the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of G is the minimum cardinality of a 2-outer-independent dominating set of G. This paper characterizes all trees with the double domination number equal to the 2-outer-independent domination number plus one.

Keywords Double domination, 2-Outer-independent domination, 2-Domination, Tree
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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G, we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf, we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We say that a subset of $V(G)$ is independent if there is no edge between every two of its vertices. We denote the path on n vertices by P_n. Let T be a tree, and v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the tree resulting from T by removing the edge vx, which contains the vertex x, is a path P_n. By a star, we mean a connected graph in which exactly one vertex has degree greater than one. By a double star, we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves. Given trees T_1 and T_2 such that T_2 is an induced subgraph of T_1, and by $T_1 - T_2$, we mean the tree obtained from T_1 by removing all vertices of T_2.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination (2-domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_2(G)$,

Manuscript received September 9, 2010. Revised September 5, 2011.
[1]Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland. E-mail: marcin.krzywkowski@gmail.com

respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [5], and further studied for example in [2–3, 6–7, 12, 14]. For a comprehensive survey of domination in graphs, see [10–11].

A subset $D \subseteq V(G)$ is a 2-outer-independent dominating set (2OIDS) of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D, and the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of G, denoted by $\gamma_2^{\mathrm{oi}}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of G. A 2-outer-independent dominating set of G of minimum cardinality is called a $\gamma_2^{\mathrm{oi}}(G)$-set. The study of 2-outer-independent domination in graphs was initiated in [13].

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a double dominating set (DDS) of G if every vertex of G is dominated by at least two vertices of D. The double domination number of G, denoted by $\gamma_{\mathrm{d}}(G)$, is the minimum cardinality of a double dominating set of G. A double dominating set of G of minimum cardinality is called a $\gamma_{\mathrm{d}}(G)$-set. Double domination in graphs was introduced by Harary and Haynes [9], and further studied for example in [1, 4, 8].

We characterize all trees with the double domination number equal to the 2-outer-independent domination number plus one.

2 Results

Since the one-vertex graph does not have a double dominating set, in this paper, by a tree, we mean only a connected graph with no cycle, which has at least two vertices.

We begin with the following three straightforward observations.

Observation 2.1 Every leaf of a graph G is in every $\gamma_2^{\mathrm{oi}}(G)$-set.

Observation 2.2 Every leaf of a graph G is in every $\gamma_{\mathrm{d}}(G)$-set.

Observation 2.3 Every support vertex of a graph G is in every $\gamma_{\mathrm{d}}(G)$-set.

It is easy to see that $\gamma_{\mathrm{d}}(P_2) = \gamma_2^{\mathrm{oi}}(P_2)$. Now we prove that for every tree different from P_2, the double domination number is greater than the 2-outer-independent domination number.

Lemma 2.1 *For every tree $T \neq P_2$, we have $\gamma_{\mathrm{d}}(T) > \gamma_2^{\mathrm{oi}}(T)$.*

Proof Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,m}$. We have $\gamma_{\mathrm{d}}(T) = m + 1 > m = \gamma_2^{\mathrm{oi}}(T)$. Now assume that $\mathrm{diam}(T) = 3$. Thus T is a double star. We have $\gamma_{\mathrm{d}}(T) = n > n - 1 = \gamma_2^{\mathrm{oi}}(T)$.

Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. We will obtain the result by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First, assume that some support vertex of T, say x, is strong, let y and z mean leaves adjacent to x. Let $T' = T - y$, and let D' be any $\gamma_2^{\mathrm{oi}}(T')$-set. Of course, $D' \cup \{y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\mathrm{oi}}(T) \leq \gamma_2^{\mathrm{oi}}(T') + 1$. Now let D be any $\gamma_{\mathrm{d}}(T)$-set. By Observations 2.2

and 2.3, we have $x, y, z \in D$. It is easy to see that $D \setminus \{y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \leq \gamma_{\mathrm{d}}(T) - 1$. Now we get $\gamma_{\mathrm{d}}(T) \geq \gamma_{\mathrm{d}}(T') + 1 > \gamma_2^{\mathrm{oi}}(T') + 1 \geq \gamma_2^{\mathrm{oi}}(T)$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of the maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at the maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x, let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $\mathrm{d}_T(u) = 2$. Let $T' = T - T_v$, and let D' be any $\gamma_2^{\mathrm{oi}}(T')$-set. By Observation 2.1, we have $u \in D'$. It is easy to see that $D' \cup \{t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\mathrm{oi}}(T) \leq \gamma_2^{\mathrm{oi}}(T') + 1$. Now let us observe that there exists a $\gamma_{\mathrm{d}}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $t, v \in D$. Let us observe that $D \cup \{u\} \setminus \{v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \leq \gamma_{\mathrm{d}}(T) - 1$. Now we get $\gamma_{\mathrm{d}}(T) \geq \gamma_{\mathrm{d}}(T') + 1 > \gamma_2^{\mathrm{oi}}(T') + 1 \geq \gamma_2^{\mathrm{oi}}(T)$.

Now assume that $\mathrm{d}_T(u) \geq 3$. First assume that u is adjacent to a leaf, say x. Let $T' = T - T_v$, and let D' be any $\gamma_2^{\mathrm{oi}}(T')$-set. Of course, $D' \cup \{v, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\mathrm{oi}}(T) \leq \gamma_2^{\mathrm{oi}}(T') + 2$. Now let D be any $\gamma_{\mathrm{d}}(T)$-set. By Observations 2.2 and 2.3, we have $t, x, v, u \in D$. It is easy to see that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \leq \gamma_{\mathrm{d}}(T) - 2$. Now we get $\gamma_{\mathrm{d}}(T) \geq \gamma_{\mathrm{d}}(T') + 2 > \gamma_2^{\mathrm{oi}}(T') + 2 \geq \gamma_2^{\mathrm{oi}}(T)$.

Now assume that every descendant of u is a support vertex. Let x mean a descendant of u different from v. We denote the leaf adjacent to x by y. Let $T' = T - T_v$, and let us observe that there exists a $\gamma_2^{\mathrm{oi}}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that $D' \cup \{t\}$ is a 2OIDS of the tree T. Thus, $\gamma_2^{\mathrm{oi}}(T) \leq \gamma_2^{\mathrm{oi}}(T') + 1$. Now let us observe that there exists a $\gamma_{\mathrm{d}}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $t, v \in D$. Let us observe that $D \cup \{u\} \setminus \{v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \leq \gamma_{\mathrm{d}}(T) - 1$. Now we get $\gamma_{\mathrm{d}}(T) \geq \gamma_{\mathrm{d}}(T') + 1 > \gamma_2^{\mathrm{oi}}(T') + 1 \geq \gamma_2^{\mathrm{oi}}(T)$.

We characterize all trees with the double domination number equal to the 2-outer-independent domination number plus one. For this purpose, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_3, P_4, P_5\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations:

(Operation $\mathcal{O}_1$) Attach a vertex by joining it to any support vertex of T_k;

(Operation $\mathcal{O}_2$) Attach a path P_3 by joining one of its leaves to a vertex of $T_k \neq P_4$ adjacent to a path P_3;

(Operation $\mathcal{O}_3$) Attach a path P_3 by joining one of its leaves to any support vertex of T_k;

(Operation $\mathcal{O}_4$) Attach a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_4;

(Operation $\mathcal{O}_5$) Attach a vertex by joining it to a vertex of T_k adjacent to a path P_4;

(Operation $\mathcal{O}_6$) Attach a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a support vertex of degree two, and to a vertex of degree two the other neighbor of which is a support vertex.

Now we prove that for every tree of the family $\mathcal{T}$, the double domination number is equal to the 2-outer-independent domination number plus one.

Lemma 2.2 *If $T \in \mathcal{T}$, then $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$.*

Proof We use the induction on the number k of operations performed to construct the tree T. If $T = P_3$, then obviously $\gamma_{\rm d}(T) = 3 = 2 + 1 = \gamma_2^{\rm oi}(T) + 1$. If $T = P_4$, then $\gamma_{\rm d}(T) = 4 = 3 + 1 = \gamma_2^{\rm oi}(T) + 1$. If $T = P_5$, then also $\gamma_{\rm d}(T) = 4 = 3 + 1 = \gamma_2^{\rm oi}(T) + 1$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We denote the attached vertex by x, and denote its neighbor by y. Let D' be any $\gamma_{\rm d}(T')$-set. By Observation 2.3, we have $y \in D'$. It is easy to see that $D' \cup \{x\}$ is a DDS of the tree T. Thus $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 1$. Now let D be any $\gamma_2^{\rm oi}(T)$-set. By Observation 2.1, we have $x \in D$. If $y \in D$, then it is easy to see that $D \setminus \{x\}$ is a 2OIDS of the tree T'. Now assume that $y \notin D$. Let a and b mean neighbors of y different from x. The set $V(T) \setminus D$ is independent, and thus $a, b \in D$. Let us observe that now also $D \setminus \{x\}$ is a 2OIDS of the tree T' as the vertex y has at least two neighbors in $D \setminus \{x\}$. Therefore, $\gamma_2^{\rm oi}(T') \leq \gamma_2^{\rm oi}(T) - 1$. Now we get $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 1 = \gamma_2^{\rm oi}(T') + 2 \leq \gamma_2^{\rm oi}(T) + 1$. On the other hand, by Lemma 2.1, we have $\gamma_{\rm d}^{\rm oi}(T) \geq \gamma_2^{\rm oi}(T) + 1$. This implies that $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We denote by x the vertex to which P_3 is attached. Let $v_1v_2v_3$ mean the attached path, and let v_1 be joined to x. We denote the path P_3 adjacent to x and different from $v_1v_2v_3$ by abc. Let a be adjacent to x, and let us observe that there exists a $\gamma_{\rm d}(T')$-set that does not contain the vertex a. Let D' be such a set. The vertex a has to be dominated twice, and thus $x \in D'$. It is easy to see that $D' \cup \{v_2, v_3\}$ is a DDS of the tree T. Thus $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 2$. Now let us observe that there exists a $\gamma_2^{\rm oi}(T)$-set that contains the vertex v_1. Let D be such a set. By Observation 2.1, we have $v_3 \in D$. The set D is minimal, and thus $v_2 \notin D$. If $x \in D$, then it is easy to see that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T'. Now assume that $x \notin D$. Let k mean a neighbor of x different from v_1 and a. The set $V(T) \setminus D$ is independent, and thus $a, k \in D$. Let us observe that now also $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T' as the vertex x has at least two neighbors in $D \setminus \{v_1, v_3\}$. Therefore, $\gamma_2^{\rm oi}(T') \leq \gamma_2^{\rm oi}(T) - 2$. Now we get $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 2 = \gamma_2^{\rm oi}(T) + 3 \leq \gamma_2^{\rm oi}(T) + 1$. This implies that $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We denote by x the vertex to which P_3 is attached. Let $v_1v_2v_3$ mean the attached path, and let v_1 be joined to x, let y mean a leaf adjacent to x, and let D' be any $\gamma_{\rm d}(T')$-set. By Observation 2.3, we have $x \in D'$. It is easy to see that $D' \cup \{v_2, v_3\}$ is DDS of the tree T. Thus $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 2$. Now let us observe that there exists a $\gamma_2^{\rm oi}(T)$-set that contains the vertex v_1. Let D be such a set. By Observation 2.1, we have $v_3, y \in D$. The set D is minimal, and thus $v_2 \notin D$. If $x \in D$, then it is easy to see that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T'. Now assume that $x \notin D$. Let k mean a neighbor of x different from y. The set $V(T) \setminus D$ is independent, and thus $k \in D$. Let us observe that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T' as the vertex x has at least two neighbors in $D \setminus \{v_1, v_3\}$. Therefore, $\gamma_2^{\rm oi}(T') \leq \gamma_2^{\rm oi}(T) - 2$. Now we get $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 2 = \gamma_2^{\rm oi}(T') + 3 \leq \gamma_2^{\rm oi}(T) + 1$. This implies $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_4$. We denote by x the vertex to which P_3 is attached. Let $v_1v_2v_3$ mean the attached path, let v_1 be joined to x, let $abcd$ mean a path P_4 adjacent to x, and let x and a be adjacent. Let us observe that there exists a

$\gamma_{\rm d}(T')$-set that does not contain the vertex b. Let D' be such a set. The vertex a has to be dominated twice, and thus $x \in D'$. It is easy to see that $D' \cup \{v_2, v_3\}$ is a DDS of the tree T. Thus $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 2$. Now let us observe that there exists a $\gamma_2^{\rm oi}(T)$-set that contains the vertices v_1, b, and x. Let D be such a set. By Observation 2.1, we have $v_3 \in D$. The set D is minimal, and thus $v_2 \notin D$. It is easy to see that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T'. Therefore, $\gamma_2^{\rm oi}(T') \leq \gamma_2^{\rm oi}(T) - 2$. Now we get $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 2 = \gamma_2^{\rm oi}(T') + 3 \leq \gamma_2^{\rm oi}(T') + 1$. This implies $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_5$. Let x mean the attached vertex, y mean its neighbor, $abcd$ mean a path P_4 adjacent to x, and let x and a be adjacent. Let us observe that there exists a $\gamma_{\rm d}(T')$-set that does not contain the vertex b. Let D' be such a set. The vertex a has to be dominated twice, and thus $x \in D$. It is easy to see that $D' \cup \{y\}$ is a DDS of the tree T. Thus $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 1$. Now let us observe that there exists a $\gamma_2^{\rm oi}(T)$-set that contains the vertices b and x. Let D be such a set. By Observation 2.1, we have $y \in D$. It is easy to see that $D \setminus \{y\}$ is a 2OIDS of the tree T'. Therefore, $\gamma_2^{\rm oi}(T') \leq \gamma_2^{\rm oi}(T) - 1$. Now we get $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 1 = \gamma_2^{\rm oi}(T') + 2 \leq \gamma_2^{\rm oi}(T) + 1$. This implies $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_6$. We denote by x the vertex to which P_3 is attached. Let $v_1v_2v_3$ mean the attached path. Let v_1 be joined to x. Let y mean a vertex of degree two adjacent to x, the other neighbor of which is a support vertex. Let us observe that there exists a $\gamma_{\rm d}(T')$-set that does not contain the vertex y. Let D' be such a set. The vertex y has to be dominated twice, and thus $x \in D'$. It is easy to see that $D' \cup \{v_2, v_3\}$ is a DDS of the tree T. Thus $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 2$. Now let us observe that there exists a $\gamma_2^{\rm oi}(T)$-set that contains the vertices v_1 and x. Let D be such a set. By Observation 2.1, we have $v_3 \in D$. The set D is minimal, and thus $v_2 \notin D$. It is easy to see that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T'. Therefore, $\gamma_2^{\rm oi}(T') \leq \gamma_2^{\rm oi}(T) - 2$. Now we have $\gamma_{\rm d}(T) \leq \gamma_{\rm d}(T') + 2 = \gamma_2^{\rm oi}(T') + 3 \leq \gamma_2^{\rm oi}(T) + 1$. This implies $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$.

Now we prove that if the double domination number of a tree is equal to its 2-outer-independent domination number plus one, then the tree belongs to the family $\mathcal{T}$.

Lemma 2.3 *Let T be a tree. If $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$, then $T \in \mathcal{T}$.*

Proof Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,m}$. If $T = P_3$, then $T \in \mathcal{T}$. If T is a star different from P_3, then it can be obtained from P_3 by a proper number of operations $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Now assume that $\mathrm{diam}(T) = 3$. Thus T is a double star. If $T = P_4$, then $T \in \mathcal{T}$. If T is a double star different from P_4, then T can be obtained from P_4 by proper numbers of operations $\mathcal{O}_1$ performed on the support vertices. Thus $T \in \mathcal{T}$.

Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. We obtain the result by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First, assume that some support vertex of T, say x, is strong. Let y and z mean leaves adjacent to x, let $T' = T - y$, and let D' be any $\gamma_2^{\rm oi}(T')$-set. Of course, $D' \cup \{y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \leq \gamma_2^{\rm oi}(T') + 1$. Now let D be any $\gamma_{\rm d}(T)$-set. By Observations 2.2 and 2.3, we have $x, y, z \in D$. It is easy to observe that $D \setminus \{y\}$ is a DDS of the tree T'. Therefore,

$\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 1$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 1 = \gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 1$. On the other hand, by Lemma 2.1, we have $\gamma_{\rm d}(T') \ge \gamma_2^{\rm oi}(T') + 1$. This implies that $\gamma_{\rm d}(T') = \gamma_2^{\rm oi}(T') + 1$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of the maximum eccentricity $\text{diam}(T)$. Let t be a leaf at the maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. If $\text{diam}(T) \ge 5$, then let d be the parent of w. If $\text{diam}(T) \ge 6$, then let e be the parent of d. If $\text{diam}(T) \ge 7$, then let f be the parent of e. By T_x, let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that among the descendants of u, there is a support vertex, say x, different from v. We denote by y the leaf adjacent to x. Assume that there exists a $\gamma_{\rm d}(T)$-set in which the vertex u is dominated at least thrice. Let D be such a set. By Observations 2.2 and 2.3, we have $t, v \in D$. Let $T' = T - T_v$. Let us observe that $D \setminus \{v, t\}$ is a DDS of the tree T' as the vertex u is dominated at least twice. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2$. Now let us observe that there exists a $\gamma_2^{\rm oi}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that $D' \cup \{t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 1$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2 = \gamma_2^{\rm oi}(T) - 1 \le \gamma_2^{\rm oi}(T')$. This is a contradiction, since by Lemma 2.1, we have $\gamma_{\rm d}(T') > \gamma_2^{\rm oi}(T')$. Therefore, in every $\gamma_{\rm d}(T)$-set, the vertex u is dominated only twice. This implies that $\text{d}_T(u) = 3$ as all leaves and support vertices belong to every $\gamma_{\rm d}(T)$-set. Let $T'' = T - T_u$. Let D'' be any $\gamma_2^{\rm oi}(T'')$-set. It is easy to observe that $D'' \cup \{u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T'') + 3$. Now let D be any $\gamma_{\rm d}(T)$-set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. The vertex u is dominated only twice, and thus $u \notin D$. Observe that $D \setminus \{v, t, x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 4$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 4 = \gamma_2^{\rm oi}(T) - 3 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Thus v is the only one support vertex among the descendants of u. Moreover, we have $\text{d}_T(u) = 3$. We denote by x the leaf adjacent to u. First assume that there is a descendant of w, say k, such that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibilities when T_k is isomorphic to T_u, or T_k is a path P_3. First assume that T_k is isomorphic to T_u. We denote by l the descendant of l which is a support vertex, denote by m the leaf adjacent to l, and denote by p the leaf adjacent to k. Let $T' = T - T_u - T_l - p$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $k \in D'$. It is easy to observe that $D' \cup \{u, t, x, m, p\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 5$. Now let D be any $\gamma_{\rm d}(T)$-set. By Observations 2.2 and 2.3, we have $t, x, m, p, v, u, l, k \in D$. If $w \in D$, then it is easy to observe that $D \setminus \{u, v, t, x, l, m, p\}$ is a DDS of the tree T'. Now assume that $w \notin D$. Let us observe that $D \cup \{w\} \setminus \{u, v, t, x, l, m, p\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 6$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 6 = \gamma_2^{\rm oi}(T) - 5 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that T_k is a path P_3, say klm. Let $T' = T - T_v - x$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $u \in D'$. It is easy to observe that $D' \cup \{t, x\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 2$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex k. Let D be such a set. By Observations 2.2 and 2.3, we have $t, x, v, u \in D$. The vertex k has to be dominated twice, and thus $w \in D$. It is easy to observe that $D \setminus \{v, t, x\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 3$. Now we get

$\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 3 = \gamma_2^{\rm oi}(T) - 2 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Assume that there exists a $\gamma_{\rm d}(T)$-set in which the vertex w is dominated at least thrice. Let D be such a set. By Observations 2.2 and 2.3, we have $t, x, v, u \in D$. Let $T' = T - T_u$. Let us observe that $D \setminus \{u, v, t, x\}$ is a DDS of the tree T' as the vertex w is dominated at least twice. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 4$. Now let D' be any $\gamma_2^{\rm oi}(T')$-set. It is easy to observe that $D' \cup \{u, t, x\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 3$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 4 = \gamma_2^{\rm oi}(T) - 3 \le \gamma_2^{\rm oi}(T')$, a contradiction. Therefore, in every $\gamma_{\rm d}(T)$-set, the vertex w is dominated only twice. This implies that $d_T(w) = 3$ as all leaves and support vertices belong to every $\gamma_{\rm d}(T)$-set. Moreover, the descendant of w different from u, say k, is a support vertex of degree two. We denote by l the leaf adjacent to k. Let $T' = T - T_w$. If $T' = P_2$, then $\gamma_{\rm d}(T) = 8 = 6 + 2 = \gamma_2^{\rm oi}(T) + 2 > \gamma_2^{\rm oi}(T) + 1$, a contradiction. Now assume that $T' \ne P_2$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. It is easy to observe that $D' \cup \{w, u, t, x, l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 5$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex w. Let D be such a set. By Observations 2.2 and 2.3, we have $t, x, l, v, u, k \in D$. Observe that $D \setminus \{u, v, t, x, k, l\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 6$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 6 = \gamma_2^{\rm oi}(T) - 5 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that $d_T(u) = 2$. First assume that there is a descendant of w, say x, such that the distance of w to the most distant vertex of T_x is three. It suffices to consider only the possibility when T_k is a path P_3. Let $T' = T - T_u$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. It is easy to see that $D' \cup \{u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 2$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $t, v \in D$. Observe that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2 = \gamma_2^{\rm oi}(T) - 1 \le \gamma_2^{\rm oi}(T') + 1$. This implies that $\gamma_{\rm d}(T') = \gamma_2^{\rm oi}(T') + 1$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that some descendant of w, say x, is a leaf. Let $T' = T - T_u$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. It is easy to see that $D' \cup \{u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 2$. Now let D be any $\gamma_{\rm d}(T)$-set. By Observations 2.2 and 2.3, we have $t, x, v, w \in D$. The set D is minimal, thus $u \notin D$. Observe that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2 = \gamma_2^{\rm oi}(T) - 1 \le \gamma_2^{\rm oi}(T') + 1$. This implies that $\gamma_{\rm d}(T') = \gamma_2^{\rm oi}(T') + 1$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that there is a descendant of w, say x, such that the distance of w to the most distant vertex of T_x is two. It suffices to consider only the possibility when x is a support vertex of degree two. We denote by y the leaf adjacent to x. First assume that $d_T(w) \ge 4$. Thus there is a descendant of w, say k, which is a support vertex of degree two different from x. Let $T' = T - T_x$. Let us observe that there exists a $\gamma_2^{\rm oi}(T')$-set that contains the vertex w. Let D' be such a set. It is easy to see that $D' \cup \{y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 1$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $x, y, k \in D$. The vertex u has to be dominated twice, and thus $w \in D$. It is easy to observe that $D \setminus \{x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2 = \gamma_2^{\rm oi}(T) - 1 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that $d_T(w) = 3$. First assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is four. It suffices to consider only the possibilities when T_k is isomorphic to T_w, or T_k is a path P_4. First assume that T_k is isomorphic to T_w. We denote by lmp the path P_3 adjacent to k, and denote by qs the path P_2 adjacent to k. Let l and q be adjacent to k, let $T' = T - T_w - T_l - T_q$, and let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $k \in D'$. It is easy to observe that $D' \cup \{w, u, t, y, l, p, s\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 7$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and l. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, p, s, v, x, l, q \in D$. Each one of the vertices u and l has to be dominated twice, and thus $w, k \in D$. If $d \in D$, then it is easy to observe that $D \setminus \{w, v, t, x, y, m, p, q, s\}$ is a DDS of the tree T'. Now assume that $d \notin D$. Let us observe that $D \cup \{d\} \setminus \{w, v, t, x, y, m, p, q, s\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 8$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 8 = \gamma_2^{\rm oi}(T) - 7 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that T_k is a path P_4, say $klmp$. Let $T' = T - T_u - T_x$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $w \in D'$. It is easy to observe that $D' \cup \{u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 3$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and l. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, v, x \in D$. Each one of the vertices u and k has to be dominated twice, and thus $w, d \in D$. It is easy to observe that $D \setminus \{v, t, x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 4$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 4 = \gamma_2^{\rm oi}(T) - 3 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_u - T_x$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $w \in D'$. It is easy to observe that $D' \cup \{u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 3$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and k. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, v, x \in D$. Each one of the vertices u and k has to be dominated twice, and thus $w, d \in D$. It is easy to observe that $D \setminus \{v, t, x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 4$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 4 = \gamma_2^{\rm oi}(T) - 3 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. We denote by l the leaf adjacent to k. First assume that $d_T(d) \ge 4$. Let m mean a descendant of d different from w and k. It suffices to consider only the possibility when m is a support vertex of degree two. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_2^{\rm oi}(T')$-set that contains the vertex d. Let D' be such a set. It is easy to see that $D' \cup \{l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 1$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $l, k, m \in D$. The vertex m has to be dominated twice, and thus $w \in D$. It is easy to observe that $D \setminus \{k, l\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2 = \gamma_2^{\rm oi}(T) - 1 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that $d_T(d) = 3$. Let $T' = T - T_d$. If $T' = P_2$, then $\gamma_{\rm d}(T) = 9 = 7 + 2 = \gamma_2^{\rm oi}(T) + 2 > \gamma_2^{\rm oi}(T) + 1$, a contradiction. Now assume that $T' \neq P_2$. Let D' be any $\gamma_2^{\rm oi}(T')$-set.

It is easy to observe that $D' \cup \{d, w, u, t, y, l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\mathrm{oi}}(T) \le \gamma_2^{\mathrm{oi}}(T') + 6$. Now let us observe that there exists a $\gamma_{\mathrm{d}}(T)$-set that does not contain the vertex d. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, l, v, x, k \in D$. The vertex u has to be dominated twice, and thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y, k, l\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 7$. Now we get $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 7 = \gamma_2^{\mathrm{oi}}(T) - 6 \le \gamma_2^{\mathrm{oi}}(T')$, a contradiction.

Now assume that some descendant of d, say k, is a leaf. Let $T' = T - T_w$. Let D' be any $\gamma_2^{\mathrm{oi}}(T')$-set. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\mathrm{oi}}(T) \le \gamma_2^{\mathrm{oi}}(T') + 4$. Now let us observe that there exists a $\gamma_{\mathrm{d}}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, k, v, x, d \in D$. The vertex u has to be dominated twice, and thus $w \in D$. It is easy to observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 5$. Now we get $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 5 = \gamma_2^{\mathrm{oi}}(T) - 4 \le \gamma_2^{\mathrm{oi}}(T')$, a contradiction.

We now turn to the possibility $\mathrm{d}_T(w) = 2$. First assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is a path P_4, say $klmp$. Let $T' = T - T_w$. Let us observe that there exists a $\gamma_2^{\mathrm{oi}}(T')$-set that contains the vertices l and d. Let D' be such a set. It is easy to observe that $D' \cup \{u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\mathrm{oi}}(T) \le \gamma_2^{\mathrm{oi}}(T') + 2$. Now let us observe that there exists a $\gamma_{\mathrm{d}}(T)$-set that does not contain the vertices u and l. Let D be such a set. By Observations 2.2 and 2.3, we have $t, p, v, m \in D$. Each one of the vertices w and k has to be dominated twice, and thus $w, d, k \in D$. It is easy to observe that $D \setminus \{w, v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 3$. Now we get $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 3 = \gamma_2^{\mathrm{oi}}(T) - 2 \le \gamma_2^{\mathrm{oi}}(T')$, a contradiction.

Now assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_k$. Let D' be any $\gamma_2^{\mathrm{oi}}(T')$-set. It is easy to see that $D' \cup \{k, m\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\mathrm{oi}}(T) \le \gamma_2^{\mathrm{oi}}(T') + 2$. Now let us observe that there exists a $\gamma_{\mathrm{d}}(T)$-set that does not contain the vertex k. Let D be such a set. By Observations 2.2 and 2.3, we have $m, l \in D$. Observe that $D \setminus \{l, m\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 2$. Now we get $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 2 = \gamma_2^{\mathrm{oi}}(T) - 1 \le \gamma_2^{\mathrm{oi}}(T') + 1$. This implies that $\gamma_{\mathrm{d}}(T') = \gamma_2^{\mathrm{oi}}(T') + 1$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Now assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. We denote by l the leaf adjacent to k. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_2^{\mathrm{oi}}(T')$-set that contains the vertices u and d. Let D' be such a set. It is easy to see that $D' \cup \{l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\mathrm{oi}}(T) \le \gamma_2^{\mathrm{oi}}(T') + 1$. Now let us observe that there exists a $\gamma_{\mathrm{d}}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $l, k \in D$. The vertex w has to be dominated twice, and thus $w, d \in D$. It is easy to observe that $D \setminus \{k, l\}$ is a DDS of the tree T'. Therefore, $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 2$. Now we get $\gamma_{\mathrm{d}}(T') \le \gamma_{\mathrm{d}}(T) - 2 = \gamma_2^{\mathrm{oi}}(T) - 1 \le \gamma_2^{\mathrm{oi}}(T')$, a contradiction.

Now assume that some descendant of d, say k, is a leaf. Let $T' = T - k$. Let D' be any

$\gamma_2^{\rm oi}(T')$-set. Of course, $D' \cup \{k\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 1$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $k, d \in D$. The vertex w has to be dominated twice, and thus $w \in D$. It is easy to observe that $D \setminus \{k\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 1$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 1 = \gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 1$. This implies that $\gamma_{\rm d}(T') = \gamma_2^{\rm oi}(T') + 1$. The tree T can be obtained from T' by operation $\mathcal{O}_5$. Thus $T \in \mathcal{T}$.

If ${\rm d}_T(d) = 1$, then $T = P_5 \in \mathcal{T}$. We now turn to the possibility ${\rm d}_T(w) = 3$. Assume that ${\rm d}_T(d) = 2$. First assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is five. It suffices to consider only the possibilities when T_k is isomorphic to $T_{\rm d}$, or T_k is a path P_5. First assume that T_k is isomorphic to $T_{\rm d}$. Let l mean the descendant of k. We denote by mpq the path P_3 adjacent to l, and denote by ab the path P_2 adjacent to l. Let m and a be adjacent to l. Let $T' = T - T_{\rm d}$. Let us observe that there exists a $\gamma_2^{\rm oi}(T')$-set that contains the vertices m, l and e. Let D' be such a set. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 4$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and d. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, v, x \in D$. The vertex w has to be dominated twice, and thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 5$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 5 = \gamma_2^{\rm oi}(T) - 4 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that T_k is a path P_5, say $klmpq$. Let $T' = T - T_{\rm d} - q$. Let us observe that there exists a $\gamma_2^{\rm oi}(T')$-set that contains the vertices l and e. Let D' be such a set. It is easy to observe that $D' \cup \{w, u, t, y, q\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 5$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u, d and m. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, q, v, x, p \in D$. Each one of the vertices d and l has to be dominated twice, and thus $w, e, k, l \in D$. Let us observe that $D \cup \{m\} \setminus \{w, v, t, x, y, l, q\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 6$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 6 = \gamma_2^{\rm oi}(T) - 5 \le \gamma_{\rm d}^{\rm oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is four. It suffices to consider only the possibilities when T_k is isomorphic to T_w, or T_k is a path P_4. First assume that T_k is isomorphic to T_w. We denote by lmp the path P_3 adjacent to k, and denote by qs the path P_2 adjacent to k. Let l and q be adjacent to k. Let $T' = T - T_{\rm d} - T_l - T_q$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1 we have $k \in D'$. If $e \in D'$, then it is easy to observe that $D' \cup \{w, u, t, y, l, p, s\}$ is a 2OIDS of the tree T. Now assume that $e \notin D'$. Let us observe that $D' \cup \{e, w, u, t, y, l, p, s\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 8$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u, d and l. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, p, s, v, x, m, q \in D$. Each one of the vertices d and l has to be dominated twice, thus $w, e, k \in D$. It is easy to observe that $D \setminus \{w, v, t, x, y, m, p, q, s\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 9$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 9 = \gamma_2^{\rm oi}(T) - 8 \le \gamma^{\rm oi}(T')$, a contradiction.

Now assume that T_k is a path P_4, say $klmp$. Let $T' = T - T_{\rm d}$. Let us observe that there exists a $\gamma_2^{\rm oi}(T')$-set that contains the vertices l and e. Let D' be such a set. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 4$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and d. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, v, x \in D$. The vertex k has to be dominated twice, and thus $e, k \in D$. It is easy to observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 5$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 5 = \gamma_2^{\rm oi}(T) - 4 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. Let $T' = T - T_{\rm d}$. Let us observe that there exists a $\gamma_2^{\rm oi}(T')$-set that contains the vertex e. Let D' be such a set. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Therefore, $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 4$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and d. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, v, x \in D$. The vertex w has to be dominated twice, and thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 5$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 5 = \gamma_2^{\rm oi}(T) - 4 \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that some descendant of e, say k, is a leaf. Let $T' = T - T_u$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. It is easy to see that $D' \cup \{u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 2$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $t, v \in D$. Observe that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2 = \gamma_2^{\rm oi}(T) - 1 \le \gamma_2^{\rm oi}(T') + 1$. This implies that $\gamma_{\rm d}(T') = \gamma_2^{\rm oi}(T') + 1$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_6$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that no descendant of e is a leaf.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let us observe that we can assume that for every descendant of e different from d, say k, the tree T_k is a path P_3. Let $k_1, k_2, \cdots, k_{{\rm d}_T(e)-2}$ mean the descendants of e different from d. We denote by l_i the descendant of k_i, and denote by m_i the descendant of l_i. Let $T' = T - T_{\rm d} - T_{k_1} - T_{k_2} - \cdots - T_{k_{{\rm d}_T(e)-2}}$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $e \in D'$. It is easy to observe that $D' \cup \{w, u, t, y, k_1, m_1, k_2, m_2, \cdots, k_{{\rm d}_T(e)-2}, m_{{\rm d}_T(e)-2}\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 2{\rm d}_T(e)$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices $u, d, k_1, k_2, \cdots, k_{{\rm d}_T(e)-2}$. Let D be such a set. By Observations 2.2 and 2.3, we have $t, v, y, x, m_1, l_1, m_2, l_2, \cdots, m_{{\rm d}_T(e)-2}, l_{{\rm d}_T(e)-2}$. The vertex w has to be dominated twice, and thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y, l_1, m_1, l_2, m_2, \cdots, l_{{\rm d}_T(e)-2}, m_{{\rm d}_T(e)-2}\}$. Therefore, $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2{\rm d}_T(e) - 1$. Now we get $\gamma_{\rm d}(T') \le \gamma_{\rm d}(T) - 2{\rm d}_T(e) - 1 = \gamma_2^{\rm oi}(T) - 2{\rm d}_T(e) \le \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that ${\rm d}_T(e) = 2$. Let $T' = T - T_{\rm d}$. If $T' = P_1$, then $\gamma_{\rm d}(T) = 7 = 5 + 2 = \gamma_2^{\rm oi}(T) + 2 > \gamma_2^{\rm oi}(T) + 1$, a contradiction. If $T' = P_2$, then let $T'' = T - T_u = P_6$. By the inductive hypothesis, we have $T'' \in \mathcal{T}$ as $\gamma_{\rm d}(P_6) = 5 = 4 + 1 = \gamma_2^{\rm oi}(P_6) + 1$. The tree T can be obtained from T'' by operation $\mathcal{O}_6$. Thus $T \in \mathcal{T}$. Now assume that $T' \ne P_1, P_2$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $e \in D'$. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \le \gamma_2^{\rm oi}(T') + 4$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and d. Let D be such a set. By Observations 2.2 and 2.3, we have $t, y, v, x \in D$. The vertex w has to be dominated twice, and thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 5$. Now we get $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 5 = \gamma_2^{\rm oi}(T) - 4 \leq \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that $d_T(w) = 2$. Assume that $d_T(d) = 2$. First assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is five. It suffices to consider only the possibility when T_k is a path P_5, say $klmpq$. Let $T' = T - T_{\rm d} - T_l$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $k \in D'$. It is easy to observe that $D' \cup \{d, u, t, m, q\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \leq \gamma_2^{\rm oi}(T') + 5$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and m. Let D be such a set. By Observations 2.2 and 2.3, we have $t, q, v, p \in D$. Each one of the vertices w and l has to be dominated twice, and thus $w, d, l, k \in D$. If $e \in D$, then it is easy to observe that $D \setminus \{d, w, v, t, l, p, q\}$ is a DDS of the tree T'. Now assume that $e \notin D$. Let us observe that $D \cup \{e\} \setminus \{d, w, v, t, l, p, q\}$ is DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 6$. Now we get $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 6 = \gamma_2^{\rm oi}(T) - 5 \leq \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is a path P_4, say $klmp$. Let $T' = T - T_{\rm d}$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. It is easy to observe that $D' \cup \{d, u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \leq \gamma_2^{\rm oi}(T') + 3$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and l. Let D be such a set. By Observations 2.2 and 2.3, we have $t, v \in D$. Each one of the vertices w and k has to be dominated twice, and thus $w, d, k, e \in D$. It is easy to observe that $D \setminus \{d, w, v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 4$. Now we get $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 4 = \gamma_2^{\rm oi}(T) - 3 \leq \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_w - T_k$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $d \in D'$. It is easy to observe that $D' \cup \{u, t, k, m\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \leq \gamma_2^{\rm oi}(T') + 4$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertices u and k. Let D be such a set. By Observations 2.2 and 2.3, we have $t, m, v, l \in D$. Each one of the vertices w and k has to be dominated twice, and thus $w, d, e \in D$. It is easy to observe that $D \setminus \{w, v, t, l, m\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 5$. Now we get $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 5 = \gamma_2^{\rm oi}(T) - 4 \leq \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that some descendant of e, say k, is a leaf. Let $T' = T - T_w$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $d \in D'$. It is easy to observe that $D' \cup \{u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \leq \gamma_2^{\rm oi}(T') + 2$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $t, k, v, e \in D$. The vertex w has to be dominated twice, and thus $w, d \in D$. It is easy to observe that $D \setminus \{w, v, t\}$ is a DDS of the tree T'. Therefore, $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 3$. Now we get $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 3 = \gamma_2^{\rm oi}(T) - 2 \leq \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. We denote by l the leaf adjacent to k. First assume that $d_T(e) \geq 4$. Thus there is a descendant of e, say a, which is a support vertex of degree two, and is different from k. We denote by b the leaf adjacent to a. Let $T' = T - T_k$. Let us observe that there exists a

$\gamma_2^{\rm oi}(T')$-set that contains the vertex e. It is easy to see that $D' \cup \{l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \leq \gamma_2^{\rm oi}(T') + 1$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $l, k, a \in D$. The vertex w is dominated twice, and thus $d \in D$. It is easy to observe that $D \setminus \{k, l\}$ is a DDS of the tree T' as the vertex e is still dominated at least twice. Therefore, $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 2$. Now we get $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 2 = \gamma_2^{\rm oi}(T) - 1 \leq \gamma_2^{\rm oi}(T')$, a contradiction.

Now assume that $d_T(e) = 3$. Let $T' = T - T_e$. If $T' = P_1$, then $\gamma_{\rm d}(T) = 8 = 6 + 2 = \gamma_2^{\rm oi}(T)$ $+2 > \gamma_2^{\rm oi}(T) + 1$, a contradiction. If $T' = P_2$, then also $\gamma_{\rm d}(T) = 8 = 6 + 2 = \gamma_2^{\rm oi}(T) + 2$ $> \gamma_2^{\rm oi}(T) + 1$, a contradiction. Now assume that $T' \neq P_1, P_2$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. It is easy to observe that $D' \cup \{e, d, u, t, l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \leq \gamma_2^{\rm oi}(T') + 5$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $t, l, v, k \in D$. The vertex w has to be dominated twice, and thus $w, d \in D$. If $e \notin D$, then observe that $D \setminus \{d, w, v, t, k, l\}$ is a DDS of the tree T'. Now assume that $e \in D$. If $f \notin D$, then let us observe that $D \cup \{f\} \setminus \{e, d, w, v, t, k, l\}$ is a DDS of the tree T'. Now assume that $f \in D$. Let z mean a neighbor of f different from e. We have $z \notin D$, otherwise $D \setminus \{e\}$ is a DDS of the tree T, a contradiction to the minimality of D. Let us observe that $D \cup \{z\} \setminus \{e, d, w, v, t, k, l\}$ is a DDS of the tree T'. Now we conclude that $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 6$. We get $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 6 = \gamma_2^{\rm oi}(T) - 5 \leq \gamma_2^{\rm oi}(T')$, a contradiction.

If $d_T(e) = 1$, then $T = P_6$. Let $T' = T - e = P_5 \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_5$. Now assume that $d_T(e) = 2$. Let $T' = T - T_{\rm d}$. Let D' be any $\gamma_2^{\rm oi}(T')$-set. By Observation 2.1, we have $e \in D'$. It is easy to observe that $D' \cup \{d, u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{\rm oi}(T) \leq \gamma_2^{\rm oi}(T') + 3$. Now let us observe that there exists a $\gamma_{\rm d}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3, we have $t, v \in D$. The vertex w has to be dominated twice, and thus $w, d \in D$. If $e \notin D$, then observe that $D \cup \{e\} \setminus \{d, w, v, t\}$ is a DDS of the tree T'. Now assume that $e \in D$. If $f \in D$, then it is easy to see that $D \setminus \{d, w, v, t\}$ is a DDS of the tree T'. Now assume that $f \notin D$. Let us observe that $D \cup \{f\} \setminus \{d, w, v, t\}$ is a DDS of the tree T'. Now we conclude that $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 3$. We get $\gamma_{\rm d}(T') \leq \gamma_{\rm d}(T) - 3 = \gamma_2^{\rm oi}(T) - 2 \leq \gamma_2^{\rm oi}(T')$, a contradiction.

As an immediate consequence of Lemmas 2.2 and 2.3, we have the following characterization of the trees with the double domination number equal to the 2-outer-independent domination number plus one.

Theorem 2.1 *Let T be a tree. Then $\gamma_{\rm d}(T) = \gamma_2^{\rm oi}(T) + 1$ if and only if $T \in \mathcal{T}$.*

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