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Backward Doubly Stochastic Differential Equations with Jumps and Stochastic Partial Differential-Integral Equations*

Qingfeng ZHU[1] Yufeng SHI[2]

Abstract Backward doubly stochastic differential equations driven by Brownian motions and Poisson process (BDSDEP) with non-Lipschitz coefficients on random time interval are studied. The probabilistic interpretation for the solutions to a class of quasilinear stochastic partial differential-integral equations (SPDIEs) is treated with BDSDEP. Under non-Lipschitz conditions, the existence and uniqueness results for measurable solutions to BDSDEP are established via the smoothing technique. Then, the continuous dependence for solutions to BDSDEP is derived. Finally, the probabilistic interpretation for the solutions to a class of quasilinear SPDIEs is given.

Keywords Backward doubly stochastic differential equations, Stochastic partial differential-integral equations, Random measure, Poisson process
2000 MR Subject Classification 60H10, 60G55, 60H15

1 Introduction

Nonlinear backward stochastic differential equations with Brownian motions as noise sources (BSDEs) were independently introduced by Pardoux and Peng [7] and Duffie and Epstein [4]. By virtue of BSDEs, Peng [9] gave a probabilistic interpretation (nonlinear Feynman-Kac formula) for the solutions to semilinear parabolic partial differential equations (PDEs). In [9], Peng also gave an existence and uniqueness result of BSDEs with random terminal time. And then Darling and Pardoux [3] proved an existence and uniqueness result for BSDEs with random terminal time under different assumptions. They used their result to construct a continuous viscosity solution to a class of semilinear elliptic PDEs.

A class of backward doubly stochastic differential equations (BDSDEs) was introduced by Pardoux and Peng [8] in 1994, in order to provide a probabilistic interpretation for the solutions to a class of semilinear stochastic partial differential equations (SPDEs). They proved the existence and uniqueness of solutions to BDSDEs under uniformly Lipschitz conditions. Since

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[1]School of Mathematics, Shandong University, Jinan 250100, China; School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan 250014, China. E-mail: zhuqf508@sohu.com
[2]Corresponding author. School of Mathematics, Shandong University, Jinan 250100, China. E-mail: yfshi@sdu.edu.cn
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then, Shi et al. [11] relaxed the Lipschitz assumptions to linear growth conditions. Bally and Matoussi [1] gave a probabilistic interpretation of the solutions in Sobolev spaces to semilinear parabolic SPDEs in terms of BDSDEs. Zhang and Zhao [16] proved the existence and uniqueness of solution to BDSDEs on infinite horizons, and described the stationary solutions to SPDEs by virtue of the solutions to BDSDEs on infinite horizons.

BSDEs driven by Brownian motions and Poisson process (BSDEP) were first discussed by Tang and Li [14]. After then Situ [12] obtained an existence and uniqueness result for BSDEP with non-Lipschitz coefficients, so as to give a probabilistic interpretation for solutions to partial differential-integral equations (PDIEs). Barles et al. [2] and Yin and Mao [15] discussed viscosity solutions to a system of PDIEs in terms of BSDEs with jumps. Recently BDSDEs driven by Brownian motions and Poisson process (BDSDEP) with Lipschitzian coefficients on a fixed time interval were discussed by Sun and Lu [15].

Because of their great significance to SPDEs, it is necessary to give an intensive investigation to the theory of BDSDEs. In this paper, we study BDSDEP with non-Lipschitzian coefficients on random time interval. Here the coefficients are assumed to be weaker than linear growth, jointly continuous and to satisfy some weak “monotone” condition. BDSDEP can provide more extensive frameworks for the probabilistic interpretations (so-called nonlinear stochastic Feynman-Kac formula) for the solutions to a class of quasilinear stochastic partial differential-integral equations (SPDIEs). First, we establish the existence and uniqueness results for measurable solutions to BDSDEP based on the smoothing technique. Then we discuss the continuous dependence for solutions to BDSDEP. Finally, by virtue of BDSDEP, we show the probabilistic interpretation for the solutions to a class of quasilinear SPDIEs.

The paper is organized as follows. In Section 2, the basic assumptions are given. In Section 3, the existence and uniqueness of solutions to BDSDEP with non-Lipschitz coefficients on random time interval is proved. In Section 4, the continuous dependence for solutions to BDSDEP is discussed. Finally, in Section 5, the probabilistic interpretation for the solutions to a class of quasilinear SPDIEs is given by virtue of this class of BDSDEP.

2 Setting of the Problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $[0, T]$ be an arbitrarily large fixed time duration throughout this paper. We suppose that $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the following three mutually independent processes:

(i) Let $\{W_t; 0 \leq t \leq T\}$ and $\{B_t; 0 \leq t \leq T\}$ be two standard Brownian motions defined on $(\Omega, \mathcal{F}, P)$, with values respectively in $\mathbb{R}^d$ and in $\mathbb{R}^l$.

(ii) Let N be a Poisson random measure, on $\mathbb{R}_+ \times Z$, where $Z \subset \mathbb{R}^r$ is a nonempty open set equipped with its Borel field $\mathcal{B}(Z)$, with compensator $\widehat{N}(\mathrm{d}z\mathrm{d}t) = \lambda(\mathrm{d}z)\mathrm{d}t$, such that $\widetilde{N}(A \times [0, t]) = (N - \widehat{N})(A \times [0, t])_{t\geq 0}$ is a martingale for all $A \in \mathcal{B}(Z)$ satisfying $\lambda(A) < \infty$. λ is assumed to be a σ-finite measure on $(Z, \mathcal{B}(Z))$ and is called the characteristic measure.

Let $\mathcal{N}$ denote the class of P-null elements of $\mathcal{F}$. For each $t \in [0, T]$, we define $\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^N$, where for any process $\{\eta_t\}$, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. Note that the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a classical filtration.

Let $\tau = \{\tau(\omega)\}$ be an $\mathcal{F}_t$-measurable time on $[0, T]$, that is, $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$, $\forall t \in [0, T]$, and let $\mathcal{F}_\tau = \{A \in \mathcal{F}_T^W \vee \mathcal{F}_T^B \vee \mathcal{F}_T^N : A \cap \{\tau \le t\} \in \mathcal{F}_t, \forall t \in [0,T]\}$. We introduce the following notations:

$$\begin{aligned}
S^2([0,\tau];\mathbb{R}^n) = \Big\{ & v_t, 0 \le t \le \tau : v_t \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-measurable process} \\
& \text{such that } \mathbb{E}\Big(\sup_{0\le t\le\tau} |v_t|^2\Big) < \infty\Big\}, \\
M^2(0,\tau;\mathbb{R}^n) = \Big\{ & v_t, 0 \le t \le \tau : v_t \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-measurable process} \\
& \text{such that } \mathbb{E}\int_0^\tau |v_t|^2 \mathrm{d}t < \infty\Big\}, \\
F_N^2(0,\tau;\mathbb{R}^n) = \Big\{ & k_t, 0 \le t \le \tau : k_t \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-measurable process} \\
& \text{such that } \mathbb{E}\int_0^\tau \int_Z |k_t(z)|^2 \lambda(\mathrm{d}z)\mathrm{d}t < \infty\Big\}, \\
L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n) = \Big\{ & k(z) :\ k(z) \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{B}(Z)\text{-measurable function} \\
& \text{such that } \|k\| = \Big(\int_Z |k(z)|^2\lambda(\mathrm{d}z)\Big)^{\frac{1}{2}} < \infty\Big\}, \\
L^2(\Omega,\mathcal{F}_\tau,P;\mathbb{R}^n) = \{ & \xi :\ \xi \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{F}_\tau\text{-measurable random variable} \\
& \text{such that } \mathbb{E}|\xi|^2 < \infty\}.
\end{aligned}$$

Consider the following BDSDEP:

$$\begin{aligned}
P_t = \xi &+ \int_{t\wedge\tau}^{\tau} f(s,P_s,Q_s,K_s)\mathrm{d}s + \int_{t\wedge\tau}^{\tau} g(s,P_s,Q_s,K_s)\,\overleftarrow{\mathrm{d}B_s} \\
&- \int_{t\wedge\tau}^{\tau} Q_s \mathrm{d}W_s - \int_{t\wedge\tau}^{\tau}\int_Z K_s(z)\widetilde{N}(\mathrm{d}z\mathrm{d}s), \quad t \ge 0, \qquad (2.1)
\end{aligned}$$

where

$$\begin{aligned}
f &: \Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n) \to \mathbb{R}^n, \\
g &: \Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n) \to \mathbb{R}^{n\times l}.
\end{aligned}$$

We note that the integral with respect to $\{B_t\}$ is a "backward Itô integral" and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral (see [8]). We use the usual inner product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$ in $\mathbb{R}^n$, $\mathbb{R}^{n\times l}$ and $\mathbb{R}^{n\times d}$. All the equalities and inequalities mentioned in this paper are in the sense of $\mathrm{d}t\times\mathrm{d}P$ almost surely on $[0,\tau]\times\Omega$.

Definition 2.1 *A solution to* BDSDEP (2.1) *is a triple of $\mathcal{F}_t$-measurable stochastic processes (P,Q,K) which belongs to the space $S^2([0,\tau];\mathbb{R}^n)\times M^2(0,\tau;\mathbb{R}^{n\times d})\times F_N^2(0,\tau;\,\mathbb{R}^n)$ and satisfies* BDSDEP (2.1).

We assume that

(H1) $\xi \in L^2(\Omega,\mathcal{F}_\tau,P;\mathbb{R}^n)$;

(H2) $f(t,p,q,k)$, $g(t,p,q,k)$ are continuous in $(p,q,k)\in\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)$;

(H3) $f = f_1+f_2$, $f_i = f_i(t,p,q,k) : \Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)\to\mathbb{R}^n$, $i=1,2$, and $g(t,p,q,k)$ are $\mathcal{F}_t$-measurable processes, such that for all $t\in[0,T]$, $p\in\mathbb{R}^n$, $q\in\mathbb{R}^{n\times d}$,

$k \in L^2_{\lambda(\cdot)}(\mathbb{R}^n)$,

$$\begin{aligned}
&|f_1(t,p,q,k)| \le \mu(t),\\
&|f_2(t,p,q,k)| \le \mu(t)(1+|p|+|q|+\|k\|),\\
&|g(t,p,q,k)| \le \mu(t),
\end{aligned}$$

where $\mu(t) \ge 0$ is a real and non-random function such that $\overline{\mu} = \int_0^T \mu^2(t)\mathrm{d}t < \infty$;

(H4) for all $t \in [0,T]$, $p, p_1, p_2 \in \mathbb{R}^n$, $q, q_1, q_2 \in \mathbb{R}^{n\times d}$, $k, k_1, k_2 \in L^2_{\lambda(\cdot)}(\mathbb{R}^n)$, such that

$$\begin{aligned}
\langle p_1 - p_2, f_1(t,p_1,q_1,k_1) - f_1(t,p_2,q_2,k_2)\rangle &\le \mu(\rho(|p_1-p_2|^2) + |p_1-p_2|(|q_1-q_2| + \|k_1-k_2\|)),\\
|f_1(t,p,q,k_1) - f_1(t,p,q,k_2)| &\le \mu\|k_1-k_2\|,\\
|f_2(t,p_1,q_1,k_1) - f_2(t,p_2,q_2,k_2)| &\le \mu(|p_1-p_2| + |q_1-q_2| + \|k_1-k_2\|),\\
|g(t,p_1,q_1,k_1) - g(t,p_2,q_2,k_2)|^2 &\le \mu(|p_1-p_2|^2 + |p_1-p_2|(|q_1-q_2| + \|k_1-k_2\|)),
\end{aligned}$$

where $\mu > 0$ is a constant, and $\rho(\cdot)$ is a nondecreasing, continuous and concave function from R_+ to R_+ such that $\rho(0) = 0$, $\rho(u) > 0$, as $u > 0$, and $\int_{0^+} \frac{\mathrm{d}u}{\rho(u)} = +\infty$.

Remark 2.1 Here the coefficients are assumed to be weaker than linear growth, jointly continuous and to satisfy some weak "monotone" condition as follows:

(i) If $\mu(t) = \mu > 0$ is a constant, then (H3) implies that f is less than the linear growth condition.

(ii) To see the generality of our result, let us give a few examples of the function $\rho(\cdot)$. Let $K > 0$, and let $\delta \in (0,1)$ be sufficiently small. Define

$$\begin{aligned}
\rho_1(u) &= Ku, \quad u \ge 0,\\
\rho_2(u) &= \begin{cases} u\ln(u^{-1}), & 0 \le u \le \delta,\\ \delta\ln(\delta^{-1}) + \rho_2'(\delta-)(u-\delta), & u > \delta,\end{cases}\\
\rho_3(u) &= \begin{cases} u\ln(u^{-1})\ln\ln(u^{-1}), & 0 \le u \le \delta,\\ \delta\ln(\delta^{-1})\ln\ln(\delta^{-1}) + \rho_3'(\delta-)(u-\delta), & u > \delta.\end{cases}
\end{aligned}$$

They are all concave nondecreasing functions satisfying $\int_{0^+} \frac{\mathrm{d}u}{\rho_i(u)} = +\infty$. In particular, we see clearly that if $\rho(u) = Ku$, then the condition of f_1 in (H4) reduces to the monotone condition. In other words, the condition of f_1 in (H4) can be referred to as a weak "monotonicity condition".

(iii) Since ρ is concave and $\rho(0) = 0$, one can find a pair of positive constants a and b such that

$$\rho(u) \le a + bu \quad \text{for all } u \ge 0.$$

3 Existence and Uniqueness of Solutions to BDSDEP with Non-Lipschitz Coefficients

In order to prove the existence and uniqueness results of solutions to BDSDEP with non-Lipschitz coefficients on random time interval, we introduce the following lemmas and theorems.

Lemma 3.1 (A Priori Estimate) *Under the assumption* (H3). *If* (P_t, Q_t, K_t) *is a solution to* BDSDEP (2.1)*, then*

$$\mathbb{E}\Big(\sup_{0\le t\le\tau} |P_t|^2 + \int_0^\tau |Q_t|^2\mathrm{d}t + \int_0^\tau \|K_t\|^2\mathrm{d}t\Big) \le C_T < \infty,$$

where $C_T \geq 0$ is a constant depending on T, $\mu(t)$ and $\mathbb{E}|\xi|^2$ only.

Proof From (H3), we easily have

$$\langle p, f(t,p,q,k)\rangle \leq \mu(t)(1+2|p|^2+|p|(|q|+\|k\|)),$$

where $\mu(t)$ has the property stated in (H3). Applying Itô's formula to $|P_t|^2$, we have

$$\begin{aligned}
&\mathbb{E}\Big(|P_{t\wedge\tau}|^2+\int_{t\wedge\tau}^{\tau}|Q_s|^2\mathrm{d}s+\int_{t\wedge\tau}^{\tau}\|K_s\|^2\mathrm{d}s\Big)\\
&=\mathbb{E}|\xi|^2+2\mathbb{E}\int_{t\wedge\tau}^{\tau}\langle P_s, f(s,P_s,Q_s,K_s)\rangle\mathrm{d}s+\mathbb{E}\int_{t\wedge\tau}^{\tau}|g(s,P_s,Q_s,K_s)|^2\mathrm{d}s\\
&\leq \mathbb{E}|\xi|^2+2\mathbb{E}\int_{t\wedge\tau}^{\tau}\mu(s)(1+2|P_s|^2+|P_s|(|Q_s|+\|K_s\|))\mathrm{d}s+\overline{\mu}.
\end{aligned}$$

We deduce

$$\begin{aligned}
&\mathbb{E}\Big(|P_{t\wedge\tau}|^2+\frac{1}{2}\int_{t\wedge\tau}^{\tau}|Q_s|^2\mathrm{d}s+\frac{1}{2}\int_{t\wedge\tau}^{\tau}\|K_s\|^2\mathrm{d}s\Big)\\
&\leq \mathbb{E}|\xi|^2+T+2\overline{\mu}+\mathbb{E}\int_{t\wedge\tau}^{T}(4\mu(s)+2\mu^2(s))|P_s|^2\mathrm{d}s.
\end{aligned}$$

By Gronwall inequality, we have

$$\mathbb{E}\Big(|P_{t\wedge\tau}|^2+\frac{1}{2}\int_{t\wedge\tau}^{\tau}|Q_s|^2\mathrm{d}s+\frac{1}{2}\int_{t\wedge\tau}^{\tau}\|K_s\|^2\mathrm{d}s\Big)\leq \widetilde{C}_T,$$

where

$$\widetilde{C}_T=(\mathbb{E}|\xi|^2+T+2\overline{\mu})\exp\Big(\int_0^T(4\mu(s)+2\mu^2(s))\mathrm{d}s\Big).$$

In particular,

$$\mathbb{E}\Big(|P_0|^2+\frac{1}{2}\int_0^{\tau}|Q_s|^2\mathrm{d}s+\frac{1}{2}\int_0^{\tau}\|K_s\|^2\mathrm{d}s\Big)\leq \widetilde{C}_T.$$

Applying Itô's formula to $|P_t|^2$ on $[0,t\wedge\tau]$, we have

$$\begin{aligned}
|P_{t\wedge\tau}|^2=&|P_0|^2-2\int_0^{t\wedge\tau}\langle P_s, f(s,P_s,Q_s,K_s)\rangle\mathrm{d}s-2\int_0^{t\wedge\tau}\langle P_s, g(s,P_s,Q_s,K_s)\rangle\,\overleftarrow{\mathrm{d}B}_s\\
&+2\int_0^{t\wedge\tau}\langle P_s,Q_s\rangle\mathrm{d}W_s+2\int_0^{t\wedge\tau}\int_Z\langle P_s,K_s(z)\rangle\widetilde{N}(\mathrm{d}z\mathrm{d}s)\\
&-\int_0^{t\wedge\tau}|g(s,P_s,Q_s,K_s)|^2\mathrm{d}s+\int_0^{t\wedge\tau}|Q_s|^2\mathrm{d}s+\int_0^{t\wedge\tau}\|K_s\|^2\mathrm{d}s.
\end{aligned}$$

Taking supremum and expectation, we get

$$\begin{aligned}
\mathbb{E}\sup_{t\leq\tau}|P_{t\wedge\tau}|^2\leq&\,\mathbb{E}|P_0|^2+2\mathbb{E}\int_0^{\tau}\mu(s)(1+2|P_s|^2+|P_s|(|Q_s|+\|K_s\|))\mathrm{d}s+\int_0^T\mu^2(s)\mathrm{d}s\\
&+\mathbb{E}\int_0^{\tau}|Q_s|^2\mathrm{d}s+\mathbb{E}\int_0^{\tau}\|K_s\|^2\mathrm{d}s+2\mathbb{E}\sup_{t\leq\tau}\Big|\int_0^{t\wedge\tau}\langle P_s, g(s,P_s,Q_s,K_s)\rangle\,\overleftarrow{\mathrm{d}B}_s\Big|\\
&+2\mathbb{E}\sup_{t\leq\tau}\Big|\int_0^{t\wedge\tau}\langle P_s,Q_s\rangle\mathrm{d}W_s\Big|+2\mathbb{E}\sup_{t\leq\tau}\Big|\int_0^{t\wedge\tau}\int_Z\langle P_s,K_s(z)\rangle\widetilde{N}(\mathrm{d}z\mathrm{d}s)\Big|.
\end{aligned}$$

By Burkholder-Davis-Gundy's inequality, we deduce

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\leq\tau}\Big|\int_0^{t\wedge\tau}\langle P_s, g(s,P_s,Q_s,K_s)\rangle\, \overleftarrow{\mathrm{d}B_s}\,\Big|\Big)\\
&\leq c\mathbb{E}\Big(\int_0^{\tau}|P_{s\wedge\tau}|^2\cdot|g(s,P_s,Q_s,K_s)|^2\mathrm{d}s\Big)^{\frac{1}{2}}\\
&\leq c\mathbb{E}\Big(\Big(\sup_{t\leq\tau}|P_{t\wedge\tau}|^2\Big)^{\frac{1}{2}}\Big(\int_0^{\tau}|g(s,P_s,Q_s,K_s)|^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big)\\
&\leq \frac{1}{8}\mathbb{E}\sup_{t\leq\tau}|P_{t\wedge\tau}|^2+2c^2\int_0^T\mu^2(s)\mathrm{d}s.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\leq\tau}\Big|\int_0^{t\wedge\tau}\langle P_s,Q_s\rangle\mathrm{d}W_s\Big|\leq\frac{1}{8}\mathbb{E}\sup_{t\leq\tau}|P_{t\wedge\tau}|^2+2c^2\mathbb{E}\int_0^{\tau}|Q_s|^2\mathrm{d}s,\\
&\mathbb{E}\sup_{t\leq\tau}\Big|\int_0^{t\wedge\tau}\int_Z\langle P_s,K_s(z)\rangle\widetilde{N}(\mathrm{d}z\mathrm{d}s)\Big|\leq\frac{1}{8}\mathbb{E}\sup_{t\leq\tau}|P_{t\wedge\tau}|^2+2c^2\mathbb{E}\int_0^{\tau}\|K_s\|^2\mathrm{d}s.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}\sup_{t\leq\tau}|P_{t\wedge\tau}|^2&\leq 4\mathbb{E}|P_0|^2+4T+8\overline{\mu}+4\mathbb{E}\int_0^{\tau}(4\mu(s)+2\mu^2(s))|P_{s\wedge\tau}|^2\mathrm{d}s\\
&\quad+4(1+2c^2)\mathbb{E}\int_0^{\tau}(|Q_s|^2+\|K_s\|^2)\mathrm{d}s\leq C_T<\infty.
\end{aligned}$$

The proof of Lemma 3.1 is completed.

As a preparation for the study of BDSDEP (2.1), we first discuss a simpler BDSDEP as follows:

$$\begin{aligned}
P_t&=\xi+\int_{t\wedge\tau}^{\tau}f(s)\mathrm{d}s+\int_{t\wedge\tau}^{\tau}g(s)\,\overleftarrow{\mathrm{d}B_s}-\int_{t\wedge\tau}^{\tau}Q_s\mathrm{d}W_s\\
&\quad-\int_{t\wedge\tau}^{\tau}\int_Z K_s(z)\widetilde{N}(\mathrm{d}z\mathrm{d}s),\quad t\geq 0.
\end{aligned}\tag{3.1}$$

Lemma 3.2 *Given* $\xi\in L^2(\Omega,\mathcal{F}_\tau,P;\mathbb{R}^n)$, $f(t)\in M^2(0,\tau;\mathbb{R}^n)$ *and* $g(t)\in M^2(0,\tau;\,\mathbb{R}^{n\times l})$, *then* BDSDEP (3.1) *has a unique solution in* $S^2([0,\tau];\mathbb{R}^n)\times M^2(0,\tau;\mathbb{R}^{n\times d})\times F_N^2(0,\tau;\mathbb{R}^n)$.

Proof Uniqueness Let (P^1,Q^1) and (P^2,Q^2) be two solutions to (3.1). Applying Itô's formula to $|P_t^1-P_t^2|^2$, we have

$$\mathbb{E}|P_{t\wedge\tau}^1-P_{t\wedge\tau}^2|^2+\mathbb{E}\int_{t\wedge\tau}^{\tau}|Q_s^1-Q_s^2|^2\mathrm{d}s+\mathbb{E}\int_{t\wedge\tau}^{\tau}\|K_s^1-K_s^2\|^2\mathrm{d}s=0.$$

Then

$$\mathbb{E}|P_{t\wedge\tau}^1-P_{t\wedge\tau}^2|^2=0,\quad\mathbb{E}\int_{t\wedge\tau}^{\tau}|Q_s^1-Q_s^2|^2\mathrm{d}s=0$$

and

$$\mathbb{E}\int_{t\wedge\tau}^{\tau}\|K_s^1-K_s^2\|^2\mathrm{d}s=0,\quad 0\leq t\leq T.$$

Hence $P_t^1 = P_t^2$, $Q_t^1 = Q_t^2$ and $K_t^1 = K_t^2$ a.s. The uniqueness is obtained.

Existence We define the filtration $(\mathcal{G}_t)_{0\leq t\leq T}$ by

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B \vee \mathcal{F}_t^N$$

and the $\mathcal{G}_t$-square integrable martingale

$$M_t = \mathbb{E}\Big[\Big(\xi + \int_0^\tau f(s)\mathrm{d}s + \int_0^\tau g(s)\,\overleftarrow{\mathrm{d}B_s}\,\Big)\Big|\mathcal{G}_t\Big].$$

An obvious extension of Itô's martingale representation theorem (see [6]) yields the existence of $\mathcal{G}_t$-progressively measurable process (Q_t, K_t) such that

$$M_t = M_0 + \int_0^t Q_s\mathrm{d}W_s + \int_0^t\int_Z K_s(z)\widetilde{N}(\mathrm{d}z\mathrm{d}s)$$

and

$$\mathbb{E}\int_0^T (|Q_s|^2 + \|K_s\|^2)\mathrm{d}s < \infty.$$

Hence

$$M_\tau = M_{t\wedge\tau} + \int_{t\wedge\tau}^\tau Q_s\mathrm{d}W_s + \int_{t\wedge\tau}^\tau\int_Z K_s(z)\widetilde{N}(\mathrm{d}z\mathrm{d}s).$$

Replacing M_τ and $M_{t\wedge\tau}$ by their defining formulas and subtracting $\int_0^{t\wedge\tau} f(s)\mathrm{d}s+\int_0^{t\wedge\tau} g(s)\,\overleftarrow{\mathrm{d}B_s}$ from both sides of the equality yields

$$P_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^\tau f(s)\mathrm{d}s + \int_{t\wedge\tau}^\tau g(s)\,\overleftarrow{\mathrm{d}B_s} - \int_{t\wedge\tau}^\tau Q_s\mathrm{d}W_s - \int_{t\wedge\tau}^\tau\int_Z K_s(z)\widetilde{N}(\mathrm{d}z\mathrm{d}s),$$

where

$$P_{t\wedge\tau} = \mathbb{E}\Big[\Big(\xi + \int_{t\wedge\tau}^\tau f(s)\mathrm{d}s + \int_{t\wedge\tau}^\tau g(s)\,\overleftarrow{\mathrm{d}B_s}\,\Big)\Big|\mathcal{G}_{t\wedge\tau}\Big].$$

This implies that (P_t, Q_t, K_t) solves (3.1).

By the similar arguments as Proposition 1.2 in [8], we show that $\{P_t\}$, $\{Q_t\}$ and $\{K_t\}$ are $\mathcal{F}_t$-measurable. For P_t, this is obvious since for each t,

$$\begin{aligned} P_{t\wedge\tau} &= \mathbb{E}\Big[\Big(\xi + \int_{t\wedge\tau}^\tau f(s)\mathrm{d}s + \int_{t\wedge\tau}^\tau g(s)\,\overleftarrow{\mathrm{d}B_s}\,\Big)\Big|\mathcal{G}_{t\wedge\tau}\Big] \\ &= \mathbb{E}(\Theta|\mathcal{F}_{t\wedge\tau} \vee \mathcal{F}_{t\wedge\tau}^B), \end{aligned}$$

where $\Theta = \xi + \int_{t\wedge\tau}^\tau f(s)\mathrm{d}s + \int_{t\wedge\tau}^\tau g(s)\,\overleftarrow{\mathrm{d}B_s}$ is $\mathcal{F}_\tau^W \vee \mathcal{F}_{t\wedge\tau,T}^B \vee \mathcal{F}_\tau^N$-measurable. Hence, $\mathcal{F}_{t\wedge\tau}^B$ is independent of $\mathcal{F}_{t\wedge\tau} \vee \sigma(\Theta)$, and

$$P_{t\wedge\tau} = \mathbb{E}(\Theta|\mathcal{F}_{t\wedge\tau}).$$

Now

$$\int_{t\wedge\tau}^\tau Q_s\mathrm{d}W_s + \int_{t\wedge\tau}^\tau\int_Z K_s(z)\widetilde{N}(\mathrm{d}z\mathrm{d}s) = \xi + \int_{t\wedge\tau}^\tau f(s)\mathrm{d}s + \int_{t\wedge\tau}^\tau g(s)\,\overleftarrow{\mathrm{d}B_s} - P_{t\wedge\tau},$$

and the right-hand side is $\mathcal{F}_\tau^W \vee \mathcal{F}_{t\wedge\tau,T}^B \vee \mathcal{F}_\tau^N$-measurable. Hence from Itô's martingale representation theorem, $\{(Q_s, K_s) : t\wedge\tau < s < \tau\}$ is $\mathcal{F}_s^W \vee \mathcal{F}_{t\wedge\tau,T}^B \vee \mathcal{F}_s^N$-adapted. Consequently, (Q_s, K_s) is $\mathcal{F}_s^W \vee \mathcal{F}_{t\wedge\tau,T}^B \vee \mathcal{F}_s^N$-measurable for any $t\wedge\tau < s < T$, so it is $\mathcal{F}_s^W \vee \mathcal{F}_{s,T}^B \vee \mathcal{F}_s^N$-measurable, that is, (Q_t, K_t) is $\mathcal{F}_t$-measurable. The proof of Lemma 3.2 is completed.

In the following of this section, we derive the existence and uniqueness results for solutions to BDSDEP on random time interval with Lipschitzian and non-Lipschitzian coefficients. The first one, that is Theorem 3.1, deals with the case where f is Lipschitz continuous.

Theorem 3.1 *Under the assumptions* (H1)– (H4)*, if* $f_1 = 0$*,* BDSDEP (2.1) *has a unique solution* (P_t, Q_t, K_t) *in* $S^2([0,\tau];\mathbb{R}^n) \times M^2(0,\tau;\, \mathbb{R}^{n\times d}) \times F_N^2(0,\tau;\mathbb{R}^n)$.

Proof We define recursively a sequence $\{(P_t^i, Q_t^i, K_t^i)\}_{i=0,1,\cdots}$ as follows. Let $P_t^0 = 0$, $Q_t^0 = 0$, $K_t^0 = 0$. By Lemma 3.2, for any $(P_t^i, Q_t^i, K_t^i) \in S^2([0,\tau];\mathbb{R}^n) \times M^2(0,\tau;\mathbb{R}^{n\times d}) \times F_N^2(0,\tau;\mathbb{R}^n)$, $i = 0, 1, \cdots$, there exists a unique $(P_t^{i+1}, Q_t^{i+1}, K_t^{i+1})$, satisfying

$$\begin{aligned} P_t^{i+1} &= \xi + \int_{t\wedge\tau}^{\tau} f(s, P_s^i, Q_s^i, K_s^i)\mathrm{d}s + \int_{t\wedge\tau}^{\tau} g(s, P_s^i, Q_s^i, K_s^i)\, \overleftarrow{\mathrm{d}B}_s \\ &\quad - \int_{t\wedge\tau}^{\tau} Q_s^{i+1}\mathrm{d}W_s - \int_{t\wedge\tau}^{\tau}\int_Z K_s^{i+1}(z)\widetilde{N}(\mathrm{d}z\mathrm{d}s), \quad t \geq 0. \end{aligned}$$

Moreover, by Lemma 3.2, $(P_t^{i+1}, Q_t^{i+1}, K_t^{i+1}) \in S^2([0,\tau];\mathbb{R}^n) \times M^2(0,\tau;\mathbb{R}^{n\times d}) \times F_N^2(0,\tau;\, \mathbb{R}^n)$. Applying the Itô's formula to $|P_t^{i+1} - P_t^i|^2\mathrm{e}^{-\beta t}$, we have

$$\begin{aligned} &\mathbb{E}|P_{t\wedge\tau}^{i+1} - P_{t\wedge\tau}^i|^2\mathrm{e}^{-\beta t} + \beta\mathbb{E}\int_{t\wedge\tau}^{\tau} |P_s^{i+1} - P_s^i|^2\mathrm{e}^{-\beta s}\mathrm{d}s \\ &\quad + \mathbb{E}\int_{t\wedge\tau}^{\tau} |Q_s^{i+1} - Q_s^i|^2\mathrm{e}^{-\beta s}\mathrm{d}s + \mathbb{E}\int_{t\wedge\tau}^{\tau} \|K_s^{i+1} - K_s^i\|^2\mathrm{e}^{-\beta s}\mathrm{d}s \\ &= 2\mathbb{E}\int_{t\wedge\tau}^{\tau} \langle P_s^{i+1} - P_s^i, f(s, P_s^i, Q_s^i, K_s^i) - f(s, P_s^{i-1}, Q_s^{i-1}, K_s^{i-1})\rangle\mathrm{e}^{-\beta s}\mathrm{d}s \\ &\quad + \mathbb{E}\int_{t\wedge\tau}^{\tau} |g(s, P_s^i, Q_s^i, K_s^i) - g(s, P_s^{i-1}, Q_s^{i-1}, K_s^{i-1})|^2\mathrm{e}^{-\beta s}\mathrm{d}s \\ &\leq 2\mu\mathbb{E}\int_{t\wedge\tau}^{\tau} |P_s^{i+1} - P_s^i|(|P_s^i - P_s^{i-1}| + |Q_s^i - Q_s^{i-1}| + \|K_s^i - K_s^{i-1}\|)\mathrm{e}^{-\beta s}\mathrm{d}s \\ &\quad + \mu\mathbb{E}\int_{t\wedge\tau}^{\tau} (|P_s^i - P_s^{i-1}|^2 + |P_s^i - P_s^{i-1}|(|Q_s^i - Q_s^{i-1}| + \|K_s^i - K_s^{i-1}\|))\mathrm{e}^{-\beta s}\mathrm{d}s \\ &\leq \frac{1}{4}\mathbb{E}\int_{t\wedge\tau}^{\tau} (|P_s^i - P_s^{i-1}|^2 + |Q_s^i - Q_s^{i-1}|^2 + \|K_s^i - K_s^{i-1}\|^2)\mathrm{e}^{-\beta s}\mathrm{d}s \\ &\quad + 12\mu\mathbb{E}\int_{t\wedge\tau}^{\tau} |P_s^{i+1} - P_s^i|^2\mathrm{e}^{-\beta s}\mathrm{d}s + 3\mu\mathbb{E}\int_{t\wedge\tau}^{\tau} |P_s^i - P_s^{i-1}|^2\mathrm{e}^{-\beta s}\mathrm{d}s \\ &\quad + \frac{1}{4}\mathbb{E}\int_{t\wedge\tau}^{\tau} (|Q_s^i - Q_s^{i-1}|^2 + \|K_s^i - K_s^{i-1}\|^2)\mathrm{e}^{-\beta s}\mathrm{d}s, \end{aligned}$$

we deduce

$$\begin{aligned} &\mathbb{E}|P_{t\wedge\tau}^{i+1} - P_{t\wedge\tau}^i|^2\mathrm{e}^{-\beta t} + (\beta - 12\mu)\mathbb{E}\int_{t\wedge\tau}^{\tau} |P_s^{i+1} - P_s^i|^2\mathrm{e}^{-\beta s}\mathrm{d}s \\ &+ \mathbb{E}\int_{t\wedge\tau}^{\tau} |Q_s^{i+1} - Q_s^i|^2\mathrm{e}^{-\beta s}\mathrm{d}s + \mathbb{E}\int_{t\wedge\tau}^{\tau} \|K_s^{i+1} - K_s^i\|^2\mathrm{e}^{-\beta s}\mathrm{d}s \end{aligned}$$

$$\leq \frac{1}{2}\mathbb{E}\int_{t\wedge\tau}^{\tau}(|Q_s^i - Q_s^{i-1}|^2 + \|K_s^i - K_s^{i-1}\|^2)\mathrm{e}^{-\beta s}\mathrm{d}s$$
$$+\Big(\frac{1}{4}+3\mu\Big)\mathbb{E}\int_{t\wedge\tau}^{\tau}|P_s^i - P_s^{i-1}|^2\mathrm{e}^{-\beta s}\mathrm{d}s.$$

Now choose $\beta = 12\mu + \frac{1+12\mu}{2}$, and define $\overline{c} = \frac{1+12\mu}{2}$.

$$\mathbb{E}|P_{t\wedge\tau}^{i+1} - P_{t\wedge\tau}^{i}|^2\mathrm{e}^{-\beta t} + \mathbb{E}\int_{t\wedge\tau}^{\tau}(\overline{c}|P_s^{i+1} - P_s^i|^2 + |Q_s^{i+1} - Q_s^i|^2 + \|K_s^{i+1} - K_s^i\|^2)\mathrm{e}^{-\beta s}\mathrm{d}s$$
$$\leq \frac{1}{2}\mathbb{E}\int_{t\wedge\tau}^{\tau}(\overline{c}|P_s^i - P_s^{i-1}|^2 + |Q_s^i - Q_s^{i-1}|^2 + \|K_s^i - K_s^{i-1}\|^2)\mathrm{e}^{-\beta s}\mathrm{d}s.$$

It follows immediately that

$$\mathbb{E}\int_{t\wedge\tau}^{\tau}(\overline{c}|P_s^{i+1} - P_s^i|^2 + |Q_s^{i+1} - Q_s^i|^2 + \|K_s^{i+1} - K_s^i\|^2)\mathrm{e}^{-\beta s}\mathrm{d}s$$
$$\leq \frac{1}{2}\mathbb{E}\int_{t\wedge\tau}^{\tau}(\overline{c}|P_s^i - P_s^{i-1}|^2 + |Q_s^i - Q_s^{i-1}|^2 + \|K_s^i - K_s^{i-1}\|^2)\mathrm{e}^{-\beta s}\mathrm{d}s,$$

and $\{(P_t^i, Q_t^i, K_t^i)\}_{i=0,1,\cdots}$ is a Cauchy sequence in $S^2([0,\tau];\mathbb{R}^n)\times M^2(0,\tau;\mathbb{R}^{n\times d})\times F_N^2(0,\tau;\mathbb{R}^n)$, and that

$$\{(P_t, Q_t, K_t)\} = \lim_{i\to\infty}\{(P_t^i, Q_t^i, K_t^i)\}$$

solves (2.1). The proof of Theorem 3.1 is completed.

The next theorem is the main result of this section, which generalizes the result of Theorem 3.1 to the case where f is continuous but not Lipschitz continuous.

Theorem 3.2 *Under the assumptions* (H1)–(H4), BDSDEP (2.1) *has a unique solution* (P_t, Q_t, K_t).

Proof Uniqueness Let (P_s^1, Q_s^1, K_s^1) and (P_s^2, Q_s^2, K_s^2) be two solutions to (2.1). Applying Itô's formula to $|P_s^1 - P_s^2|^2$, we obtain

$$\mathbb{E}\Big(|P_{t\wedge\tau}^1 - P_{t\wedge\tau}^2|^2 + \int_{t\wedge\tau}^{\tau}|Q_s^1 - Q_s^2|^2\mathrm{d}s + \int_{t\wedge\tau}^{\tau}\|K_s^1 - K_s^2\|^2\mathrm{d}s\Big)$$
$$= 2\mathbb{E}\int_{t\wedge\tau}^{\tau}\langle P_s^1 - P_s^2, f(s, P_s^1, Q_s^1, K_s^1) - f(s, P_s^2, Q_s^2, K_s^2)\rangle\mathrm{d}s$$
$$+\mathbb{E}\int_{t\wedge\tau}^{\tau}|g(s, P_s^1, Q_s^1, K_s^1) - g(s, P_s^2, Q_s^2, K_s^2)|^2\mathrm{d}s$$
$$\leq 2\mu\mathbb{E}\int_{t\wedge\tau}^{\tau}(\rho(|P_s^1 - P_s^2|^2) + |P_s^1 - P_s^2|(|Q_s^1 - Q_s^2| + \|K_s^1 - K_s^2\|))\mathrm{d}s$$
$$+\mu\mathbb{E}\int_{t\wedge\tau}^{\tau}(|P_s^1 - P_s^2|^2 + |P_s^1 - P_s^2|(|Q_s^1 - Q_s^2| + \|K_s^1 - K_s^2\|))\mathrm{d}s.$$

From (H4), we have

$$X_t = \mathbb{E}\Big(|P_{t\wedge\tau}^1 - P_{t\wedge\tau}^2|^2 + \frac{1}{2}\int_{t\wedge\tau}^{\tau}|Q_s^1 - Q_s^2|^2\mathrm{d}s + \frac{1}{2}\int_{t\wedge\tau}^{\tau}\|K_s^1 - K_s^2\|^2\mathrm{d}s\Big)$$
$$\leq \mu\mathbb{E}\int_{t\wedge\tau}^{\tau}(2\rho(|P_s^1 - P_s^2|^2) + 11|P_s^1 - P_s^2|^2)\mathrm{d}s$$
$$\leq 11\mu\int_t^T \rho_1(X_s)\mathrm{d}s,$$

where $\rho_1(u) = 2\rho(u) + 16u$.

By the Bahari's inequality, we obtain

$$\mathbb{E}\Big(|P^1_{t\wedge\tau} - P^2_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^{\tau} |Q^1_s - Q^2_s|^2 \mathrm{d}s + \int_{t\wedge\tau}^{\tau} \|K^1_s - K^2_s\|^2 \mathrm{d}s\Big) = 0$$

for all $t \in [0, T]$.

It implies that for all $t \in [0, T]$,

$$\mathbb{E}|P^1_{t\wedge\tau} - P^2_{t\wedge\tau}|^2 = 0, \quad \mathbb{E}\int_0^{\tau} |Q^1_s - Q^2_s|^2 \mathrm{d}s = 0, \quad \mathbb{E}\int_0^{\tau} \|K^1_s - K^2_s\|^2 \mathrm{d}s = 0.$$

Existence For simplicity, we assume that $f_2 = 0$. (In the case $f_2 \neq 0$, we can just smooth out f_1 and proceed as follows.) Let us smooth out f to get f^m, i.e., let

$$f^m(t, P, Q, K) = \int_{\mathbb{R}^{n+n\times l}} f(t, P - m^{-1}\overline{P}, Q - m^{-1}\overline{Q}, K) J(\overline{P}, \overline{Q}) \mathrm{d}\overline{P} \mathrm{d}\overline{Q},$$

where $J(P, Q) = J_1(P)J_2(Q)$ and $J_1(P)$ is defined for all $P \in \mathbb{R}^n$,

$$J_1(P) = \begin{cases} c_0 \exp(-(1 - |P|^2)^{-1}), & \text{as } |P| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

such that the constant c_0 satisfies $\int_{\mathbb{R}^n} J_1(x)\mathrm{d}x = 1$. $J_2(Q)$ is similarly defined for any $Q \in \mathbb{R}^{n\times d}$. It is easy to check that

$$|f^m(t, , P_1, Q_1, K_1) - f^m(t, P_2, Q_2, K_2)| \leq C_m \mu(|P_1 - P_2| + |Q_1 - Q_2| + \|K_1 - K_2\|),$$

as $(P_i, Q_i, K_i) \in \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)$, $i = 1, 2$. Hence by Theorem 3.1, for each $m = 1, 2, \cdots$, there exists a unique solution (P^m_t, Q^m_t, K^m_t) to solve the following BDSDEP:

$$\begin{aligned} P^m_{t\wedge\tau} = \xi &+ \int_{t\wedge\tau}^{\tau} f^m(s, P^m_s, Q^m_s, K^m_s)\mathrm{d}s + \int_{t\wedge\tau}^{\tau} g(s, P^m_s, Q^m_s, K^m_s)\, \overleftarrow{\mathrm{d}B}_s \\ &- \int_{t\wedge\tau}^{\tau} Q^m_s \mathrm{d}W_s - \int_{t\wedge\tau}^{\tau}\int_Z K^m_s(z)\widetilde{N}(\mathrm{d}z\mathrm{d}s). \end{aligned} \tag{3.2}$$

Applying Itô's formula to $|P^m_t - P^j_t|^2$, we have

$$\begin{aligned} &|P^m_{t\wedge\tau} - P^j_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^{\tau} |Q^m_s - Q^j_s|^2 \mathrm{d}s + \int_{t\wedge\tau}^{\tau} \|K^m_s - K^j_s\|^2 \mathrm{d}s \\ =\,& 2\int_{t\wedge\tau}^{\tau} \langle P^m_s - P^j_s, f^m(s, P^m_s, Q^m_s, K^m_s) - f^j(s, P^j_s, Q^j_s, K^j_s)\rangle \mathrm{d}s \\ &+ \int_{t\wedge\tau}^{\tau} |g(s, P^m_s, Q^m_s, K^m_s) - g(s, P^j_s, Q^j_s, K^j_s)|^2 \mathrm{d}s - 2\int_{t\wedge\tau}^{\tau} \langle P^m_s - P^j_s, Q^m_s - Q^j_s\rangle \mathrm{d}W_s \\ &+ 2\int_{t\wedge\tau}^{\tau} \langle P^m_s - P^j_s, g(s, P^m_s, Q^m_s, K^m_s) - g(s, P^j_s, Q^j_s, K^j_s)\rangle\, \overleftarrow{\mathrm{d}B}_s \\ &- 2\int_{t\wedge\tau}^{\tau}\int_Z \langle P^m_s - P^j_s, K^m_s - K^j_s\rangle \widetilde{N}(\mathrm{d}z\mathrm{d}s) \\ =\,& \sum_{i=1}^{5} \mathrm{I}_i. \end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{I}_1 &= 2\int_{t\wedge\tau}^{\tau}\Big\langle P_s^m - P_s^j, \int_{\mathbb{R}^{n+n\times l}}(f(s, P_s^m - m^{-1}\overline{P}, Q_s^m - m^{-1}\overline{Q}, K_s^m) \\
&\quad - f(s, P_s^j - j^{-1}\overline{P}, Q_s^j - j^{-1}\overline{Q}, K_s^j))J(\overline{P},\overline{Q})\mathrm{d}\overline{P}\mathrm{d}\overline{Q}\Big\rangle \mathrm{d}s \\
&\leq \mu\int_{t\wedge\tau}^{\tau}\int_{\mathbb{R}^{n+n\times l}}((\rho(|P_s^m - P_s^j - (m^{-1} - j^{-1})\overline{P}|^2) \\
&\quad + |P_s^m - P_s^j - (m^{-1} - j^{-1})\overline{P}| \times (|Q_s^m - Q_s^j - (m^{-1} - j^{-1})\overline{Q}| \\
&\quad + \|K_s^m - K_s^j\|))\mu(s) + |m^{-1} - j^{-1}||\overline{P}|2\mu(s))J(\overline{P},\overline{Q})\mathrm{d}\overline{P}\mathrm{d}\overline{Q}\mathrm{d}s.
\end{aligned}$$

Since by Lemma 3.1 for all m,

$$\mathbb{E}\Big(\sup_{t\leq\tau}|P_t^m|^2 + \int_0^{\tau}|Q_t^m|^2\mathrm{d}t + \int_0^{\tau}\|K_t^m\|^2\mathrm{d}t\Big) \leq C_T < \infty.$$

Hence

$$\begin{aligned}
&\mathbb{E}\Big(|P_{t\wedge\tau}^m - P_{t\wedge\tau}^j|^2 + \int_{t\wedge\tau}^{\tau}|Q_s^m - Q_s^j|^2\mathrm{d}s + \int_{t\wedge\tau}^{\tau}\|K_s^m - K_s^j\|^2\mathrm{d}s\Big) \\
&\leq \overline{C}_T(\mu^2+\mu)\int_t^T\int_{\mathbb{R}^{n+n\times l}}(\rho(\mathbb{E}|P_{s\wedge\tau}^m - P_{s\wedge\tau}^j - (m^{-1} - j^{-1})\overline{P}|^2) \\
&\quad + \mathbb{E}|P_{s\wedge\tau}^m - P_{s\wedge\tau}^j|^2)J(\overline{P},\overline{Q})\mathrm{d}\overline{P}\mathrm{d}\overline{Q}\mathrm{d}s + \overline{C}_T(m^{-1} + j^{-1}).
\end{aligned}$$

Note that

$$\rho(2\mathbb{E}|P_{s\wedge\tau}^m - P_{s\wedge\tau}^j|^2 + 2(m^{-1} - j^{-1})^2|\overline{P}|^2) \leq \rho(4C_T + 2|\overline{P}|^2).$$

But by the assumption, it yields that

$$\int\rho(4C_T + 2|\overline{P}|^2)J(\overline{P},\overline{Q})\mathrm{d}\overline{P}\mathrm{d}\overline{Q} \leq \rho(4C_T + 2) < \infty.$$

Hence by Lemma 3.1 and by the Fatou lemma, it is easy to see that

$$\begin{aligned}
&\limsup_{m,j\to\infty}\mathbb{E}|P_{t\wedge\tau}^m - P_{t\wedge\tau}^j|^2 + \limsup_{m,j\to\infty}\mathbb{E}\int_{t\wedge\tau}^{\tau}(|Q_s^m - Q_s^j|^2 + \|K_s^m - K_s^j\|^2)\mathrm{d}s \\
&\leq \widehat{C}_T(\mu^2+\mu)\int_t^T\rho_1\Big(\limsup_{m,j\to\infty}2\mathbb{E}|P_{s\wedge\tau}^m - P_{s\wedge\tau}^j|^2\Big)\mathrm{d}s,
\end{aligned}$$

where $\rho_1(u) = \rho(u) + u$. By the Bahari's inequality, we obtain

$$\limsup_{m,j\to\infty}\mathbb{E}|P_{t\wedge\tau}^m - P_{t\wedge\tau}^j|^2 = 0 \quad \text{for all } t\in[0,T]$$

and

$$\limsup_{m,j\to\infty}\mathbb{E}\int_0^{\tau}(|Q_s^m - Q_s^j|^2 + \|K_s^m - K_s^j\|^2)\mathrm{d}s = 0.$$

These, together with the Burkholder-Davis-Gundy's inequality, yield

$$\lim_{m,j\to\infty}\mathbb{E}\sup_{0\leq t\leq\tau}|P_t^m - P_t^j|^2 = 0.$$

By the completeness of Banach space, we know that there exists a unique $(P,Q,K)\in S^2([0,\tau];\mathbb{R}^n)\times M^2(0,\tau;\mathbb{R}^{n\times d})\times F_N^2(0,\tau;\mathbb{R}^n)$, such that as $m\to\infty$,

$$\mathbb{E}\sup_{0\le t\le\tau}|P_t^m-P_t|^2\to 0,\quad \mathbb{E}\int_0^\tau|Q_s^m-Q_s|^2\mathrm{d}s\to 0,\quad \mathbb{E}\int_0^\tau\|K_s^m-K_s\|^2\mathrm{d}s\to 0.$$

Therefore, we can take a subsequence $\{m_k\}$ of $\{m\}$, denote it by $\{m\}$ again such that almost surely for $(t,\omega)\in[0,T]\times\Omega$,

$$(P_t^m,Q_t^m,K_t^m)\to(P_t,Q_t,K_t),\quad \text{in } \mathbb{R}^n\times\mathbb{R}^{n\times l}\times L_{\lambda(\,\cdot\,)}(\mathbb{R}^n).$$

Hence, by the continuity of f in (P,Q,K), (H3), Lemma 3.1 and the Lebesgue domination convergence theorem, we have that

$$\mathbb{E}\int_0^\tau|f^m(s,P_s^m,Q_s^m,K_s^m)-f(s,P_s,Q_s,K_s)|\mathrm{d}s\to 0,\quad m\to\infty.$$

It is easy to check that (P,Q,K) is a solution to (2.1) by taking the limit on both sides of (3.2). The proof of Theorem 3.2 is completed.

4 Continuous Dependence for Solutions of BDSDEP

In this section, we discuss the continuous dependence for solutions to BDSDEP (2.1). By the similar method in the proof of Theorems 3.1 and 3.2, we have the following theorem.

Theorem 4.1 *For $m=0,1,2,\cdots$, we have*

(i) $f^m=f^m(t,p,q,k):[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)\to\mathbb{R}^n$ *are $\mathcal{F}_t$-measurable, such that P-a.s.*

$$\langle p,f^m(t,p,q,k)\rangle\le\mu(t)(1+|p|^2+|p|(|q|+\|k\|)),$$

where $\mu(t)$ has the property stated in (H3);

(ii) *for all $p_1,p_2\in\mathbb{R}^n$, $q_1,q_2\in\mathbb{R}^{n\times d}$, $k_1,k_2\in L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)$, such that P-a.s.*

$$\begin{aligned}&\langle p_1-p_2,f^0(t,p_1,q_1,k_1)-f^0(t,p_2,q_2,k_2)\rangle\\ &\le\mu(t)(\rho(|p_1-p_2|^2)+|p_1-p_2|(|q_1-q_2|+\|k_1-k_2\|)),\end{aligned}$$

where $\rho(\,\cdot\,)$ has the property stated in (H4);

(iii) $g^m=g^m(t,p,q,k):[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)\to\mathbb{R}^{n\times l}$ *are $\mathcal{F}_t$-measurable, such that P-a.s.*

$$\begin{aligned}&|g^m(t,p,q,k)|\le\mu(t),\\ &|g^m(t,p_1,q_1,k_1)-g^m(t,p_2,q_2,k_2)|^2\le\mu(t)(|p_1-p_2|^2+|p_1-p_2|(|q_1-q_2|+\|k_1-k_2\|)),\end{aligned}$$

where $\mu(t)$ has the property stated in (H3);

(iv) $\lim\limits_{m\to\infty}\sup\limits_{\substack{p\in\mathbb{R}^n\\ q\in\mathbb{R}^{n\times d}\\ k\in L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)}}\int_0^T|f^m(t,p,q,k)-f^0(t,p,q,k)|^2\mathrm{d}t=0,$

$$\lim_{m\to\infty} \sup_{\substack{p\in\mathbb{R}^n \\ q\in\mathbb{R}^{n\times d} \\ k\in L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)}} \int_0^T |g^m(t,p,q,k)-g^0(t,p,q,k)|^2\mathrm{d}t = 0;$$

(v) ξ^m *is* $\mathcal{F}_\tau$*-measurable and*

$$\mathbb{E}|\xi^m-\xi^0|^2 \to 0, \quad \textit{as } m\to\infty,\ \mathbb{E}|\xi^m|^2<\infty.$$

If (P_t^m, Q_t^m, K_t^m) *are solutions to the following* BDSDEP: *as* $0\le s\le T$*,*

$$P^m_{s\wedge\tau} = \xi^m + \int_{s\wedge\tau}^{T\wedge\tau} f^m(r,P^m_r,Q^m_r,K^m_r)\mathrm{d}r + \int_{s\wedge\tau}^{T\wedge\tau} g^m(r,P^m_r,Q^m_r,K^m_r)\,\mathrm{d}\overleftarrow{B_r}$$
$$- \int_{s\wedge\tau}^{T\wedge\tau} Q^m_r\mathrm{d}W_r - \int_{s\wedge\tau}^{T\wedge\tau}\int_Z K^m_r(z)\widetilde{N}(\mathrm{d}z\mathrm{d}r), \quad m=0,1,2,\cdots,$$

then for all $0\le s\le T$*,*

$$\lim_{m\to\infty}\mathbb{E}\Big(\sup_{s\le r\le T}|P^m_{r\wedge\tau}-P^0_{r\wedge\tau}|^2 + \int_{s\wedge\tau}^{T\wedge\tau}(|Q^m_r-Q^0_r|^2+\|K^m_r-K^0_r\|^2)\mathrm{d}r\Big)=0.$$

We also have other useful continuous dependence for solutions to BDSDEP as follows.

Theorem 4.2 *For* $m=0,1,2,\cdots,$ *we have*

(i) $f^m(t,p,q,k)$ *are* $\mathcal{F}_t$*-measurable, such that* P*-a.s.*

$$|f^m(t,p,q,k)|\le C_0(1+|p|+|q|+\|k\|),$$

where $C_0\le 0$ *is a constant;*

(ii) $\forall p_1,p_2\in\mathbb{R}^n,\ q_1,q_2\in\mathbb{R}^{n\times l},\ k_1,k_2\in L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n)$*, such that*

$$\langle p_1-p_2, f^m(t,p_1,q_1,k_1)-f^m(t,p_2,q_2,k_2)\rangle$$
$$\le \mu(t)\rho(|p_1-p_2|^2)+|p_1-p_2|(|q_1-q_2|+\|k_1-k_2\|),$$

where $\mu(t)$ *has the property stated in* (H3) *and* $\rho(\,\cdot\,)$ *has the property stated in* (H4);

(iii) *the same as* (iii) *in Theorem* 4.1;

(iv) $\lim_{m\to\infty} f^m(t,p,q,k)=f^0(t,p,q,k)$*,* P*-a.s.,* $\lim_{m\to\infty} g^m(t,p,q,k)=g^0(t,p,q,k)$*,* P*-a.s.*;

(v) *the same as* (v) *in Theorem* 4.1*.*

Then the conclusion of Theorem 4.1 *still holds.*

5 The Probabilistic Interpretation of SPDIEs

The connection of BDSDEs and systems of second-order quasilinear SPDEs was observed by Pardoux and Peng [8]. This can be regarded as a stochastic version of the well-known Feynman-Kac formula which gives a probabilistic interpretation for second-order SPDEs of parabolic types. Thereafter, this subject has attracted many mathematicians (refer to [1, 5, 10, 16]). In [5], the authors got a probabilistic interpretation for the solution to a semilinear SPDIE, via BDSDEs with Lévy process for a fixed terminal time under the Lipschitzian assumption. This section can be viewed as a continuation of such a theme, and will exploit the above theory of BDSDEP with non-Lipschitzian coefficients and random terminal time in order to provide a probabilistic formula for the solution to a quasilinear SPDIE.

Let D be a bounded domain in $\mathbb{R}^m$ with the boundary $\partial D = S$, $D^c = \mathbb{R}^m \setminus D$.

First, consider the following forward SDE with Poisson jumps in $\mathbb{R}^m$ for any given $(t,x) \in [0,T] \times D$,

$$\begin{aligned} X_s = x &+ \int_t^s b(r, X_r)\mathrm{d}r + \int_t^s \sigma(r, X_r)\mathrm{d}W_r \\ &+ \int_t^s \int_Z h(r_-, X_{r_-}, z)\widetilde{N}(\mathrm{d}z\mathrm{d}r), \quad t \le s \le T, \end{aligned} \tag{5.1}$$

where

$$b : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m, \quad \sigma : [0,T] \times \mathbb{R}^m \to \mathbb{R}^{m \times d}, \quad h : [0,T] \times \mathbb{R}^m \times Z \to \mathbb{R}^m.$$

It is known that, if the coefficients are less than linear increasing, and satisfy the Lipschitz condition, then SDE (5.1) has a unique solution (see [12]).

Now for any $(t,x) \in [0,T] \times D$, let

$$\tau = \tau_x = \inf\{s > t : X_s^{t,x} \notin D\} \quad \text{and} \quad \tau = \tau_x = T \quad \text{for } \inf\{\phi\}.$$

Consider the following BDSDEP (for simplicity, denote $X_s = X_s^{t,x}$):

$$\begin{aligned} P_s = \Phi(X_\tau) &+ \int_{s\wedge\tau}^{\tau} f(r, X_r, P_r, Q_r, K_r)\mathrm{d}r + \int_{s\wedge\tau}^{\tau} g(r, X_r, P_r, Q_r, K_r)\, \overleftarrow{\mathrm{d}B_r} \\ &+ \int_{s\wedge\tau}^{\tau} Q_r \mathrm{d}W_r + \int_{s\wedge\tau}^{\tau}\int_Z K_{r_-}(z)\widetilde{N}(\mathrm{d}z\mathrm{d}r), \quad t \le s \le T, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} &f : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n) \to \mathbb{R}^n, \\ &g : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2_{\lambda(\,\cdot\,)}(\mathbb{R}^n) \to \mathbb{R}^{n\times l}, \\ &\Phi : \mathbb{R}^m \to \mathbb{R}^n. \end{aligned}$$

Suppose that $f(t,x,\cdot,\cdot,\cdot)$ and $g(t,x,\cdot,\cdot,\cdot)$ satisfy the conditions in Theorem 3.2 uniformly for t and x, and suppose that $\mathbb{E}|\Phi(X_\tau)|^2 < \infty$, then by Theorem 3.2, BDSDEP (5.2) has a unique solution $(P_t, Q_t, K_t) \in S^2([0,\tau];\mathbb{R}^n) \times M^2(0,\tau;\mathbb{R}^{n\times d}) \times F_N^2(0,\tau;\mathbb{R}^n)$.

We now relate BDSDEP (5.2) to the following system of quasilinear second-order parabolic SPDIE:

$$\begin{cases} \mathcal{L}u(t,x)\mathrm{d}t \\ = f(t,x,u(t,x),\nabla u(t,x)\sigma(t,x),u(t,x+h(t,x,\,\cdot\,)) - u(t,x))\mathrm{d}t \\ \quad +g(t,x,u(t,x),\nabla u(t,x)\sigma(t,x),u(t,x+h(t,x,\,\cdot\,)) - u(t,x))\, \overleftarrow{\mathrm{d}B_t}, \\ \hfill \forall (t,x) \in [0,T] \times D, \\ u(T,x) = \Phi(x), \quad \forall x \in D, \\ u(t,x)|_{D^c} = \Psi(t,x), \quad \Psi(T,x) = \Phi(x)|_{D^c}, \end{cases} \tag{5.3}$$

where $u : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^n$,

$$\mathcal{L}u = \begin{pmatrix} Lu_1 \\ \vdots \\ Lu_n \end{pmatrix}$$

with

$$
\begin{aligned}
&Lu_k(t,x)\\
&=\frac{\partial u_k}{\partial t}(t,x)+\sum_{i=1}^{n}b_i(t,x)\frac{\partial u_k}{\partial x_i}(t,x)+\frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2 u_k}{\partial x_i\partial x_j}(t,x)\\
&\quad+\int_Z\Big(u_k(t,x+h(t,x,z))-u_k(t,x)-\sum_{i=1}^{n}h_i(t,x,z)\frac{\partial u_k}{\partial x_i}(t,x)\Big)\lambda(\mathrm{d}z),\quad k=1,\cdots,n.
\end{aligned}
$$

Now assume that σ is uniformly non-degenerate, i.e., there exists a constant $\beta>0$, such that

$$
\frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^*)_{ij}(t,x)\xi_i\xi_j\geq\beta|\xi|^2\quad\text{for all }\xi\in\mathbb{R}^m\text{ and }(t,x)\in[0,T]\times\overline{D},
$$

where $\overline{D}=$ the closure of D. Hence, SPDIE (5.3) is a true quasilinear type equation.

We can assert that

Theorem 5.1 *Under the above related conditions, b, σ, h, f and g are of class C^3, and Φ is of class C^2. Suppose that* SPDIE (5.3) *has a unique solution $u(t,x)\in C^{1,2}(\Omega\times[0,T]\times\mathbb{R}^m;\mathbb{R}^n)$. Then, for any given (t,x), $u(t,x)$ has the following interpretation:*

$$
u(t,x)=P_t,\tag{5.4}
$$

where P_t is determined uniquely by (5.1) *and* (5.2).

Proof Applying Itô's formula to $u(t,X_t)$ (see [12, Theorem 6]) on $[s\wedge\tau,\tau]$, we obtain

$$
\begin{aligned}
&u(\tau,X_\tau)-u(s\wedge\tau,X_{s\wedge\tau})\\
&=\int_{s\wedge\tau}^{\tau}\frac{\partial u}{\partial r}(r,X_r)\mathrm{d}r+\int_{s\wedge\tau}^{\tau}\sum_{i=1}^{m}b_i(r,X_r)\frac{\partial u}{\partial x_i}(r,X_r)\mathrm{d}r\\
&\quad+\int_{s\wedge\tau}^{\tau}\nabla u(r,X_r)\sigma(r,X_r)\mathrm{d}W_r+\int_{s\wedge\tau}^{\tau}\frac{1}{2}\sum_{i,j=1}^{m}(\sigma\sigma^*)_{ij}(r,X_r)\frac{\partial^2 u}{\partial x_i\partial x_j}(r,X_r)\mathrm{d}r\\
&\quad+\int_{s\wedge\tau}^{\tau}\int_Z(u(r,X_r+h(r,X_r,z))-u(r,X_r))\widetilde{N}(\mathrm{d}z\mathrm{d}r)\\
&\quad+\int_{s\wedge\tau}^{\tau}\int_Z\Big(u(r,X_r+h(r,X_r,z))-u(r,X_r)-\sum_{i=1}^{m}h_i(r,X_r,z)\frac{\partial u}{\partial x_i}(r,X_r)\Big)\lambda(\mathrm{d}z)\mathrm{d}r.
\end{aligned}
$$

Because $u(t,x)$ satisfies SPDIE (5.3), it holds that

$$
\begin{aligned}
&\Phi(X_\tau)-u(s\wedge\tau,X_{s\wedge\tau})\\
&=\int_{s\wedge\tau}^{\tau}f(r,X_r,u(r,X_r),\nabla u(r,X_r)\sigma(r,X_r,u(r,X_r)),u(r,X_r+h(r,X_r,\cdot))-u(r,X_r))\mathrm{d}r\\
&\quad+\int_{s\wedge\tau}^{\tau}g(r,X_r,u(r,X_r),\nabla u(r,X_r)\sigma(r,X_r,u(r,X_r)),\\
&\quad u(r,X_r+h(r,X_r,\cdot))-u(r,X_r))\,\mathrm{d}\overleftarrow{B_r}\\
&\quad+\int_{s\wedge\tau}^{\tau}\nabla u(r,X_r)g(r,X_r)\mathrm{d}W_r\\
&\quad+\int_{s\wedge\tau}^{\tau}\int_Z(u(r,X_r+h(r,X_r,z))-u(r,X_r))\widetilde{N}(\mathrm{d}z\mathrm{d}r).
\end{aligned}
$$

It is easy to check that $(u(t,X_t),\nabla u(t,X_t)\sigma(t,X_t),u(t,X_t+h(t,X_t,\cdot))-u(t,X_t))$ coincides with the unique solution of BDSDEP (5.2). It follows that

$$u(t,x)=P_t.$$

The proof of Theorem 5.1 is completed.

Remark 5.1 (5.4) can be called as a stochastic Feynman-Kac formula for SPDIE (5.3), which is a useful tool in the study of SPDIEs.

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