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Movement of Intransitive Permutation Groups Having Maximum Degree

Mehdi ALAEIYAN[1] Mehdi REZAEI[1]

Abstract Let G be a permutation group on a set Ω with no fixed points in Ω, and m be a positive integer. Then the movement of G is defined as move(G):=$\sup\limits_{\Gamma}\{|\Gamma^g \setminus \Gamma| \mid g \in G\}$. It was shown by Praeger that if $\text{move}(G) = m$, then $|\Omega| \leqslant 3m + t - 1$, where t is the number of G-orbits on Ω. In this paper, all intransitive permutation groups with degree $3m + t - 1$ which have maximum bound are classified. Indeed, a positive answer to her question that whether the upper bound $|\Omega| = 3m + t - 1$ for $|\Omega|$ is sharp for every $t > 1$ is given.

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1 Introduction

Let G be a permutation group on a set Ω with no fixed points in Ω, and m be a positive integer. If for a subset Γ of Ω the size $|\Gamma^g - \Gamma|$ is bounded, for $g \in G$, we define the movement of Γ as

$$\text{move}(\Gamma) := \sup_{g \in G} |\Gamma^g - \Gamma|.$$

If $\text{move}(\Gamma) \le m$ for all $\Gamma \subseteq \Omega$, then G is said to have bounded movement m and the movement of G is defined as

$$\text{move}(G) := \sup_{\Gamma, g} |\Gamma^g - \Gamma|,$$

where Γ ranges over all subsets of Ω and g ranges over all elements of G.

This notion was introduced in [4, 7]. By [7, Theorem 1], if G has bounded movement m, then Ω is finite. Moreover, both the number of G-orbits in Ω and the length of each G-orbit are bounded above by linear functions of m. In particular, each G-orbit has length at most $3m$, $t \le 2m - 1$ and $n := |\Omega| \le 3m + t - 1 \le 5m - 2$, where t is the number of G-orbits on Ω. In [3], it was shown that $n = 5m - 2$ if and only if $n = 3$ and G is transitive. But in [5], this bound was refined further and it was shown that $n \le \frac{9m-3}{2}$. Moreover, if $n = \frac{9m-3}{2}$ then either $n = 3$ and $G = S_3$ or G is an elementary abelian 3-group and all its orbits have length 3. Also this upper bound was improved to the bound $n \le 4m - p$ in [1, 2], where $p \ge 5$ is the least odd prime dividing $|G|$.

Throughout this paper, m is a positive integer and G is an intransitive permutation group on a set Ω of size n with movement m and $t(\ge 2)$ orbits, such that $n = 3m + t - 1$. The purpose

Manuscript received November 21, 2009. Revised June 19, 2010.
[1]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran.
E-mail: Alaeiyan@iust.ac.ir m_rezaei@iust.ac.ir

of this paper is to classify all intransitive permutation groups of degree $n = 3m + t - 1$ which attains the upper bound $n \leq \lfloor \frac{9m-3}{2} \rfloor$ for the degree n, obtained in [5]. In [6], Praeger asked in Question 1.5 that whether the upper bound of $|\Omega|$ by $3m + t - 1$ is sharp for any value of t greater than 1. In this paper, we give a positive answer to this question. We note that for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the integer part of x.

Theorem 1.1 *Let G be an intransitive permutation group on a set Ω with t (≥ 2) orbits which have no fixed points in Ω. Suppose further that m is a positive integer such that* $\text{move}(G) = m$ *and* $n = |\Omega| = 3m + t - 1$. *Then* $n \leq \lfloor \frac{9m-3}{2} \rfloor$, *where the equality holds if and only if G is one of the following:*

(i) *G is an elementary abelian 3-group and all its orbits have length 3.*

(ii) *G is the semidirect product of $\mathbb{Z}_2^2$ by $\mathbb{Z}_3$ with normal subgroup $\mathbb{Z}_2^2$.*

The "if" part is verified and illustrated by Examples 2.1 and 2.2.

2 Examples and Preliminaries

Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has t (t is a positive integer) nontrivial cycles of lengths $l_1, \cdots, l_t$, say. We might represent g as

$$g = (a_1 a_2 \cdots a_{l_1})(b_1 b_2 \cdots b_{l_2}) \cdots (z_1 z_2 \cdots z_{l_t}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor \frac{l_i}{2} \rfloor$ points from the ith cycle, for each i, chosen in such a way that $\Gamma(g)^g \cap \Gamma(g) = \emptyset$. For example, we could choose

$$\Gamma(g) = \{a_2, a_4, \cdots, a_{k_1}, b_2, b_4, \cdots, b_{k_2}, \cdots, z_2, z_4, \cdots, z_{k_t}\},$$

where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written. For any set $\Gamma(g)$ of this kind, we say that $\Gamma(g)$ consists of every second point of every cycle of g. From the definition of $\Gamma(g)$, we see that

$$|\Gamma(g)^g \backslash \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t} \left\lfloor \frac{l_i}{2} \right\rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g \backslash \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1 (see [4, Lemma 2.1]) *Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g \backslash \Gamma| \leq \sum_{i=1}^{t} \lfloor \frac{l_i}{2} \rfloor$, where l_i is the length of the ith cycle of g and t is the number of nontrivial cycles of g in its disjoint cycle representation. This upper bound is attained for $\Gamma = \Gamma(g)$ defined above.*

This immediately implies the following formula for move(G) for a finite group G acting on a finite set Ω:

$$\text{move}(G) := \underset{\Omega}{\text{move}}(G) = \max_{g \in G} \underset{\Omega}{\text{move}}(g) = \max_{g \in G} \sum_{l=1}^{n} \left\lfloor \frac{l}{2} \right\rfloor t_l(g),$$

where $t_l(g)$ denotes the number of l-cycles in the disjoint cycle representation of g.

Now we show that there certainly is an infinite family of 3-groups for which the maximum bound mentioned in Theorem 1.1 holds.

Example 2.1 Let d be a positive integer, $G := \mathbb{Z}_3^d$, $t := \frac{3^d-1}{2}$, and let $H_1, \cdots, H_t$ be all subgroups of index 3 in G. Define Ω_i to be the right coset space $\{H_i g \mid g \in G\}$ of H_i and $\Omega := \Omega_1 \cup \cdots \cup \Omega_t$. Consider G as a permutation group on Ω by the right multiplication, that is, $x \in G$ is identified with the composite of permutation $H_i g \mapsto H_i g x$ $(i = 1, \cdots, t)$ on Ω_i for $i = 1, \cdots, t$. If $g \in G - \{1\}$, then g lies in $\frac{3^{d-1}-1}{2}$ groups H_i and therefore acts on Ω as a permutation with $\frac{3(3^{d-1}-1)}{2}$ fixed points and 3^{d-1} orbits of length 3. Taking every second point from each of these 3-cycles to form a set Γ, we see that $\mathrm{move}(g) = m \geq 3^{d-1}$ and it is not hard to prove that in fact $\mathrm{move}(g) = m = 3^{d-1}$. It is easy to see that $n = \frac{9m-3}{2} = 3m + t - 1$.

The above example is one of the family of groups meeting the upper bound of size $n = 3m + t - 1$ with $t = \frac{3m-1}{2}$ and this is a partial answer to Question 1.5 in [6].

Remark 2.1 Let g be an element of a permutation group G on a set Ω. Assume that the set Ω is the disjoint union of G-invariant sets Ω_1 and Ω_2. Then every subset Γ of Ω is a disjoint union of subsets $\Gamma_i = \Gamma \cap \Omega_i$ for $i = 1, 2$. Let g_i be the permutation on Ω_i induced by g for $i = 1, 2$. Since $|\Gamma^g - \Gamma| = |\Gamma_1^{g_1} - \Gamma_1| + |\Gamma_2^{g_2} - \Gamma_2|$, we have

$$\underset{\Omega}{\mathrm{move}}(g) = \sum_{i=1}^{2} \max\{|\Gamma_i^{g_i} \setminus \Gamma_i| \mid \Gamma_i \subseteq \Omega_i\} = \underset{\Omega_1}{\mathrm{move}}(g_1) + \underset{\Omega_2}{\mathrm{move}}(g_2).$$

We start with constructing families of groups having movement m which attain the maximum bound $n = \lfloor \frac{9m-3}{2} \rfloor$. We see later that these families are the only examples meeting the bound. The second example is as follows. We note that $H \rtimes K$ is a semi-direct product of H by K with normal subgroup H.

Example 2.2 Let G be a permutation group on a set $\Omega := \Omega_1 \cup \Omega_2$ of size 7, such that $\Omega_1 = \{1, 2, 3, 4\}$ and $\Omega_2 = \{1', 2', 3'\}$. Moreover, suppose that $G^{\Omega_1} \cong \mathbb{Z}_2^2$ is an elementary abelian group of order 4 acting regularly on Ω_1 but fixing every point of Ω_2 and $G^{\Omega_2} \cong \mathbb{Z}_3$, where $\mathbb{Z}_3 \cong \langle (123)(1'2'3') \rangle$. Then the semidirect product $G = \mathbb{Z}_2^2 \rtimes \mathbb{Z}_3$ with normal subgroup $\mathbb{Z}_2^2$ has $t = 2$ orbits, and since each non-identity element of G is the product of two cycles of length 2 or two cycles of length 3, so $m = \mathrm{move}(G) = 2$. It follows that $|\Omega| = 3m + t - 1 = \lfloor \frac{9m-3}{2} \rfloor = 7$, which meets the upper bound in Theorem 1.1.

3 Proof of Theorem 1.1

To prove Theorem 1.1, we introduce the following notation:

$$r_3 := \text{number of } G\text{-orbits of length 3 on which } G \text{ acts as } \mathbb{Z}_3;$$
$$r_3' := \text{number of } G\text{-orbits of length 3 on which } G \text{ acts as } \mathrm{Sym}(3)$$

and

$$r_2 := \text{number of } G\text{-orbits of length 2};$$
$$r_4 := \text{number of } G\text{-orbits of length 4};$$
$$s := \text{number of } G\text{-orbits of length} \geq 5.$$

The orbits are labelled accordingly: $\Omega_1, \cdots, \Omega_{r_3}$ are those of length 3 on which G acts as $\mathbb{Z}_3$; $\Omega_{r_3+1}, \cdots, \Omega_{r_3+r_3'}$ are those of length 3 on which G acts as Sym(3); $\Omega_{r_3+r_3'+1}, \cdots, \Omega_{r_3+r_3'+r_2}$ are those of length 2, etc. Define $t := r_3 + r_3' + r_2 + r_4 + s$, $t_0 := r_3 + r_3' + r_2$, $\Sigma_4 := \bigcup_{i=t_0+1}^{t_0+r_4} \Omega_i$, and $\Sigma_5 := \bigcup_{i=t_0+r_4+1}^{t} \Omega_i$ and so $|\Omega| = n = 3r_3 + 3r_3' + 2r_2 + 4r_4 + |\Sigma_5|$.

With the above notation by [5, Lemma 3], we have

$$n < \frac{9}{2}m - \Big(\frac{3}{4}r_3' + \frac{1}{4}r_2 + \frac{5}{4}r_4 + \frac{1}{2}(|\Sigma_5| - 3s)\Big). \tag{3.1}$$

With some simple calculations, inequality (3.1) can be simplified as $n \leq \lfloor\frac{9m-3}{2}\rfloor$. Now we suppose that the equality holds. We define $\eta := 9m - 2n$. Clearly η is a positive integer and from inequality (3.1) we have

$$2\eta > 3r_3' + r_2 + 5r_4 + 2(|\Sigma_5| - 3s) \geq 0. \tag{3.2}$$

It is not hard to see that $\eta = 3$ or 4.

To prove the theorem, we suppose that $\eta = 3$ or 4 and seek to discover what configurations may occur. Then $2\eta = 6$ or 8. Since certainly $|\Sigma_5| \geq 5s$, from inequality (3.2), it follows that there are only the following possibilities:

(a) $3r_3' + r_2 + 5r_4 < 8,\ s = |\Sigma_5| = 0,$

(b) $3r_3' + 5r_4 + 2(|\Sigma_5| - 3s) < 8,\ r_2 = 0,$

(c) $3r_3' + r_2 + 2(|\Sigma_5|) - 3s < 8,\ r_4 = 0,$

(d) $r_2 + 5r_4 + 2(|\Sigma_5| - 3s) < 8,\ r_3' = 0.$

It follows easily from the maximum value of t (which is attained in Example 2.1) and the equality $9m - 2n = 3$ or 4, and also with arithmetical reasons that the three following cases will be remained:

(I) $r_3' = r_2 = |\Sigma_5| = s = 0,\ r_4 = 1,$

(II) $r_3' = r_4 = 0,\ r_2 = s = 1,\ |\Sigma_5| = 5.$

(III) $3r_3' + r_2 \leq 5,\ r_4 = |\Sigma_5| = s = 0.$

For example, in Case (a) and for $\eta = 4$ we have $3r_3' + r_2 + 5r_4 < 8,\ s = |\Sigma_5| = 0$, $9m - 2n = 4$ and $n = 3r_3 + 3r_3' + 2r_2 + 4r_4$. So there are only the following subcases:

(1) $r_3' = 1,\ r_2 = 1,\ r_4 = 0,$

(2) $r_3' = 1,\ r_2 = r_4 = 0,$

(3) $r_3' = 0,\ r_2 = 1 = r_4 = 1,$

(4) $r_3' = 0,\ r_2 = 1,\ r_4 = 0,$

(5) $r_3' = 0,\ r_2 = 0,\ r_4 = 1.$

It is obvious that the subcases (1), (2) and (4) belong to Case (III) and the subcase (5) belongs to Case (I). In the subcase (3), n must be of the form $3r_3 + 6$, and therefore with respect to the equality $9m - 2n = 4$, the integer m must be of the form $\frac{6r_3+16}{9}$ which is not possible because m is an integer number. With similar reasons, it can be shown that only the three cases (I), (II) and (III) will be remained.

We first consider the case when $\eta = 3$. With simple calculations as above cases (I) and (II) cannot arise. Also by [5] and intransitivity of G, we can see that Case (III) arises if and only if G is an elementary abelian 3-group and all of its orbits have length 3.

We now assume that $\eta = 4$. Then the group in Example 2.2 satisfies in Case (I) and we also will show that it is the only example satisfying in Case (I). Suppose that Case (I) holds. Set $\Delta := \Sigma_4$. Then the permutation group $X := G^{\Delta}$ induced on a set Δ of length 4 is a subgroup of S_4 which is transitive on Δ. As $|S_4| = 2^3 \cdot 3$, we have $|X| = 2^a$ or $2^a \cdot 3$ for some $a = 0, \cdots, 3$. By transitivity on Δ, we have $a = 2$ and so X is isomorphic to $\mathbb{Z}_2^2$ or $\mathbb{Z}_4$ or $\mathrm{Alt}(4)$. We show that the latter two cases do not occur. By similar argument used in Example 2.2, we conclude that the movement m in each case will be equal to $2 + 3^{d-1}$ (for some positive integer d), and so $n = 4 + \frac{3^{d+1}-3}{2} = \frac{3^{d+1}+5}{2}$, and $t = 1 + \frac{3^d-1}{2} = \frac{3^d+1}{2}$. Thus the number m satisfies in $n = 3m + t - 4$ which is a contradiction to the equality $n = 3m + t - 1 = \lfloor \frac{9m-3}{2} \rfloor$). Now let G be the semidirect product of G_1 by G_2 which is a permutation group on $\Omega_1 \cup \Omega_2$ with $|\Omega_1| = 4$ with the following properties:

The normal subgroup G_1 is an elementary abelian group of order 4 acting regularly on Ω_1 but fixing every point of Ω_2. The group G_2 acts nontrivially on Ω_1 and it induces on Ω_2 the permutation group H in Example 2.1. We show that the permutation group G satisfies in Case (I) if and only if $d = 1$.

We now consider that the centralizer K of G_1 in G_2, which is a subgroup of index 3 in G_2, as $G_2 \cong H$ is an elementary abelian 3-group of order 3^d which normalizes the elementary abelian group G_1 of order 4. As K is a 3-subgroup, it fixes a point of Ω_1. Since G_1 is transitive on Ω_1, the group K commuting with G_1 fixes all points of Ω_1. Thus if $d \geq 2$, there is a nontrivial element g_2 of K, and the element $g = g_1 g_2$ for every nontrivial element g_1 of G_1 is of order 6. This element g induces two 2-cycles on Ω_1, while induces 3^{d-1} 3-cycles on Ω_2. Hence, the movement of g on $\Omega := \Omega_1 \cup \Omega_2$ is

$$\underset{\Omega}{\mathrm{move}}(g) = \underset{\Omega_1}{\mathrm{move}}(g_1) + \underset{\Omega_2}{\mathrm{move}}(g_2) = 2 + 3^{d-1}.$$

The other nontrivial elements of $G_1 \rtimes G_2$ are involutions in G_1 (with movement 2), elements of order 3 in G_2 (with movement 3^{d-1}) and elements of order 3 of form $g_1 g_2$ with $1 \neq g_1 \in G_1$ and $g_2 \in G_2 - K$ (with movement $1 + 3^{d-1}$, as $g_1 g_2$ induces a 3-cycle on Ω_1). Hence we conclude $m := \underset{\Omega}{\mathrm{move}}(G_1 \rtimes G_2) = 2 + 3^{d-1}$. As $n := |\Omega| = |\Omega_1| + |\Omega_2| = 4 + \frac{3^{d+1}-3}{2} = \frac{3^{d+1}+5}{2}$, and $t = 1 + \frac{3^d-1}{2} = \frac{3^d+1}{2}$, the number m satisfies in $n = 3m + t - 4$, but not $n = 3m + t - 1$ which is a contradiction. Therefore the group in Example 2.2 is the only group satisfying in Case (I).

It is easy to see that Case (III) for arithmetical reasons cannot arise.

Finally, we show that Case (II) cannot arise. Suppose therefore that $r_3' = r_4 = 0$. Since in this case $|\Omega| = n = 3r_3 + 7$ and $m = \frac{6r_3+18}{9}$, it follows that r_3 must be a multiple of 3. Let k be a positive integer such that $r_3 = 3k$. Therefore we have $m = 2k + 2$, $t = 3k + 2$ and $n = 9k + 7$. Define $\Sigma_1 := \bigcup_{i=1}^{r_3} \Omega_i$, the union of the orbits of length 3 on which G acts as $\mathbb{Z}_3$, and $\Sigma_2 := \bigcup_{i=r_3+1}^{t} \Omega_i$, the union of those orbits of length 5 and those of length 2. Then define K_1 to be the kernel of the action of G on Σ_1 and K_2 the kernel of its action on Σ_2. Clearly, $K_1 \cap K_2 = \{1\}$ since G acts faithfully on Ω. Now let H be the subgroup of G generated by its 2-elements. Then $G^{\Sigma_2} = H^{\Sigma_2}$, that is, $G = HK_2$. But $H \leqslant K_1$ and therefore $G = K_1 K_2$, that is, $G = K_1 \times K_2$. It follows easily that if $m_1 := \mathrm{move}(G^{\Sigma_1})$ and $m_2 := \mathrm{move}(G^{\Sigma_2})$ then $m = m_1 + m_2$. Defining $n_1 := |\Sigma_1|$ and $n_2 := |\Sigma_2|$, we have from Example 2.2 that $n_1 \leq \frac{9m_1-3}{2}$. Since $n_1 = 3r_3 = 9k$

and $n_2 = 7$, therefore $m_1 \geq 2k+1$ and $m_2 > 1$ and so $m = m_1 + m_2 > 2k+2$ which is a contradiction. Now the proof of Theorem 1.1 is completed.

References

[1] Alaeiyan, M. and Yoshiara, S., Permutation groups of minimal movement, *Arch. Math.,* **85**, 2005, 211–226.

[2] Alaeiyan, M. and Tavallaee, H. A., Improvement on the bound of intransitive permutation groups with bounded movement, *Bull. Belg. Math. Soc. Simon Stevin,* **13**(3), 2006, 471–477.

[3] Cho, J. R., Kim, P. S. and Praeger, C. E., The maximal number of orbits of a permutation groups with bounded movement, *J. Algebra,* **214**, 1999, 625–630.

[4] Hassani, A., Khayaty (Alaeiyan), M., Khukhro, E. I. and Praeger, C. E., Transitive permutation groups with bounded movement having maximal degree, *J. Algebra,* **214**, 1999, 317–337.

[5] Neumann, P. M. and Praeger, C. E., On the movement of a permutation group, *J. Algebra,* **214**, 1999, 631–635.

[6] Praeger, C. E., Movement and separation of subsets of points under group actions, *J. London Math. Soc.,* **56**(2), 1997, 519–528.

[7] Praeger, C. E., On permutation group with bounded movement, *J. Algebra,* **144**, 1991, 436–442.