Chin. Ann. Math.
33B(1), 2012, 149–160
DOI: 10.1007/s11401-011-0684-x

Chinese Annals of Mathematics, Series B

Controllability of the Heat Equation with a Control Acting on a Measurable Set*

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Abstract The paper deals with the controllability of a heat equation. It is well-known that the heat equation $y_t - \Delta y = u\chi_E$ in $(0,T)\times\Omega$ with homogeneous Dirichlet boundary conditions is null controllable for any $T > 0$ and any open nonempty subset E of Ω. In this note, the author studies the case that E is an arbitrary measurable set with positive measure.

Keywords Heat equation, Measurable sets, Null controllability, Observability inequality
2000 MR Subject Classification 93B05, 93B07, 93D15

1 Introduction

Let $T > 0$, Ω be a bounded, C^∞ domain in $\mathbb{R}^n$ and $E \subset \Omega$ with $|E| > 0$ where $|\cdot| = \text{meas}(\cdot)$ denotes the Lebesgue measure. Consider the following system:

$$\begin{cases} \partial_t y - \Delta y = u\chi_E, & \text{in } (0,T)\times\Omega, \\ y = 0, & \text{on } (0,T)\times\partial\Omega, \\ y(x,0) = y_0, & \text{in } \Omega. \end{cases} \tag{1.1}$$

For any initial data $y_0 \in L^2(\Omega)$, we define $y(\cdot\,,\,\cdot\,; u, y_0)$ the solution to system (1.1) with the control u.

Then null controllability problem and approximate controllability may be addressed as follows.

Definition 1.1 *System* (1.1) *is null controllable in time T if for every initial data $y_0 \in L^2(\Omega)$, there exists a control $u \in L^2((0,T)\times E)$ such that*

$$y(T,x;u) = 0, \quad \text{a.e. } x \in \Omega.$$

Definition 1.2 *System* (1.1) *is approximate controllable in time T if for every data $y_0, y_1 \in L^2(\Omega)$ and any $\varepsilon > 0$, there exists a control $u \in L^2((0,T)\times E)$ such that*

$$\|y(T,\cdot\,;u) - y_1\|_{L^2(\Omega)} < \varepsilon.$$

Manuscript received March 30, 2009. Revised November 23, 2010.
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*Project supported by the National Natural Science Foundation of China (No. 10671040), the Foundation for the Author of National Excellent Doctoral Dissertation of China (No. 200522) and the Program for the New Century Excellent Talents in University of China (No. 06-0359).

The problems of null controllability are clearly of significant practical interest. It is well-known that any initial data in $L^2(\Omega)$ can be steered to 0 in an arbitrary time T by applying a suitable control $u \in L^2((0,T)\times E)$ if E is an open subset of Ω. Some classical references prior to 1988 are Russell [9, 15]. This field did not develop so fast until Fursikov and Imanuvilov introduced Carleman estimates to this subject in their note (see [4]). As Castro and Zuazua indicating in their paper [2], Carleman estimates have been used systematically as the most efficient tool to study controllability so that we can obtain many interesting results in this filed, such as null controllability of heat equations in high dimensions (see [6] for more details), exact controllability of wave equations under the GCC conditions (see [1]) and the relation between wave equations and heat equations (see [10, 17]).

We know that the null controllability problem for system (1.1) is equivalent to the observability inequality

$$\int_\Omega \varphi^2(x,0)\mathrm{d}x \le C\int_0^T\int_E \varphi^2(x,t)\mathrm{d}x\mathrm{d}t \quad \text{for all } \varphi_T \in L^2(\Omega), \tag{1.2}$$

where $\varphi(x,t)$ is the solution to the following adjoint system:

$$\begin{cases} \partial_t\varphi + \Delta\varphi = 0, & \text{in } \Omega\times(0,T), \\ \varphi = 0, & \text{on } \partial\Omega\times(0,T), \\ \varphi(x,T) = \varphi_T, & \text{in } \Omega. \end{cases} \tag{1.3}$$

By Carleman inequalities, (1.2) is valid when E is an open subset of Ω (see [4]). On the other hand, Lebeau and Robbiano [7] proved the null controllability by using the following inequality for all $\{a_i\} \in l^2$ and for all $\mu > 0$,

$$C_1 \mathrm{e}^{-C_2\sqrt{\mu}} \sum_{\lambda_i\le\mu} |a_i|^2 \le \int_E \Big|\sum_{\lambda_i\le\mu} a_i\phi_i(x)\Big|^2 \mathrm{d}x,$$

where λ_i and ϕ_i are the eigenvalue and the corresponding eigenfunction of $-\Delta$ with Dirichlet boundary conditions respectively. The proof of the above inequality is also based on Carleman inequalities (see also [8]) and the assumption that E is open is necessary.

In order to prove (1.2), we usually have to choose a proper weight function. In detail, we need to find a function of $C^2(\overline{\Omega})$ satisfying

$$\psi > 0 \ \text{ in } \Omega, \quad \psi = 0 \ \text{ on } \partial\Omega, \tag{1.4}$$

$$|\nabla\psi| > 0, \quad \forall x \in \overline{\Omega\backslash E} \tag{1.5}$$

(see [4]). Obviously, such functions do not always exist when E is an arbitrarily measurable set.

Indeed, let $\Omega = (0,1) \in \mathbb{R}^1$, $E = (0,1)\backslash\mathbb{Q}$. Then $\overline{\Omega\backslash E} = [0,1]$. By Rolle Theorem, there is no $f \in C^2([0,1])$ satisfying (1.4) and (1.5).

So, we cannot desire Carleman inequalities working in our case.

Recently, G. Wang [16] obtained null controllability of a heat equation with a control localized in $\omega\times G$, where G is a measurable set of $(0,T)$ and ω is an open subset of Ω. But it is totally different from the case we will discuss because Lebeau-Robbiano inequality is still valid in Wang's case. To our best knowledge, there is no results on null controllability of heat equations with the control localized in a measurable set of Ω even in one dimension. Thus controllability of heat equations on a measurable set is nontrivial.

2 Main Results

Miller [12] introduced a new idea called transmutation method to get the null controllability of heat equations from the exact controllability of wave equations. By using this method, we can obtain the null controllability of heat equations on a measurable set in one dimension.

In this paper, we firstly prove the exact controllability of wave equations with the control localized in a measurable set in one dimension by using the Fourier expansion. More precisely, we introduce the following results which can also be seen in [11].

Theorem 2.1 *Let $T' > 2$, $\Omega = [-1, 1] \subset \mathbb{R}^1$, and $E \subset \Omega$ with $|E| > 0$. The following system*

$$\begin{cases} \partial_{tt} w - \Delta w = \chi_{(0,T')\times E} f, & \text{in } (0, \infty) \times \Omega, \\ w = 0, & \text{on } (0, \infty) \times \partial\Omega, \\ w(0, x) = w_0(x), \quad w_t(0, x) = w^1(x), & \text{in } \Omega \end{cases} \tag{2.1}$$

is exact controllability, i.e., for any $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a control $f \in L^2(\Omega)$ such that

$$w(T', x) = 0 = w_t(T', x).$$

Hence, the null controllability of wave equations on measurable sets holds in one dimension. However, for the reason stated in Remark 3.1, we cannot draw the same conclusion for heat equations by using the classical method in [10] or [17].

Fortunately, we can prove the null controllability of (1.1) in one dimension by the transmutation method. Our first main result is stated as follows.

Theorem 2.2 *Let $\Omega = [-1, 1] \subset \mathbb{R}^1$, and $E \subset \Omega$ with $|E| > 0$. System* (1.1) *is null controllable.*

In the case of high dimensions our proof is not valid because Theorem 2.1 contradicts the GCC condition in high dimensions.

Approximate controllability is weaker than null controllability and thus is easier to obtain because approximate controllability is equivalent to the following general unique continuation principle of (1.3):

$$\varphi|_{E\times(0,T)} = 0 \Rightarrow \varphi_T = 0. \tag{2.2}$$

When E is an open subset, (2.2) is just a consequence of Holmgren Uniqueness Theorem (see [5]). However, it is different if E does not contain an analytical manifold. On the other hand, Poon [13] showed that if φ solves (1.3) and φ vanishes of infinite order at (x_0, t_0) (i.e., $\varphi(x, t) = O(|x - x_0|^{2K} + |t - t_0|^K)$ for all positive integers K), then $u \equiv 0$ in Ω.

As to null controllability, we can hardly obtain the observability inequality (1.2). But by Harnack inequality of parabolic equations, we may obtain our second result as follows.

Theorem 2.3 *Assuming $\varphi_T \in L^2(\Omega)$, $\varphi(x,t)$ is the solution to* (1.3). *When $\varphi \geq 0$ (or $\varphi \leq 0$), there exists a constant C only depending on T, E and Ω, such that*

$$\int_\Omega \varphi^2(x,0)\mathrm{d}x \leq C \int_0^T \int_E \varphi^2(x,t)\mathrm{d}x\mathrm{d}t. \tag{2.3}$$

This is an observability inequality with the observation over E, but the final data φ_T can not be given arbitrary in $L^2(\Omega)$.

Using Theorem 2.3, we can draw some results on controllability.

Theorem 2.4 *Assume that E is a measurable subset of Ω with positive measure. For all $u_0 \in L^2(\Omega)$, $T > 0$, there exist f_1 and $f_2 \in L^2(\Omega \times (0,T))$, such that the solution to* (1.1) *satisfies*

$$u(\cdot, T; f_1, u_0) \geq 0, \quad \text{in } \Omega,$$
$$u(\cdot, T; f_2, u_0) \leq 0, \quad \text{in } \Omega.$$

Next, we show that if the initial data satisfies a certain conditions, the solution to (1.1) can be controlled to zero. Define

$$L^2_+(\Omega) = \{w \in L^2(\Omega) : w \geq 0\},$$
$$H^+ = \{\phi(x,t) : \exists \phi_T \in L^2_+(\Omega), \text{ s.t. } \phi \text{ is the solution to (1.3) with the final data } \phi_T\}.$$

By the maximal principle of parabolic equations, for any $\phi \in H^+$, we have

$$\phi \geq 0, \quad \text{in } \Omega \times (0,T).$$

We continue defining

$$U^+ = \{u_0 \in L^2(\Omega) : \forall \varepsilon, \ \exists f^\varepsilon \in H^+, \text{ s.t. } \|u(x,T;f^\varepsilon,u_0)\|_{L^2(\Omega)} < \varepsilon\}.$$

Then we have the following theorem.

Theorem 2.5 *System* (1.1) *is null controllable if the initial data $u_0 \in U^+$ and there exists a constant C, such that for any $\varepsilon > 0$, it holds that*

$$\int_0^T \int_E |f^\varepsilon|^2 \mathrm{d}x\mathrm{d}t + \int_\Omega f^\varepsilon(x,0)u_0\mathrm{d}x < C. \tag{2.4}$$

The remaining part of this paper is organized as follows. In Section 3, we prove the null controllability in one dimension. In Section 4, we study the case in multiple dimensions. Throughout the paper, C stands for a generic constant and its value may vary from line to line.

3 Null Controllability in One Dimension

This section is devoted to prove Theorems 2.1 and 2.2. In this section, we assume $\Omega = [-1,1] \subset \mathbb{R}^1$, and $E \subset \Omega$ with $|E| > 0$.

First of all we consider the following dual system:

$$\begin{cases} \partial_{tt} z - \Delta z = 0, & \text{in } (0,T') \times \Omega, \\ z = 0, & \text{on } (0,T') \times \partial\Omega, \\ z(T',x) = z_0, \quad z_t(T',x) = z_1, & \text{in } \Omega. \end{cases} \tag{3.1}$$

The observability of system (3.1) is as follows.

Lemma 3.1 *For any $(z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and z solution to (3.1), it holds that*

$$\|(z_0, z_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^{T'} \int_E |z(t,x)|^2 \mathrm{d}x\mathrm{d}t.$$

Proof We choose $\{\sin(n\pi x)\}_{n=1}^{\infty}$ as an orthogonal basis of $L^2(\Omega)$. Then we can assume

$$z_0 = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \quad \text{in } L^2(\Omega),$$

where $\sum\limits_{n=1}^{\infty} |a_n|^2 < \infty$ and

$$z_1 = \sum_{n=1}^{\infty} b_n \sin(n\pi x), \quad \text{in } H^{-1}(\Omega),$$

where $\sum\limits_{n=1}^{\infty} \frac{|b_n|^2}{n^2} < \infty$.

Then the solution to (3.1) can be represented as

$$z(t,x) = \sum_{n=1}^{\infty} \Big[a_n \cos(n\pi(t-T')) + \frac{b_n \sin(n\pi(t-T'))}{n\pi} \Big] \sin(n\pi x), \quad \text{in } H^{-1}(\Omega).$$

Noticing that

$$\|(z_0, z_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} = \sum_{n=1}^{\infty} \Big(|a_n|^2 + \frac{|b_n|^2}{n^2\pi^2} \Big)$$

and $T' > 2$, one has

$$\begin{aligned} \int_0^{T'} \int_E |z|^2 \mathrm{d}x\mathrm{d}t &\geq \int_0^2 \int_E |z|^2 \mathrm{d}x\mathrm{d}t \\ &= \int_0^2 \int_E \Big| \sum_{n=1}^{\infty} \Big(\Big[a_n \cos(n\pi(t-2)) + \frac{b_n \sin(n\pi(t-2))}{n\pi} \Big] \sin(n\pi x) \Big) \Big|^2 \mathrm{d}x\mathrm{d}t \\ &= \sum_{n=1}^{\infty} \Big(\int_0^2 \Big[a_n \cos(n\pi(t-2)) + \frac{b_n \sin(n\pi(t-2))}{n\pi} \Big]^2 \mathrm{d}t \int_E \sin^2(n\pi x) \mathrm{d}x \Big) \\ &= \sum_{n=1}^{\infty} \Big[\Big(|a_n|^2 + \frac{|b_n|^2}{n^2\pi^2} \Big) \int_E \sin^2(n\pi x) \mathrm{d}x \Big]. \end{aligned} \tag{3.2}$$

If we denote $b_n = \int_E \sin^2(n\pi x)\mathrm{d}x$, then

$$B = \inf_{n\in\mathbb{Z}^+} b_n > 0. \tag{3.3}$$

Indeed,

$$b_n = \frac{|E|}{2} - \int_J \frac{\cos(2n\pi x)}{2}\mathrm{d}x \geq \frac{|E|}{2} - \frac{1}{2n\pi}.$$

Since $\frac{1}{2n\pi}$ tends to zero when n tends to infinity, there exists an $n_0 > 0$ such that

$$b_n \geq \frac{|E|}{2} - \frac{1}{2n\pi} \geq \frac{|E|}{4} > 0$$

for any $n > n_0$. It follows that

$$B_1 = \inf_{n\geq n_0} b_n > 0$$

and $B > 0$ since $b_n > 0$ for all n.

It completes the proof of Lemma 3.1.

Due to the relation between null controllability and observability, Theorem 2.1 follows by Lemma 3.1 immediately.

Next, we prove Theorem 2.2.

Firstly, we introduce the following lemma.

Lemma 3.2 *For all $T, T' > 0$, $\Omega' = (-T', T')$, there is a function $v \in C((0,T), H_0^1(\Omega'))$ satisfying*

$$\partial_t v - \Delta v = 0, \quad \text{in } (0,T)\times\Omega', \tag{3.4}$$

$$v(0,\cdot\,) = \delta \quad \text{and} \quad v(T,\cdot\,) = 0, \tag{3.5}$$

where δ is the Dirac delta function.

Proof We denote $v(x,t;z)$ as the solution to

$$\begin{cases} \partial_t v - \Delta v = 0, & \text{in } (0,T)\times\Omega', \\ v(t,T') = z, & \text{in } (0,T), \\ v(t,-T') = 0, & \text{in } (0,T), \\ v(0,\cdot\,) = \delta, & \text{in } \Omega'. \end{cases} \tag{3.6}$$

We set $z = 0$ for $t \leq \varepsilon T$, where $0 < \varepsilon < 1$. Note that $v_0 = v(\varepsilon T, x)$ is just the Dirac mass at the origin smoothed out by the homogeneous heat semigroup during a time εT, so that $v_0 \in L^2(\Omega')$.

Hence, by the boundary null controllability of heat equation, we can find $h \in L^2(\varepsilon T, T)$ such that $v(T,x) = 0$.

Then Lemma 3.2 follows when we choose

$$z = \begin{cases} 0, & \text{in } (0,\varepsilon T), \\ h, & \text{in } (\varepsilon T, T). \end{cases} \tag{3.7}$$

Proof of Theorem 2.2 Applying Lemma 3.2 with $w_0 = y_0 \in H_0^1(\Omega)$ and $w^1 = 0 \in L^2(\Omega)$, one can find $f \in L^2((0,\infty)\times\Omega)$, such that

$$w(T,\cdot\,;f) = w_t(T,\cdot\,;f) = 0.$$

Next, we expand v as follows:

$$\overline{v} = \begin{cases} v, & \text{in } (0,T)\times(-T',T'), \\ 0, & \text{else.} \end{cases} \tag{3.8}$$

We also define $\overline{w} \in L^2(\mathbb{R}\times\Omega)$ and $\overline{f} \in L^2(\mathbb{R}\times\Omega)$ as the extensions of w and f by reflection to $s = 0$, i.e., $\overline{w}(s,x) = w(s,x) = \overline{w}(-s,x)$ and $\overline{f}(s,x) = f(s,x) = \overline{f}(-s,x)$.

Then defining

$$y(t,x) = \int_{\mathbb{R}^1} \overline{v}(t,s)\overline{w}(s,x)\mathrm{d}s,$$
$$u(t,x) = \int_{\mathbb{R}^1} \overline{v}(t,s)\overline{f}(s,x)\mathrm{d}s,$$

we have

$$\begin{aligned} y_t - \Delta y &= \int_{\mathbb{R}^1} \overline{v}_t(t,s)\overline{w}(s,x)\mathrm{d}s - \int_{\mathbb{R}^1} \overline{v}(t,s)\Delta_x\overline{w}(s,x)\mathrm{d}s \\ &= \int_{(-T',T')} \overline{v}_{ss}(t,s)\overline{w}(s,x)\mathrm{d}s - \int_{(-T',T')} \overline{v}(t,s)\Delta_x\overline{w}(s,x)\mathrm{d}s \\ &= \int_{(-T',T')} \overline{v}(t,s)\overline{w}_{ss}(s,x)\mathrm{d}s - \int_{(-T',T')} \overline{v}(t,s)\Delta_x\overline{w}(s,x)\mathrm{d}s \\ &= \int_{(-T',T')} \overline{v}(t,s)(\overline{w}_{ss}(s,x) - \Delta_x\overline{w}(s,x))\mathrm{d}s \\ &= \int_{(-T',T')} \overline{v}(t,s)\chi_{(-T,\ T)\times E}\overline{f}\mathrm{d}s \\ &= \chi_E \int_{(-T',T')} \overline{v}(t,s)\overline{f}\mathrm{d}s, \\ &= \chi_E u. \end{aligned} \tag{3.9}$$

Noticing (3.5), it follows that

$$y(t,x) = 0, \quad \text{on } (0,T)\times\partial\Omega$$

and

$$y(0,x) = \int_{\mathbb{R}^1} \overline{v}(0,s)\overline{w}(s,x)\mathrm{d}s = y_0(x),$$

which satisfy (1.1).

Furthermore, we also have

$$y(T,x) = \int_{\mathbb{R}^1} \overline{v}(T,s)\overline{w}(s,x)\mathrm{d}s = 0.$$

Now we have proved Theorem 2.2 when $y_0 \in H_0^1(\Omega)$. However, due to the regularizing effect of the heat equation, as soon as we let the heat equation evolve freely (without control) during an arbitrarily short time interval, even if the initial data lie in $L^2(\Omega)$, the solution enters $H_0^1(\Omega)$, and then the above result applies. This completes the proof of Theorem 2.2.

Remark 3.1 In [10, 17], the authors showed that the null controllability of heat equations may be obtained as a singular limit of the exact controllability properties of singularly perturbed damped wave equations. But both of their methods depend on the Lebeau-Rabbiano inequality, i.e., for all $\{a_i\} \in l^2$ and for all $\mu > 0$,

$$C_1 \mathrm{e}^{-C_2\sqrt{\mu}} \sum_{\lambda_i \le \mu} |a_i|^2 \le \int_E \Big| \sum_{\lambda_i \le \mu} a_i \phi_i(x) \Big|^2 \mathrm{d}x, \tag{3.10}$$

where λ_i and ϕ_i are the eigenvalue and the corresponding eigenfunction of $-\Delta$ with Dirichlet boundary conditions respectively. Since the inequality (3.10) only holds when the controller E is an open subset of Ω, we cannot use this method to prove our result.

4 Case for High Dimensions

We first prove Theorem 2.3. To achieve it, we need some preparations.

By Lebesgue Lemma, almost all the points of a measurable set are Lebesgue points. Using this idea, we can easily prove the following lemma.

Lemma 4.1 *For a measurable subset E, $|E| > 0$ and $\delta \in (0,1)$, there must exist I, a rectangular subdomain of Ω, such that*

$$\frac{|E \cap I|}{|I|} > 1 - \delta. \tag{4.1}$$

The proof of this lemma can be found in any books on real analysis, for example in [14].

Next we introduce the following lemma called Harnack inequality of parabolic equations.

Lemma 4.2 $\forall\, 0 \le t_1 < t_2 < T,\ I \subset\subset \Omega,\ \ \varphi \ge 0$ *solves* (1.2) *and*

$$\sup_{x\in I} \varphi(x, t_2) \le C \inf_{x \in I} \varphi(x, t_1), \tag{4.2}$$

where C only depends on I, t_1 and t_2.

We refer to [4, p. 370] for the proof.

Remark 4.1 From the proof of Lemma 4.2, we can see C only depends on I and $|t_1 - t_2|$. So for all $t \in (0, T)$ and $\delta > 0$ satisfying $t + \delta < T$, (4.2) can be rewritten as

$$\sup_{x\in I} \varphi(x, t+\delta) \le C(I, \delta) \inf_{x\in I} \varphi(x, t). \tag{4.3}$$

Define

$$P(\Omega) := \{\varphi_T : \text{the solution to (1.2) } \varphi \ge 0\}.$$

Proof of Theorem 2.3 We only prove the theorem when $\varphi_T \in P(\Omega)$. By (4.3) and under the conditions of Lemma 4.2, we can easily get

$$\sup_{x\in I} \varphi^2(x, t+\delta) \le C(I, \delta) \inf_{x\in I} \varphi^2(x, t). \tag{4.4}$$

Integrating t from 0 to $T-\delta$, one has

$$\int_0^{T-\delta} \sup_{x\in I} \varphi^2(x,t+\delta)\mathrm{d}t \le C^2(I,\delta)\int_0^{T-\delta} \inf_{x\in I} \varphi^2(x,t)\mathrm{d}t.$$

Choosing $I \subset\subset \Omega$ satisfying (4.1) and then multiplying $|I|$ by each side and paying attention to Lemma 4.1, we find

$$\begin{aligned}|I|\int_\delta^T \sup_{x\in I}\varphi^2(x,t)\mathrm{d}t &\le |I|C^2(I,\delta)\int_0^T \inf_{x\in I}\varphi^2(x,t)\mathrm{d}t\\ &= \frac{|I|}{|E\cap I|}C^2(I,\delta)\int_{E\cap I}\mathrm{d}x\int_0^T \inf_{x\in I}\varphi^2(x,t)\mathrm{d}t\\ &\le (1-\delta)^{-1}C^2(I,\delta)\int_0^T\int_{E\cap I}\varphi^2(x,t)\mathrm{d}x\mathrm{d}t\\ &\le C(I,\delta)\int_0^T\int_E \varphi^2(x,t)\mathrm{d}x\mathrm{d}t. \qquad (4.5)\end{aligned}$$

On the other side,

$$|I|\int_\delta^T \sup_{x\in I}\varphi^2(x,t)\mathrm{d}t \ge \int_\delta^T\int_I \varphi^2(x,t)\mathrm{d}x\mathrm{d}t. \qquad (4.6)$$

By (4.5) and (4.6), we obtain

$$\int_\delta^T\int_I \varphi^2(x,t)\mathrm{d}x\mathrm{d}t \le C\int_0^T\int_E \varphi^2(x,t)\mathrm{d}x\mathrm{d}t. \qquad (4.7)$$

Because I is a rectangular region, we have the observability inequality (see [4, 6, 7] for the proof)

$$\int_\Omega \varphi^2(x,0)\mathrm{d}x \le C\int_0^T\int_I \varphi^2(x,t)\mathrm{d}x\mathrm{d}t. \qquad (4.8)$$

We also have the energy inequality

$$\int_\Omega \varphi^2(x,t_1)\mathrm{d}x \le \int_\Omega \varphi^2(x,t_2)\mathrm{d}x, \quad \forall\, 0\le t_1\le t_2\le T. \qquad (4.9)$$

According to (4.7)–(4.9), one has

$$\begin{aligned}\int_0^T\int_I \varphi^2\mathrm{d}x\mathrm{d}t &\le \int_0^\delta\int_\Omega \varphi^2\mathrm{d}x\mathrm{d}t + \int_\delta^{\mathrm{T}}\int_I \varphi^2\mathrm{d}x\mathrm{d}t\\ &\le \delta\int_\Omega \varphi^2(x,\delta)\mathrm{d}x\mathrm{d}t + \int_\delta^T\int_I \varphi^2\mathrm{d}x\mathrm{d}t\\ &\le \delta C\int_\delta^T\int_I \varphi^2\mathrm{d}x\mathrm{d}t + \int_\delta^T\int_I \varphi^2\mathrm{d}x\mathrm{d}t\\ &\le (1+\delta C)\int_\delta^T\int_I \varphi^2\mathrm{d}x\mathrm{d}t. \qquad (4.10)\end{aligned}$$

Combining (4.7)–(4.8) and (4.10), we have

$$\int_\Omega \varphi^2(x,0)\mathrm{d}x \le C(I,T)\int_0^T\int_E \varphi^2(x,t)\mathrm{d}x\mathrm{d}t,$$

as desired.

Next, let us recall the following fundamental result in the Calculus of Variations which is a consequence of the so-called Direct Method of the Calculus of Variations.

Lemma 4.3 *Let X be a reflexive Banach space, K a closed convex subset of X and φ: $K \to \mathbb{R}$ a function with the following properties:*
(1) *φ is convex*;
(2) *φ is lower semi-continuous*;
(3) *If K is unbounded, then φ is coercive, i.e.,*

$$\lim_{\|x\|\to\infty} \varphi(x) = \infty.$$

Then φ attains its minimum in K, i.e., there exists an x_0 in K such that

$$\varphi(x_0) = \min_{x\in K} \varphi(x).$$

Proof of Theorem 2.4 We only prove the first inequality.

Assuming $u_0 \in L^2(\Omega)$ for all $\varepsilon > 0$, we define

$$J_\varepsilon(\varphi_T) = \frac{1}{2}\int_0^T \int_E \varphi^2(x,t)\mathrm{d}x\mathrm{d}t + \int_\Omega \varphi(x,0)u_0(x)\mathrm{d}x + \varepsilon\|\varphi_T\|_{L^2} \tag{4.11}$$

over $L^2_+(\Omega)$.

Due to Lemmas 3.2 and 4.3, we know that J_ε has a minimizer in L^2_+. Indeed, L^2_+ is closed and it is easy to check that J_ε is continuous and strictly convex. Lemma 3.2 guarantees the coercivity of J_ε (see the proof in [16]). Hence by Lemma 4.3, the above conclusion follows.

For all $\varepsilon > 0$, we suppose that J_ε attains its minimum value at $\widehat{\varphi}^\varepsilon_T \in L^2_+$. Then, using a standard variational method, we have

$$\int_0^T \int_E \widehat{\varphi}^\varepsilon(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_\Omega \varphi(x,0)u_0(x)\mathrm{d}x \geq -\varepsilon\|\varphi_T\| \tag{4.12}$$

for any $\varphi_T \in L^2_+(\Omega)$.

Set the control

$$f^\varepsilon = \widehat{\varphi}^\varepsilon, \quad \text{in } \Omega \times (0,T).$$

By the fine regularity of the solution to the heat equation, we know

$$f^\varepsilon \in L^2(\Omega \times (0,T)).$$

Multiplying (1.1) with the solution to (1.3) associated the final data φ_T and then integrating by parts, one has

$$\int_0^T \int_E \widehat{\varphi}^\varepsilon(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_\Omega \varphi(x,0)u_0(x)\mathrm{d}x = \int_\Omega \varphi_T(x)u^\varepsilon(x,T)\mathrm{d}x \quad \text{for all } \varphi_T \in L^2_+.$$

Noticing (4.12), we deduce

$$\int_\Omega \varphi_T(x)u^\varepsilon(x,T)\mathrm{d}x \geq -\varepsilon\|\varphi_T\| \quad \text{for all } \varphi_T \in L^2_+. \tag{4.13}$$

In order to see that the controls f_ε are uniformly bounded, we have to use their structures. At the minimizer $\widehat{\varphi}_T^\varepsilon$, we have $J_\varepsilon(\widehat{\varphi}_T^\varepsilon) \le J_\varepsilon(0) = 0$. This implies that

$$\int_0^T \int_E |\widehat{\varphi}^\varepsilon|^2 \mathrm{d}x\mathrm{d}t \le 2\|u_0\|_{L^2\Omega}\|\widehat{\varphi}^\varepsilon(x,0)\|_{L^2(\Omega)}.$$

By Theorem 2.3, we have

$$\int_0^T \int_E |f^\varepsilon|^2 \mathrm{d}x\mathrm{d}t = \int_0^{\mathrm{T}} \int_E |\widehat{\varphi}^\varepsilon|^2 \mathrm{d}x\mathrm{d}t \le C\|u_0\|^2_{L^2(\Omega)}.$$

Hence, $\{f^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^2(E\times(0,T))$. By extracting subsequences, we have $f^\varepsilon \rightharpoonup f_1$ in $L^2(E\times(0,T))$. Using the continuous dependence of the solutions to the heat equation, we can show that $u^\varepsilon(x,T)$ converges to $u(x,T)$ weakly in $L^2(\Omega)$. In view of (4.13), this implies that

$$\int_\Omega \varphi_T(x)u(x,T)\mathrm{d}x \ge 0 \quad \text{for all } \varphi_T \in L^2_+.$$

Then we deduce $u(\,\cdot\,,T;f_1) \ge 0$ in Ω.

Proof of Theorem 2.5 Suppose that u_0 satisfies the conditions of the theorem. Then for all $\varepsilon > 0$, there exists an approximate control $f^\varepsilon \ge 0$, such that the solution u_ε to (1.1) satisfies the condition

$$\|u_\varepsilon\|_{L^2(\Omega)} \le \varepsilon. \tag{4.14}$$

By Theorem 2.3 and condition (2.4), we know that the approximate controls $\{f^\varepsilon\}_\varepsilon$ are uniform bounded. Using the same argument in the above proof, we can show $u^\varepsilon(x,T)$ converges to $u(x,T)$ weakly in $L^2(\Omega)$. Then (4.14) implies

$$\Big|\int_\Omega u_\varepsilon(x,T)\psi(x)\mathrm{d}x\Big| \le \|u_\varepsilon\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} \le \varepsilon\|\psi\|_{L^2(\Omega)} \quad \text{for any } \psi \in L^2(\Omega).$$

It means

$$\int_\Omega u(x,T)\psi(x)\mathrm{d}x = 0 \quad \text{for any } \psi \in L^2(\Omega).$$

This completes the proof of Theorem 2.5.

Acknowledgement The author wants to thank Professor Xu Zhang for organizing the summer school in Sichuan where the author first knew this problem. The author also acknowledges Professors Hongwei Lou and Xu Zhang for many helpful guidance and suggestions.

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