Chin. Ann. Math.
33B(2), 2012, 161–178
DOI: 10.1007/s11401-012-0705-4

Chinese Annals of Mathematics, Series B

Maximal Dimension of Invariant Subspaces to Systems of Nonlinear Evolution Equations*

Shoufeng SHEN[1] Changzheng QU[2] Yongyang JIN[1] Lina JI[3]

Abstract In this paper, the dimension of invariant subspaces admitted by nonlinear systems is estimated under certain conditions. It is shown that if the two-component nonlinear vector differential operator $\mathbb{F} = (F^1, F^2)$ with orders $\{k_1, k_2\}$ $(k_1 \geq k_2)$ preserves the invariant subspace $W^1_{n_1} \times W^2_{n_2}$ $(n_1 \geq n_2)$, then $n_1 - n_2 \leq k_2$, $n_1 \leq 2(k_1 + k_2) + 1$, where $W^q_{n_q}$ is the space generated by solutions of a linear ordinary differential equation of order n_q $(q = 1, 2)$. Several examples including the (1+1)-dimensional diffusion system and Itô's type, Drinfel'd-Sokolov-Wilson's type and Whitham-Broer-Kaup's type equations are presented to illustrate the result. Furthermore, the estimate of dimension for m-component nonlinear systems is also given.

Keywords Invariant subspace, Nonlinear PDEs, Exact solution, Symmetry, Dynamical system
2000 MR Subject Classification 37J15, 37K35, 35Q53, 35K55

1 Introduction

Symmetry related methods are powerful in the study of nonlinear partial differential equations (PDEs) (see [1–2]). Group-invariant solutions stemming from symmetries play important roles in the study of their asymptotical behavior, blow up and geometric properties etc. These solutions can also be used to justify the numerical scheme of solving PDEs. The invariant subspace method (see [3]) was found to be very effective in seeking for exact solutions of nonlinear PDEs. Indeed, various invariant subspaces to a number of nonlinear evolution equations have been obtained (see [3–6] and references therein). Recently, Galaktionov and Svirshchevskii [3] provided a systematic approach to invariant subspaces of nonlinear evolution equations, and they obtained many interesting exact solutions of nonlinear evolution equations in mechanics and physics.

Manuscript received October 14, 2011. Revised December 13, 2011.
[1]Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, China.
[2]Corresponding author. Department of Mathematics, Ningbo University, Ningbo 315211, Zhejiang, China. E-mail: quchangzheng@nbu.edu.cn
[3]Department of Information and Computational Science, Henan Agricultural University, Zhengzhou 450002, China.
*Project supported by the National Natural Science Foundation of China for Distinguished Young Scholars (No. 10925104), the National Natural Science Foundation of China (No.11001240), the Doctoral Program Foundation of the Ministry of Education of China (No. 20106101110008) and the Zhejiang Provincial Natural Science Foundation of China (Nos. Y6090359, Y6090383).

Let us give a brief account of the invariant subspace method. Consider the (1 + 1)-dimensional nonlinear evolution equation

$$u_t = F(x, u, u_x, \cdots, u_{kx}), \tag{1.1}$$

where $F(x, u, u_x, \cdots, u_{kx})$ is a given sufficiently smooth function of its arguments and

$$u_{jx} = \frac{\partial^j u}{\partial u^j}, \quad j = 1, \cdots, k.$$

Let

$$\{f_j(x) \mid j = 1, 2, \cdots, n\}$$

be a finite set of linearly independent functions and W_n denote their linear span

$$W_n = \mathcal{L}\{f_1(x), f_2(x), \cdots, f_n(x)\}.$$

The subspace W_n is said to be invariant under the given operator F, namely, F is said to preserve W_n if $F(W_n) \subseteq W_n$, which means

$$F\Big[\sum_{j=1}^{n} C_j f_j(x)\Big] = \sum_{j=1}^{n} \Psi_j(C_1, C_2, \cdots, C_n) f_j(x)$$

for any $(C_1, C_2, \cdots, C_n) \in \mathbb{R}^n$. It follows that if the linear subspace W_n is invariant with respect to F, then (1.1) has an exact solution of the form

$$u(x, t) = \sum_{j=1}^{n} C_j(t) f_j(x),$$

where the coefficients $\{C_j(t) \mid j = 1, 2, \cdots, n\}$ satisfy the n-dimensional dynamical system

$$\frac{\mathrm{d}C_j(t)}{\mathrm{d}t} = \Psi\left(C_1(t), C_2(t), \cdots, C_n(t)\right), \quad j = 1, 2, \cdots, n.$$

Assume that the invariant subspace W_n is defined as the space of solutions to a linear nth-order ordinary differential equation (ODE)

$$L[y] \equiv y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = 0.$$

Then the invariant condition with respect to F takes the form

$$L\left[F[u]\right]|_{[H]} = 0,$$

where $[H]$ denotes the equation $L[u] = 0$ and its differential consequences with respect to x. It turns out that the invariant subspace method is related to the Lie-Bäcklund symmetry and the conditional Lie-Bäcklund symmetry. Some other related approaches were given in [7–11]. By using the invariant subspace method, many different types of exact solutions to nonlinear

PDEs can be obtained. For instance in [5], Galaktionov utilized the invariant subspace method to obtain exact solutions to nonlinear evolution equations with quadratic nonlinearities. He showed that exact solutions to the quasi-linear heat equations

$$u_t = (u^{-\frac{4}{3}} u_x)_x - a u^{-\frac{1}{3}} + b u^{\frac{7}{3}} + cu \equiv \overline{F}_1[u], \quad u > 0,\ a, b, c \in \mathbb{R}$$

can be constructed in the linear subspaces of polynomial or trigonometric form, which are admitted by the spatial operator $\overline{F}_1[u]$. Interestingly, the N-soliton solutions to integrable equations such as the KdV equation, mKdV equation, nonlinear Schrödinger equation and sine-Gordon equation derived by the Hirota's bilinear method (see [12]) belong to a linear subspace of exponential functions in the sense of change of variables (see [3]). The linear superposition principle for constructing N-solitons of Hirota bilinear equations is related to the invariant subspace method (see [13–14]).

The invariant subspace method was also utilized to construct exact solutions to nonlinear systems. In [15], the authors performed a classification to the systems of nonlinear parabolic equations

$$\begin{aligned} u_t &= [f(u,v)u_x + p(u,v)v_x]_x + r(u,v), \\ v_t &= [g(u,v)u_x + q(u,v)v_x]_x + s(u,v) \end{aligned}$$

based on the invariant subspaces defined by linear ODEs. Furthermore, in [16], the nonlinear systems of the form

$$\mathbb{U}_t = \mathbb{F}[\mathbb{U}] \equiv \big(F^1[\mathbb{U}], F^2[\mathbb{U}], \cdots, F^m[\mathbb{U}]\big) \tag{1.2}$$

are considered, where

$$\mathbb{U} = (u^1, \cdots, u^m) \in \mathbb{R}^m$$

and

$$F^q[\mathbb{U}] = F^q\left(x, u^1, \cdots, u^m, \cdots, u^1_k, \cdots, u^m_k\right) \tag{1.3}$$

are given sufficiently smooth functions. Throughout the paper, we use the notations

$$u^q_0 = u^q(x,t), \quad u^q_j = \frac{\partial^j u^q(x,t)}{\partial x^j}, \quad q = 1, \cdots, m,\ j = 1, 2, \cdots .$$

Let $\mathcal{W}$ denote a new linear subspace

$$W^1_{n_1} \times \cdots \times W^m_{n_m},$$

where

$$W^q_{n_q} = \mathcal{L}\{f^q_1(x), \cdots, f^q_{n_q}(x)\} \equiv \Big\{\sum_{j=1}^{n_q} C^q_j f^q_j(x)\Big\}, \quad q = 1, \cdots, m$$

and

$$f_1^j(x), \cdots, f_{n_j}^j(x)$$

are linearly independent. If the above vector operator $\mathbb{F}$ fulfills the condition

$$\mathbb{F}: W_{n_1}^1 \times \cdots \times W_{n_m}^m \to W_{n_1}^1 \times \cdots \times W_{n_m}^m,$$

that is,

$$F^q: W_{n_1}^1 \times \cdots \times W_{n_m}^m \to W_{n_q}^q, \quad q = 1, \cdots, m$$

satisfies

$$\begin{aligned} & F^q\Big[\sum_{j=n_1}^{n_1} C_j^1 f_j^1(x), \cdots, \sum_{j=n_m}^{n_m} C_j^m f_j^m(x)\Big] \\ = & \sum_{j=1}^{n_q} \Psi_j^q(C_1^1, \cdots, C_{n_1}^1, \cdots, C_1^m, \cdots, C_{n_m}^m) f_j^q(x), \end{aligned}$$

then the vector operator $\mathbb{F}$ is said to admit the invariant subspaces $\mathcal{W}$, or $\mathcal{W}$ is said to be invariant under the given operator $\mathbb{F}$.

If the subspace $\mathcal{W}$ is admitted by the vector operator $\mathbb{F}[\mathbb{U}]$, then (1.2) possesses the exact solution of the form

$$u^q = \sum_{j=1}^{n_q} C_j^q(t) f_j^q(x), \quad q = 1, \cdots, m$$

with $C_j^q(t)$ satisfying the following system of ODEs:

$$\frac{\mathrm{d}C_j^q(t)}{\mathrm{d}t} = \Psi_j^q(C_1^1(t), \cdots, C_{n_1}^1(t), \cdots C_1^m(t), \cdots, C_{n_m}^m(t)), \quad j = 1, \cdots, n_q, \ q = 1, \cdots, m.$$

Assume that

$$W_{n_q}^q = \mathcal{L}\{f_1^q(x), \cdots, f_{n_q}^q(x)\}$$

is generated by solutions to the linear n_qth-order ODE

$$L^q[y_q] = y_q^{(n_q)} + a_{n_q-1}^q(x) y_q^{(n_q-1)} + \cdots + a_1^q(x) y_q' + a_0^q(x) y_q = 0, \quad q = 1, \cdots, m. \tag{1.4}$$

It follows that the invariant condition for the subspace $\mathcal{W}$ with respect to $\mathbb{F}$ is

$$L^q[F^q[\mathbb{U}]]|_{[H_1]\bigcap\cdots\bigcap[H_m]} = 0, \quad q = 1, \cdots, m, \tag{1.5}$$

where we denote by $[H_q]$ the equation $L^q[u^q] = 0$ and its differential consequences with respect to x.

The estimate of dimension for invariant subspaces of nonlinear systems plays an important role in the approach. Once the maximal dimension of invariant subspaces is determined, we are able to provide a complete classification to the equation and obtain the corresponding exact solutions to the equations under consideration. The problem of maximal dimension of invariant subspaces was firstly posed and solved for the scalar case in [3]. For the scalar case, the maximal dimension of invariant subspaces is not greater than $2k+1$, where k is the order of the operator F in (1.1). For the m-component nonlinear system (1.2) with (1.3) under certain conditions (see [16]), the dimension of invariant subspace

$$\mathcal{W} = W^1_{n_1} \times \cdots \times W^m_{n_m}, \quad n_1 \leq n_2 \leq \cdots \leq n_m$$

satisfies

$$n_{i+1} - n_i \leq k, \quad i = 1, \cdots, m-1, \quad n_m \leq 2mk + 1.$$

Now a question arises: What is the maximal dimension of the invariant subspaces corresponding to the following more generalized vector operator $\mathbb{F}[\mathbb{U}]$:

$$F^q[\mathbb{U}] = F^q(x, u^1, \cdots, u^m, \cdots, u^1_{k_q}, \cdots, u^m_{k_q}), \quad q = 1, \cdots, m? \tag{1.6}$$

In this paper, we estimate the dimension of invariant subspaces admitted by nonlinear systems with different orders. The outline of this paper is as follows. In Section 2, we show that if a two-component nonlinear vector operator $\mathbb{F}[\mathbb{U}]$ with orders $k_1 \geq k_2$ satisfies certain conditions and admits the invariant subspace

$$\mathcal{W} = W^1_{n_1} \times W^2_{n_2}, \quad n_1 \geq n_2,$$

then

$$n_1 - n_2 \leq k_2, \quad n_1 \leq 2(k_1 + k_2) + 1.$$

A brief proof for m-component case will be given in Appendix. In Section 3, we provide some examples for $m = 2$ to illustrate the application of the estimate. Section 4 presents some conclusive remarks on this work.

2 Estimate of Dimension for Nonlinear Systems

In this section, we estimate the dimension of invariant subspaces $\mathcal{W}$ admitted by the two-component nonlinear system

$$\begin{aligned} u^1_t &= F^1(x, u^1, u^2, \cdots, u^1_{k_1}, u^2_{k_1}), \\ u^2_t &= F^2(x, u^1, u^2, \cdots, u^1_{k_2}, u^2_{k_2}), \end{aligned} \tag{2.1}$$

where the operators F^1 and F^2 are respectively the k_1th order and k_2th order operators, namely,

$$\begin{aligned} &(F^1_{u^1_{k_1}})^2 + (F^1_{u^2_{k_1}})^2 \neq 0, \\ &(F^2_{u^1_{k_2}})^2 + (F^2_{u^2_{k_2}})^2 \neq 0. \end{aligned} \tag{2.2}$$

That the vector operator $\mathbb{F} = (F^1, F^2)$ is really coupled means

$$\begin{aligned} &(F^1_{u^2_0})^2 + (F^1_{u^2_1})^2 + \cdots + (F^1_{u^2_{k_1}})^2 \neq 0, \\ &(F^2_{u^1_0})^2 + (F^2_{u^1_1})^2 + \cdots + (F^2_{u^1_{k_2}})^2 \neq 0. \end{aligned} \tag{2.3}$$

The vector operator $\mathbb{F} = (F^1, F^2)$ is a nonlinear vector differential operator, i.e., for some $i_0, j_0, l_0 \in \{1, 2\}$, $p_0, q_0 \in \{0, 1, \cdots, k_{i_0}\}$, there holds

$$\frac{\partial F^{i_0}}{\partial u^{j_0}_{p_0} \partial u^{l_0}_{q_0}} \neq 0. \tag{2.4}$$

Without loss of generality, we assume that

$$k_1 \geq k_2.$$

If a nonlinear vector operator preserves the invariant subspace $\mathcal{W}$ specified in the introduction and each component of $\mathcal{W}$ belongs to a different scalar subspace, then we have the following theorem.

Theorem 2.1 *Let $\mathbb{F} = (F^1, F^2)$ be a vector operator with assumptions* (2.2)–(2.4)*, $k_1 \geq k_2$. If the operator $\mathbb{F}$ admits the invariant subspace $W^1_{n_1} \times W^2_{n_2}$ $(n_1 \geq n_2 > 0)$, then there hold*

$$n_1 - n_2 \leq k_2, \quad n_1 \leq 2\,(k_1 + k_2) + 1. \tag{2.5}$$

Proof We prove the theorem following the lines of the proof of Theorem 1 in [16]. We first prove

$$n_1 - n_2 \leq k_2$$

by contradiction. Assume that

$$n_1 - n_2 > k_2, \quad \text{i.e.,} \quad n_1 > n_2 + k_2$$

and $\mathbb{F}$ preserves the invariant subspace $W^1_{n_1} \times W^2_{n_2}$ defined by (1.4). Then the following identities

$$D^{n_1} F^1 = -[a^1_{n_1-1}(x) D^{n_1-1} F^1 + \cdots + a^1_1(x) D F^1 + a^1_0(x) F^1], \tag{2.6}$$

$$D^{n_2} F^2 = -[a^2_{n_2-1}(x) D^{n_2-1} F^2 + \cdots + a^2_1(x) D F^2 + a^2_0(x) F^2] \tag{2.7}$$

hold on the solution manifold (1.4). Differentiating $F^i(x, u^1, u^2, \cdots, u^1_{k_i}, u^2_{k_i})$ with respect to x and keeping the leading linear and quadratic terms yield

$$\begin{aligned} DF^i &= \sum_{j=1}^{2} u^j_{k_i+1} \frac{\partial F^i}{\partial u^j_{k_i}} + \cdots, \\ D^2 F^i &= \sum_{j=1}^{2} u^j_{k_i+2} \frac{\partial F^i}{\partial u^j_{k_i}} + \sum_{j,l=1}^{2} u^j_{k_i+1} u^l_{k_i+1} \frac{\partial^2 F^i}{\partial u^j_{k_i} \partial u^l_{k_i}} + \cdots, \\ D^3 F^i &= \sum_{j=1}^{2} u^j_{k_i+3} \frac{\partial F^i}{\partial u^j_{k_i}} + 3 \sum_{j,l=1}^{2} u^j_{k_i+2} u^l_{k_i+1} \frac{\partial^2 F^i}{\partial u^j_{k_i} \partial u^l_{k_i}} + \cdots. \end{aligned}$$

Here D denotes the operator of total derivatives with respect to x. By induction, for any $p \geq 4$, we have

$$D^p F^i = \sum_{j=1}^{2} u^j_{k_i+p} \frac{\partial F^i}{\partial u^j_{k_i}} + \sum_{j,l=1}^{2} \Big[\Big(\sum_{s=1}^{[\frac{p}{2}]-1} C^s_p u^j_{k_i+p-s} u^l_{k_i+s} + \gamma C_p^{[\frac{p}{2}]} u^j_{k_i+p-[\frac{p}{2}]} u^l_{k_i+[\frac{p}{2}]} \Big) \frac{\partial^2 F^i}{\partial u^j_{k_i} \partial u^l_{k_i}} \Big] + \cdots, \tag{2.8}$$

where $[\frac{p}{2}]$ denotes its integer part, and $\gamma = \frac{1}{2}$ for p even, $\gamma = 1$ for p odd. In particular, for $p = n_2$ and $i = 2$, we have

$$D^{n_2} F^2 = \sum_{j=1}^{2} u^j_{k_2+n_2} \frac{\partial F^2}{\partial u^j_{k_2}} + \sum_{j,l=1}^{2} \Big[\Big(\sum_{s=1}^{[\frac{n_2}{2}]-1} C^s_{n_2} u^j_{k_2+n_2-s} u^l_{k_2+s} + \gamma C_{n_2}^{[\frac{n_2}{2}]} u^j_{k_2+n_2-[\frac{n_2}{2}]} u^l_{k_2+[\frac{n_2}{2}]} \Big) \frac{\partial^2 F^2}{\partial u^j_{k_2} \partial u^l_{k_2}} \Big] + \cdots. \tag{2.9}$$

Note that the term containing the derivative of u^1 with the maximal order $k_2 + n_2 (< n_1)$ only appears in $D^{n_2} F^2$ and does not appear in the derivatives $D^p F^2$, $p < n_2$. Taking into account (2.9) and using (2.7), equating the coefficients of $u^1_{k_2+n_2}$ with zero implies that

$$\frac{\partial F^2}{\partial u^1_{k_2}} = 0.$$

In a similar way to the above computation, we have

$$\frac{\partial F^2}{\partial u^1_{k_2-1}} = 0.$$

By induction, it follows that

$$\frac{\partial F^2}{\partial u^1_j} = 0, \quad j = 0, 1, \cdots, k_2,$$

which implies that F^2 does not depend on u^1_j $(j = 0, 1, \cdots, k_2)$. This contradicts the assumption (2.3).

Next, we prove

$$n_1 \leq 2(k_1 + k_2) + 1$$

by contradiction, that is, we have

$$n_1 > 2(k_1 + k_2) + 1, \quad \text{i.e.,} \quad n_1 \geq 2(k_1 + k_2) + 2.$$

By $n_1 - n_2 \leq k_2$, we get

$$n_2 > 2k_1 + k_2 + 1, \quad \text{i.e.,} \quad n_2 \geq 2k_1 + k_2 + 2.$$

The sum in the square brackets in (2.9) is composed of the quadratic summands that exhibit the maximal total order of both derivatives

$$(k_2+n_2-s)+(k_2+s)=2k_2+n_2.$$

Keeping in (2.9) only the quadratical terms containing at least one derivative of the order not less than n_2-1 gives

$$D^{n_2}F^2=\sum_{j,l=1}^{2}\Big[\sum_{s=1}^{[\frac{n_2}{2}]}\alpha_s u^j_{k_2+n_2-s}u^l_{k_2+s}\frac{\partial^2F^2}{\partial u^j_{k_2}\partial u^l_{k_2}}\Big]+\cdots, \tag{2.10}$$

where

$$\alpha_s=C^s_{n_2},\quad s=1,2,\cdots,\Big[\frac{n_2}{2}\Big]-1,\quad \alpha_{[\frac{n_2}{2}]}=\gamma C^{[\frac{n_2}{2}]}_{n_2}.$$

By (1.4), $u^i_{n_i+k}$ for $k\ge 0$ can be linearly expressed in terms of $u^i_{n_i-1},\cdots,u^i_0$, $i=1,2$. Keeping in the square brackets in (2.10) the terms containing $u^j_{n_2-1}$, we have

$$D^{n_2}F^2=\sum_{j,l=1}^{2}\Big[\alpha_{k_2+1}u^j_{n_2-1}u^l_{2k_2+1}\frac{\partial^2F^2}{\partial u^j_{k_2}\partial u^l_{k_2}}\Big]+\cdots. \tag{2.11}$$

Clearly,

$$n_2-1<n_i,\quad 2k_2+1<n_i,\quad i=1,2.$$

There exist the following quadratical terms in (2.11) of the maximal total order $2k_2+n_2$:

$$\alpha_{k_2+1}u^j_{n_2-1}u^l_{2k_2+1}\frac{\partial^2F^2}{\partial u^j_{k_2}\partial u^l_{k_2}},\quad j,l=1,2.$$

It is easy to see that such terms do not appear in the derivatives D^pF^2 $(p<n_2)$. In order to make (2.7) valid, we should set

$$\frac{\partial^2F^2}{\partial u^j_{k_2}\partial u^l_{k_2}}=0,\quad j,l=1,2.$$

Similarly, by induction, we get

$$\frac{\partial^2F^2}{\partial u^j_p\partial u^l_q}=0,\quad j,l\in\{1,2\},\ p,q\in\{0,1,\cdots,k_2\},$$

which implies that F^2 depends on u^j_p $(j=1,2,\ p=0,1,\cdots,k_j)$ linearly.

Furthermore, for F^1, we have

$$\begin{aligned}D^{n_1}F^1=&\sum_{j=1}^{2}u^j_{k_1+n_1}\frac{\partial F^1}{\partial u^j_{k_1}}+\sum_{j,l=1}^{2}\Big[\Big(\sum_{s=1}^{[\frac{n_1}{2}]-1}C^s_{n_1}u^j_{k_1+n_1-s}u^l_{k_1+s}\\&+\gamma C^{[\frac{n_1}{2}]}_{n_1}u^j_{k_1+n_1-[\frac{n_1}{2}]}u^l_{k_1+[\frac{n_1}{2}]}\Big)\frac{\partial^2F^1}{\partial u^j_{k_1}\partial u^l_{k_1}}\Big]+\cdots,\end{aligned} \tag{2.12}$$

which can be written as

$$D^{n_1}F^1 = \sum_{j,l=1}^{2}\Big[\sum_{s=1}^{k_1+k_2+1}\beta_s u^j_{k_1+n_1-s}u^l_{k_1+s}\frac{\partial^2 F^1}{\partial u^j_{k_1}\partial u^l_{k_1}}\Big]+\cdots, \tag{2.13}$$

where

$$\beta_s = C^s_{n_1}, \quad s=1,2,\cdots,k_1+k_2$$

and

$$\beta_{k_1+k_2+1} = \begin{cases} C^{k_1+k_2+1}_{n_1}, & \text{if } n_1 > 2(k_1+k_2)+2, \\ \frac{1}{2}C^{k_1+k_2+1}_{n_1}, & \text{if } n_1 = 2(k_1+k_2)+2. \end{cases}$$

Clearly,

$$n_1-k_2-1<n_i, \quad 2k_1+k_2+1<n_i, \quad i=1,2.$$

Note that the quadratical terms in (2.13) with the maximal total order $2k_1+n_1$ are

$$\beta_{k_1+k_2+1}u^j_{n_1-k_2-1}u^l_{2k_1+k_2+1}\frac{\partial^2 F^1}{\partial u^j_{k_1}\partial u^l_{k_1}}, \quad j,l=1,2.$$

It is easy to see that such terms do not appear in the derivatives D^pF^1 $(p<n_1)$. In order to make (2.6) valid, we should set

$$\frac{\partial^2 F^1}{\partial u^j_{k_1}\partial u^l_{k_1}}=0, \quad j,l=1,2.$$

Similarly, by induction, it follows that

$$\frac{\partial^2 F^1}{\partial u^j_p\partial u^l_q}=0, \quad j,l\in\{1,2\},\ p,q\in\{0,1,\cdots,k_1\},$$

which implies that F^1 depends on u^j_p $(j=1,2,\ p=0,1,\cdots,k_j)$ linearly.

This completes the proof of Theorem 2.1.

Theorem 2.2 *Let* $\mathbb{F}[\mathbb{U}]$ *be a nonlinear vector operator* (1.6) *with different orders. Assume that*

$$k_1\geq k_2\geq\cdots\geq k_m\geq 0, \tag{2.14}$$

and there exist some $l\in\{0,1,\cdots,k_j\}$, $q\in\{i_0,\cdots,m\}$ *and* $j\in\{1,\cdots,i_0-1\}$ *such that*

$$\frac{\partial F^q}{\partial u^j_l}\neq 0. \tag{2.15}$$

If the nonlinear operator $\mathbb{F}[\mathbb{U}]$ *admits the invariant subspace*

$$W^1_{n_1}\times\cdots\times W^m_{n_m}$$

with

$$n_1 \geq n_2 \geq \cdots \geq n_m > 0, \tag{2.16}$$

then we have the following estimate:

$$n_{i-1} - n_i \leq k_i, \quad i = 2, \cdots, m, \quad n_1 \leq 2\sum_{j=1}^{m} k_j + 1. \tag{2.17}$$

The proof for Theorem 2.2 is similar to that of Theorem 2.1. We will prove Theorem 2.2 in Appendix to make the article more readable.

3 Invariant Subspaces Defined by Linear ODEs

In this section, we present several examples including the (1+1)-dimensional diffusion system and Itô's type equation, Drinfel'd-Sokolov-Wilson's type equation and Whitham-Broer-Kaup type's equation (see [3, 17–19]) to illustrate how to obtain invariant subspaces and construct exact solutions to a given nonlinear system (2.1) by using the above results.

Example 3.1 (1+1)-dimensional diffusion system.

Consider the following two-component nonlinear diffusion system:

$$\begin{aligned} u_t &= (u_x + f v v_x)_x + g v^2, \\ v_t &= u_x + p u + q v, \end{aligned} \tag{3.1}$$

where the constant coefficients f and g are not simultaneously equal to zero. Theorem 2.1 implies that to give a full description of the above system admitting an invariant subspace $W^1_{n_1} \times W^2_{n_2}$ defined by (1.4) with constant coefficients, it suffices to consider the cases for (n_1, n_2):

$$\big\{(2,2),\ (3,2),\ (3,3),\ (4,3),\ (4,4),\ (5,4),\ (5,5),\ (6,5),\ (6,6),\ (7,6),\ (7,7)\big\}.$$

First, for the invariant subspace $W^1_2 \times W^2_2$ defined by

$$L_1[y] = y'' + a_1 y' + a_0 y = 0$$

and

$$L_2[z] = z'' + b_1 z' + b_0 z = 0,$$

we have the corresponding invariant criteria

$$\begin{aligned} (D^2F + a_1 DF + a_0 F)|_{[H_1]\bigcap[H_2]} &= 0, \\ (D^2G + b_1 DG + b_0 G)|_{[H_1]\bigcap[H_2]} &= 0, \end{aligned}$$

where

$$F = (u_x + fvv_x)_x + gv^2,$$
$$G = u_x + pu + qv.$$

After substituting the expressions of F and G into the above equations and replacing u_{xx} and v_{xx} respectively by $-a_1 u_x - a_0 u$ and $-b_1 v_x - b_0 v$, we have the following equations:

$$\begin{aligned}
&7fb_1^2 + a_0 f + 2g - 4fb_0 - 3a_1 fb_1 = 0,\\
&12fb_1b_0 - a_0 fb_1 - 4a_1 fb_0 + 2a_1 g - fb_1^3 - 2gb_1 + a_1 fb_1^2 = 0,\\
&4fb_0^2 + a_0 g - fb_1^2 b_0 + a_1 fb_1 b_0 - a_0 fb_0 - 2gb_0 = 0,\\
&a_1^2 - pa_1 - a_0 + b_1 p + b_0 - b_1 a_1 = 0,\\
&a_1 a_0 - pa_0 - b_1 a_0 + b_0 p = 0.
\end{aligned} \tag{3.2}$$

Solving the system, we arrive at the following classification result:

$$\begin{aligned}
&u_t = (u_x + fvv_x)_x, \quad L_1[y] = y'' + py' = 0,\\
&v_t = u_x + pu + qv, \quad L_2[z] = z'' = 0;
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&u_t = (u_x + fvv_x)_x - \frac{fp^2}{8}v^2, \quad L_1[y] = y'' + py' = 0,\\
&v_t = u_x + pu + qv, \qquad\qquad L_2[z] = z'' + \frac{p}{2}z' = 0;
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&u_t = (u_x + fvv_x)_x - 2fb_1^2 v^2, \quad L_1[y] = y'' + b_1 y' = 0,\\
&v_t = u_x + pu + qv, \qquad\qquad L_2[z] = z'' + b_1 z' = 0;
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&u_t = (u_x + fvv_x)_x - 2fa_1^2 v^2, \quad L_1[y] = y'' + a_1 y' = 0,\\
&v_t = u_x + qv, \qquad\qquad L_2[z] = z'' - a_1^2 z = 0;
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&u_t = (u_x + fvv_x)_x - \frac{fp^2}{18}v^2, \quad L_1[y] = y'' + \frac{3p}{2}y' + \frac{p^2}{2}y = 0,\\
&v_t = u_x + pu + qv, \qquad\qquad L_2[z] = z'' + \frac{p}{3}z' - \frac{p^2}{12}z = 0;
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&u_t = (u_x + fvv_x)_x, \quad L_1[y] = y'' + \frac{3p}{2}y' + \tfrac{p^2}{2}y = 0,\\
&v_t = u_x + pu + qv, \quad L_2[z] = z'' + \frac{p}{2}z' = 0;
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&u_t = (u_x + fvv_x)_x - \frac{9fp^2}{32}v^2, \quad L_1[y] = y'' + \frac{3p}{2}y' + \frac{p^2}{2}y = 0,\\
&v_t = u_x + pu + qv, \qquad\qquad L_2[z] = z'' + \frac{3p}{4}z' + \frac{p^2}{8}z = 0;
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&u_t = (u_x + fvv_x)_x - \frac{8fp^2}{9}v^2, \quad L_1[y] = y'' + \frac{5p}{3}y' + \frac{2p^2}{3}y = 0,\\
&v_t = u_x + pu + qv, \qquad\qquad L_2[z] = z'' + pz' + \frac{2p^2}{9}z = 0;
\end{aligned} \tag{3.10}$$

$$\begin{aligned} &u_t=(u_x+fvv_x)_x-\frac{fp^2}{18}v^2, \quad L_1[y]=y''+\frac{5p}{6}y'-\frac{p^2}{6}y=0,\\ &v_t=u_x+pu+qv, \qquad\qquad\quad L_2[z]=z''+\frac{p}{3}z'-\frac{p^2}{12}z=0; \end{aligned} \tag{3.11}$$

$$\begin{aligned} u_t&=(u_x+fvv_x)_x+\frac{54287\sqrt{7081}-4851733}{2039184}fp^2v^2,\\ L_1[y]&=y''+\frac{269+\sqrt{7081}}{238}py'+\frac{31+\sqrt{7081}}{238}p^2y=0,\\ v_t&=u_x+pu+qv,\\ L_2[z]&=z''+\frac{5\left(275-\sqrt{7081}\right)}{1428}pz'-\frac{106\sqrt{7081}-5129}{42483}p^2z=0; \end{aligned} \tag{3.12}$$

$$\begin{aligned} u_t&=(u_x+fvv_x)_x-\frac{54287\sqrt{7081}+4851733}{2039184}fp^2v^2,\\ L_1[y]&=y''+\frac{269-\sqrt{7081}}{238}py'+\frac{31-\sqrt{7081}}{238}p^2y=0,\\ v_t&=u_x+pu+qv,\\ L_2[z]&=z''+\frac{5\left(275+\sqrt{7081}\right)}{1428}pz'-\frac{106\sqrt{7081}+5129}{42483}p^2z=0. \end{aligned} \tag{3.13}$$

By a similar calculation, we can give a description of the nonlinear operator $\mathbb{F}=(F,G)$ admitting the corresponding invariant subspaces in other cases.

Case $W_3^1\times W_2^2$

$$\begin{aligned} &u_t=(u_x+fvv_x)_x+gv^2, \quad L_1[y]=y'''+3b_1y''+2b_1^2y'=0,\\ &v_t=u_x+2b_1u+qv, \qquad\quad L_2[z]=z''+b_1z'=0; \end{aligned} \tag{3.14}$$

$$\begin{aligned} &u_t=(u_x+fvv_x)_x-2fb_1^2v^2, \quad L_1[y]=y'''+a_2y''+(a_2b_1-b_1^2)y'=0,\\ &v_t=u_x+(a_2-b_1)u+qv, \qquad L_2[z]=z''+b_1z'=0; \end{aligned} \tag{3.15}$$

$$\begin{aligned} &u_t=(u_x+fvv_x)_x-\frac{8fb_1^2}{9}v^2, \quad L_1[y]=y'''+2b_1y''+\frac{11b_1^2}{9}y'+\frac{2b_1^3}{9}y=0,\\ &v_t=u_x+b_1u+qv, \qquad\qquad L_2[z]=z''+b_1z'+\frac{2b_1^2}{9}z=0; \end{aligned} \tag{3.16}$$

$$\begin{aligned} &u_t=(u_x+fvv_x)_x-\frac{fb_1^2}{2}v^2, \quad L_1[y]=y'''+4b_1y''+\frac{9b_1^2}{4}y'-\frac{9b_1^3}{4}y=0,\\ &v_t=u_x+3b_1u+qv, \qquad\qquad L_2[z]=z''+b_1z'-\frac{3b_1^2}{4}z=0; \end{aligned} \tag{3.17}$$

$$\begin{aligned} &u_t=(u_x+fvv_x)_x-\frac{fb_1^2}{2}v^2, \quad L_1[y]=y'''+\frac{7b_1}{3}y''+\frac{14b_1^2}{9}y'-\frac{8b_1^3}{27}y=0,\\ &v_t=u_x+\frac{4b_1}{3}u+qv, \qquad\qquad L_2[z]=z''+b_1z'+\frac{2b_1^2}{9}z=0; \end{aligned} \tag{3.18}$$

$$\begin{aligned} &u_t = (u_x + fvv_x)_x + 2fb_0v^2, \quad && L_1[y] = y''' + b_0y' = 0, \\ &v_t = u_x + qv, && L_2[z] = z'' + b_0z = 0. \end{aligned} \tag{3.19}$$

Case $W_3^1 \times W_3^2$

$$\begin{aligned} &u_t = (u_x + fvv_x)_x + 2fa_1v^2, \quad && L_1[y] = y''' + a_1y' = 0, \\ &v_t = u_x + pu + qv, && L_2[z] = z''' + a_1z' = 0. \end{aligned} \tag{3.20}$$

Case $W_4^1 \times W_3^2$

$$\begin{aligned} &u_t = (u_x + fvv_x)_x + 2fb_1v^2, \quad && L_1[y] = y^{(4)} + a_3y''' + b_1y'' + a_3b_1y' = 0, \\ &v_t = u_x + a_3u + qv, && L_2[z] = z''' + b_1z' = 0; \end{aligned} \tag{3.21}$$

$$\begin{aligned} &u_t = (u_x + fvv_x)_x - \tfrac{8fb_2^2}{9}v^2, \quad && L_1[y] = y^{(4)} + 2b_2y''' + \tfrac{11b_2^2}{9}y'' + \tfrac{2b_2^3}{9}y' = 0, \\ &v_t = u_x + b_2u + qv, && L_2[z] = z''' + b_2z'' + \tfrac{2b_2^2}{9}z' = 0; \end{aligned} \tag{3.22}$$

$$\begin{aligned} &u_t = (u_x + fvv_x)_x - \tfrac{fb_2^2}{2}v^2, \quad && L_1[y] = y^{(4)} + 4b_2y''' + \tfrac{9b_2^2}{4}y'' - \tfrac{9b_2^3}{4}y' = 0, \\ &v_t = u_x + 3b_2u + qv, && L_2[z] = z''' + b_2z'' - \tfrac{3b_2^2}{4}z' = 0; \end{aligned} \tag{3.23}$$

$$\begin{aligned} &u_t = (u_x + fvv_x)_x - \tfrac{fb_2^2}{2}v^2, \quad && L_1[y] = y^{(4)} + \tfrac{7b_2}{3}y''' + \tfrac{14b_2^2}{9}y'' + \tfrac{8b_2^3}{27}y' = 0, \\ &v_t = u_x + \tfrac{4b_2}{3}u + qv, && L_2[z] = z''' + b_2z'' + \tfrac{2b_2^2}{9}z' = 0. \end{aligned} \tag{3.24}$$

Case $W_5^1 \times W_4^2$

$$\begin{aligned} &u_t = (u_x + fvv_x)_x, \quad && L_1[y] = y^{(5)} = 0, \\ &v_t = u_x + qv, && L_2[z] = z^{(4)} = 0. \end{aligned} \tag{3.25}$$

Similarly, for other equations, we can get the following results by using Theorem 2.1. There exist more systematical solutions of the above kind, for example, complexiton solutions, to nonlinear evolution equations including typical soliton equations (see [14]).

Example 3.2 Itô's type equation

$$\begin{aligned} &u_t = -u_{xxx} - 3(uv)_x, \quad && L_1[y] = y''' = 0, \\ &v_t = u_x, && L_2[z] = z'' = 0. \end{aligned} \tag{3.26}$$

Example 3.3 Drinfel'd-Sokolov-Wilson's type equation

$$\begin{aligned} &u_t = fvu_x - guv_x - pu_{xxx}, \quad && L_1[y] = y'' = 0, \\ &v_t = -uu_x, && L_2[z] = z'' = 0; \end{aligned} \tag{3.27}$$

$$\begin{aligned} &u_t = 2gvu_x - guv_x - pu_{xxx}, \quad && L_1[y] = y'' + \tfrac{b_0}{4}y = 0, \\ &v_t = -uu_x, && L_2[z] = z'' + b_0z = 0. \end{aligned} \tag{3.28}$$

Example 3.4 Whitham-Broer-Kaup's type equation

$$\begin{aligned} &u_t = fu_{xx} - gu_{xxx} - (uv)_x, \quad && L_1[y] = y''' = 0, \\ &v_t = -vv_x - u_x - fv_{xx}, && L_2[z] = z'' = 0. \end{aligned} \tag{3.29}$$

Next, we give two examples to illustrate how to construct exact solutions to corresponding nonlinear systems. Firstly, we consider the system in (3.16)

$$\begin{aligned} &u_t=(u_x+vv_x)_x-\tfrac{8}{9}v^2, \quad L_1[y]=y'''+2y''+\tfrac{11}{9}y'+\tfrac{2}{9}y=0,\\ &v_t=u_x+u+v, \qquad\qquad L_2[z]=z''+z'+\tfrac{2}{9}z=0. \end{aligned} \tag{3.30}$$

From

$$L_1[y]=0 \quad \text{and} \quad L_2[z]=0,$$

we get the invariant subspace

$$W_3^1\times W_2^2=\mathcal{L}\{\mathrm{e}^{-\frac{1}{3}x},\mathrm{e}^{-\frac{2}{3}x},\mathrm{e}^{-x}\}\times\mathcal{L}\{\mathrm{e}^{-\frac{1}{3}x},\mathrm{e}^{-\frac{2}{3}x}\}.$$

Hence, we get an exact solution of the form

$$\begin{aligned} &u=C_1(t)\mathrm{e}^{-\frac{1}{3}x}+C_2(t)\mathrm{e}^{-\frac{2}{3}x}+C_3(t)\mathrm{e}^{-x},\\ &v=D_1(t)\mathrm{e}^{-\frac{1}{3}x}+D_2(t)\mathrm{e}^{-\frac{2}{3}x}. \end{aligned} \tag{3.31}$$

Substituting the solution into (3.30), we have the following dynamical system:

$$\begin{aligned} &C_1'=\frac{C_1}{9}, \quad C_2'=\frac{4}{9}C_2-\frac{2}{3}D_1^2, \quad C_3'=C_3-\frac{7}{9}D_1D_2,\\ &D_1'=\frac{2}{3}C_1+D_1, \quad D_2'=\frac{C_2}{3}+D_2. \end{aligned}$$

Solving this system, we obtain the exact solution (3.31) with

$$\begin{aligned} C_1&=c_1\mathrm{e}^{\frac{t}{9}},\\ C_2&=\frac{27}{16}c_1^2\mathrm{e}^{\frac{2}{9}t}+\frac{3}{2}c_1c_2\mathrm{e}^{\frac{10}{9}t}-\frac{3}{7}c_2^2\mathrm{e}^{2t}+c_3\mathrm{e}^{\frac{4}{9}t},\\ C_3&=\Big(\frac{81}{128}c_1^3\mathrm{e}^{-\frac{2}{3}t}+\frac{459}{32}c_1^2c_2\mathrm{e}^{\frac{2}{9}t}-\frac{129}{40}c_1c_2^2\mathrm{e}^{\frac{10}{9}t}+\frac{63}{80}c_1c_3\mathrm{e}^{-\frac{4}{9}t}\\ &\quad+\frac{21}{4}c_1c_4\mathrm{e}^{\frac{t}{9}}+\frac{1}{18}c_2^3\mathrm{e}^{2t}+\frac{21}{20}c_2c_3\mathrm{e}^{\frac{4}{9}t}-\frac{7}{9}c_2c_4\mathrm{e}^{t}+c_5\Big)\mathrm{e}^{t},\\ D_1&=\Big(c_2-\frac{3}{4}c_1\mathrm{e}^{-\frac{8}{9}t}\Big)\mathrm{e}^{t},\\ D_2&=\Big(\frac{9}{2}c_1c_2\mathrm{e}^{\frac{t}{9}}-\frac{81}{112}c_1^2\mathrm{e}^{-\frac{7}{9}t}-\frac{1}{7}c_2^2\mathrm{e}^{t}-\frac{3}{5}c_3\mathrm{e}^{-\frac{5}{9}t}+c_4\Big)\mathrm{e}^{t}, \end{aligned}$$

where c_j $(j=1,\cdots,5)$ are arbitrary constants.

Furthermore, we consider the system from (3.19)

$$\begin{aligned} &u_t=(u_x+vv_x)_x+2v^2, \quad L_1[y]=y'''+y'=0,\\ &v_t=u_x+v, \qquad\qquad L_2[z]=z''+z=0. \end{aligned} \tag{3.32}$$

From $L_1[y]=0$ and $L_2[z]=0$, we get the invariant subspace

$$W_3^1\times W_2^2=\mathcal{L}\{1,\cos x,\sin x\}\times\mathcal{L}\{\cos x,\sin x\}.$$

Thus, we obtain an exact solution of the form

$$\begin{aligned} u &= C_1(t) + C_2(t)\cos x + C_3(t)\sin x, \\ v &= D_1(t)\cos x + D_2(t)\sin x. \end{aligned} \tag{3.33}$$

Substituting the solution into (3.32), we arrive at the following dynamical system:

$$\begin{aligned} &C_1' = D_1^2 + D_2^2, \quad C_2' = -C_2, \quad C_3' = -C_3, \\ &D_1' = C_3 + D_1, \quad D_2' = D_2 - C_2. \end{aligned}$$

Solving the system, we obtain the exact solution (3.33) with

$$\begin{aligned} C_1 &= \frac{1}{2}(c_4^2 + c_3^2)\mathrm{e}^{2t} - \frac{1}{8}(c_1^2 + c_2^2)\mathrm{e}^{-2t} + (c_1c_4 - c_2c_3)t + c_5, \\ C_2 &= c_1\mathrm{e}^{-t}, \\ C_3 &= c_2\mathrm{e}^{-t}, \\ D_1 &= c_3\mathrm{e}^{t} - \frac{1}{2}c_2\mathrm{e}^{-t}, \\ D_2 &= \frac{1}{2}c_1\mathrm{e}^{-t} + c_4\mathrm{e}^{t}, \end{aligned}$$

where $c_j,\ j = 1, \cdots, 5$, are arbitrary constants.

4 Concluding Remarks

In this paper, the dimension of invariant subspaces admitted by systems of nonlinear PDEs is estimated. It is shown that if the two-component nonlinear operators (F^1, F^2) with orders $\{k_1, k_2\}$ $(k_1 \geq k_2)$ satisfy certain conditions (2.2)–(2.4) and admit the invariant subspace $W_{n_1}^1 \times W_{n_2}^2$ $(n_1 \geq n_2)$ defined by (1.4), then

$$n_1 - n_2 \leq k_2 \quad \text{and} \quad n_1 \leq 2(k_1 + k_2) + 1.$$

The maximal dimension for the m-component vector operators is also determined. The invariant subspaces determined by linear ODEs for some nonlinear systems are presented.

At the end of this paper, we would like to pose two questions regarding this work. One is how to describe the invariant subspace $W = \{1, x^{j_1}, \cdots, x^{j_m}\}$, where $j_i - j_{i-1} = r,\ r \in \mathbb{Z}$ and how to classify the scalar equation (1.1) admitting such an invariant subspace. The other one is how to determine such an invariant subspace for m-component systems, namely, $W = W_{n_1}^1 \times \cdots \times W_{n_m}^m$, where $W_{n_i}^i = \{1, x^{i_1}, \cdots, x^{i_{n_i}}\},\ i_j - i_{j-1} = r_i,\ 2 \leq j \leq n_i,\ r_i \in \mathbb{Z}$. It is of interest to extend the approach in this paper to studying nonlinear evolution differential-difference equations.

References

[1] Bluman, G. W. and Kumei, S., Symmetries and Differential Equations, Springer-Verlag, New York, 1989.

[2] Olver, P. J., Applications of Lie Groups to Differential Equations, 2nd ed., Springer-Verlag, New York, 1993.

[3] Galaktionov, V. A. and Svirshchevskii, S. R., Exact Solutions and Invarinat Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics, Chapman and Hall/CRC, London, 2007.

[4] King, J. R., Exact polynomial solutions to some nonlinear diffusion equations, *Phys. D*, **64**, 1993, 35–65.

[5] Galaktionov, V. A., Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities, *Proc. Roy. Soc. Endin. Sect. A*, **125**, 1995, 225–246.

[6] Svirshchevskii, S. R., Lie-Bäcklund symmetries of linear ODEs and generalised separation of variables in nonlinear equations, *Phys. Lett. A*, **199**, 1995, 344–348.

[7] Fokas, A. S. and Liu, Q. M., Nonlinear interaction of travelling waves of nonintegrable equations, *Phys. Rev. Lett.*, **72**, 1994, 3293–3296.

[8] Zhdanov, R. Z., Conditional Lie-Bäcklund symmetry and reductions of evolution equations, *J. Phys. A: Math. Gen.*, **28**, 1995, 3841–3850.

[9] Qu, C. Z., Group classification and generalized conditional symmetry reduction of the nonlinear diffusion-convection equation with a nonlinear source, *Stud. Appl. Math.*, **99**, 1997, 107–136.

[10] Qu, C. Z., Ji, L. N. and Wang, L. Z., Conditional Lie-Bäcklund symmetries and sign-invarints to quasi-linear diffusion equations, *Stud. Appl. Math.*, **119**, 2007, 355–391.

[11] Ji, L. N. and Qu, C. Z., Conditional Lie-Bäcklund symmetries and solutions to $(n+1)$-dimensional nonlinear diffusion equations, *J. Math. Phys.*, **48**, 2007, 103509, 23 pages.

[12] Hirota, R., Grammaticos, B. and Ramani, A., Soliton structure of the Drinfel'd-Sokolov-Wilson equation, *J. Math. Phys.*, **27**, 1986, 1499–1505.

[13] Ma, W. X. and Fan, E. G., Linear superposition principle applying to Hirota bilinear equations, *Comput. Math. Appl.*, **61**, 2011, 950–959.

[14] Ma, W. X., Complexiton solutions to integrable equations, *Nonlinear Anal.*, **63**, 2005, e2461–e2471.

[15] Qu, C. Z. and Zhu, C. R., Classification of coupled systems with two-component nonlinear diffusion equations by the invariant subspace method, *J. Phys. A: Math. Theor.*, **42**, 2009, 475201, 27 pages.

[16] Zhu, C. R. and Qu, C. Z., Maximal dimension of invariant subspaces admitted by nonlinear vector differential operators, *J. Math. Phys.*, **52**, 2011, 043507, 15 pages.

[17] Hirota, R., Hu, X. B. and Yang, X. Y., A vector potential KdV equation and vector Itô equation: soliton solutions, bilinear Bäcklund transformations and Lax pairs, *J. Math. Anal. Appl.*, **288**, 2003, 326–348.

[18] Makhankov, V. G., Myrsakulov, R. and Pashaev, O. K., Gauge equivalence, supersymmetry and classical solutions of the ospu (1,1/1) Heisenberg model and the nonlinear Schrödinger equation, *Lett. Math. Phys.*, **16**, 1988, 83–92.

[19] Kupershmidt, B. A., Mathematics of dispersive water waves, *Comm. Math. Phys.*, **99**, 1985, 51–73.

Appendix The proof of Theorem 2.2

First, we note that if the nonlinear operator $\mathbb{F}[\mathbb{U}]$ admits the invariant subspace $W^1_{n_1} \times \cdots \times W^m_{n_m}$, then the following identities

$$D^{n_q}F^q = -[a^q_{n_q-1}(x)D^{n_q-1}F^q + \cdots + a^q_1(x)DF^q + a^q_0(x)F^q], \quad q = 1, \cdots, m \tag{A.1}$$

hold on the solution manifold (1.4).

Now we prove (2.17) by contradiction. Assume that there exists an $i_0 \in \{2, \cdots, m\}$ such that

$$n_{i_0-1} - n_{i_0} > k_{i_0}.$$

Then

$$n_1 \geq n_2 \geq \cdots \geq n_{i_0-1} > n_{i_0} + k_{i_0}.$$

Differentiating $F^q(x, u^1, \cdots, u^m, \cdots, u^1_{k_q}, \cdots, u^m_{k_q})$ and keeping the leading linear and quadratic terms yield

$$\begin{aligned} D^{n_q}F^q = {} & \sum_{j=1}^{m} u^j_{k_q+n_q} \frac{\partial F^q}{\partial u^j_{k_q}} + \sum_{j,l=1}^{m} \Big[\Big(\sum_{s=1}^{[\frac{n_q}{2}]-1} C^s_{n_q} u^j_{k_q+n_q-s} u^l_{k_q+s} \\ & + \gamma C^{[\frac{n_q}{2}]}_{n_q} u^j_{k_q+n_q-[\frac{n_q}{2}]} u^l_{k_q+[\frac{n_q}{2}]} \Big) \frac{\partial^2 F^q}{\partial u^j_{k_q} \partial u^l_{k_q}} \Big] + \cdots, \quad q = i_0, \cdots, m. \end{aligned}$$

Clearly,

$$k_q + n_q < n_j, \quad j = 1, \cdots, i_0 - 1, \ q = i_0, \cdots, m.$$

It follows from (A.1) that the term containing the derivative of u^j with the maximal order k_q+n_q only appears in $D^{n_q}F^q$ and does not appear in the derivatives D^pF^q $(p < n_q)$. Equating the coefficients of $u^j_{k_q+n_q}$ $(j = 1, \cdots, i_0 - 1)$ with zero leads to

$$\frac{\partial F^q}{\partial u^j_{k_q}} = 0, \quad q = i_0, \cdots, m, \ j = 1, \cdots, i_0 - 1.$$

Taking into account (A.1) and doing computation in a similar way as above, we find

$$\frac{\partial F^q}{\partial u^j_{k_q-1}} = 0, \quad q = i_0, \cdots, m, \ j = 1, \cdots, i_0 - 1.$$

By induction, we immediately have

$$\frac{\partial F^q}{\partial u^j_l} = 0, \quad l = 0, 1, \cdots, k_j, \ q = i_0, \cdots, m, \ j = 1, \cdots, i_0 - 1.$$

This contradicts the assumption.

Furthermore, by using the first formula in (2.17) we get

$$n_m \geq n_q - (k_m + k_{m-1} + \cdots + k_{q+1}), \quad q = 1, \cdots, m$$

and

$$n_q \geq n_1 - (k_2 + k_3 + \cdots + k_q), \quad q = 1, \cdots, m.$$

Similarly, we verify the second formula $n_1 \leq 2\sum\limits_{j=1}^{m} k_j + 1$ by contradiction. Assume that

$$n_1 > 2\sum_{j=1}^{m} k_j + 1, \quad \text{i.e.,} \quad n_1 \geq 2\sum_{j=1}^{m} k_j + 2.$$

It implies that

$$n_q \geq 2k_1 + k_2 + \cdots + k_q + 2(k_{q+1} + \cdots + k_m) + 2, \quad q = 2, \cdots, m.$$

Notice that the sum in the square brackets in

$$D^{n_q} F^q = + \sum_{j,l=1}^{m} \Big[\Big(\sum_{s=1}^{[\frac{n_q}{2}]-1} C_{n_q}^{s} u^j_{k_q+n_q-s} u^l_{k_q+s} + \gamma C_{n_q}^{[\frac{n_q}{2}]} u^j_{k_q+n_q-[\frac{n_q}{2}]} u^l_{k_q+[\frac{n_q}{2}]} \Big) \frac{\partial^2 F^q}{\partial u^j_{k_q} \partial u^l_{k_q}} \Big] + \cdots$$

is composed of the quadratical terms that exhibit the maximal total order of both derivatives

$$(k_q + n_q - s) + (k_q + s) = 2k_q + n_q.$$

For $s = k_q + k_{q+1} + \cdots + k_m + 1$, we have

$$k_q + n_q - s = n_q - (k_{q+1} + \cdots + k_m + 1) \leq n_m - 1 < n_j, \quad j = 1, \cdots, m.$$
$$k_q + s = 2k_q + k_{q+1} + \cdots + k_m + 1 < n_m \leq n_j, \quad j = 1, \cdots, m.$$

Thus, from (A.1), the terms

$$\alpha_{k_q+k_{q+1}+\cdots+k_m+1} u^j_{n_q-(k_{q+1}+\cdots+k_m+1)} u^l_{2k_q+k_{q+1}+\cdots+k_m+1} \frac{\partial^2 F^q}{\partial u^j_{k_q} \partial u^l_{k_q}} \quad \text{for } j, l = 1, \cdots, m$$

do not appear in the derivatives $D^p F^q$ $(p < n_q)$. In order for (A.1) to be valid, we must have

$$\frac{\partial^2 F^q}{\partial u^j_{k_q} \partial u^l_{k_q}} = 0, \quad j, l = 1, \cdots, m.$$

In a similar way to the above computation, we find that the functions $F^q(\mathbb{U})$, $q = 1, \cdots, m$ are linear in its all arguments.

This completes the proof of Theorem 2.2.