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The Exact Traveling Wave Solutions to Two Integrable KdV6 Equations*

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Abstract The exact explicit traveling solutions to the two completely integrable sixth-order nonlinear equations KdV6 are given by using the method of dynamical systems and Cosgrove's work. It is proved that these traveling wave solutions correspond to some orbits in the 4-dimensional phase space of two 4-dimensional dynamical systems. These orbits lie in the intersection of two level sets defined by two first integrals.

Keywords KdV6 equation, Exact traveling wave solution, Solitary wave solution, Quasi-periodic wave solution
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1 Introduction

More recent years, Karasu-Kalkanli and his co-workers derived a new sixth-order nonlinear wave equation (KdV6) in [1] given by

$$u_{xxxxxx} + au_x u_{xxxx} + bu_{xx}u_{xxx} + cu_x^2 u_{xx} + du_{tt} + eu_{xxxt} + fu_x u_{xt} + gu_t u_{xx} = 0, \tag{1.1}$$

where a, b, c, d, e, f and g are arbitrary parameters. They found that there are four distinct cases of relations between the parameters for (1.1) to passing the Painlevé test. Three of them were the well-known integrable systems: a bidirectional version of the Sawada-Kotera-Caudrey-Dodd-Gibbon equation (see [2–3]), the Kaup-Kupershmidt equation (see [4–5]) as well as the Drinfeld-Sokolov-Hirota-Satsuma equation (see [6–7]). But the fourth was a new integrable equation. As to the integrability of (1.1), Kupershmidt [8] studied the Hamiltonian structures and conservation laws. Further, Yao and Zeng [9] showed that the KdV6 equation is equivalent to the Rosochatius deformation of KdV equation with self-consistent sources recently presented in [10].

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By using the Cole-Hopf transformation method combined with the Hirota's bilinear sense, for the following three distinct integrable cases of (1.1):

(i) $a=20,\ b=40,\ c=120,\ d=0,\ e=\frac{1}{2}(f+g)=1,\ f=8,\ g=4,\ f=2g,$

$$\begin{aligned} &u_{xxxxxxx}+20u_xu_{xxxxx}+40u_{xx}u_{xxx}+120u_x^2u_{xx}+u_{xxxt}+8u_xu_{xt}+4u_tu_{xx}\\ &=(\partial_x^3+8u_x\partial_x+4u_{xx})(u_t+u_{xxx}+6u_x^2)=0; \end{aligned} \tag{1.2}$$

(ii) $a=18,\ b=36,\ c=72,\ f=g=0,\ e=1,\ d=-2e^2+\frac{1}{2}ef-\frac{1}{36}f^2=-2,$

$$u_{xxxxxxx}+18u_xu_{xxxxx}+36u_{xx}u_{xxx}+72u_x^2u_{xx}-2u_{tt}+u_{xxxt}=0; \tag{1.3}$$

(iii) $a=30,\ b=30,\ c=180,\ f=g=6,\ d=-\frac{1}{180}g^2=-\frac{1}{5},\ e=\frac{1}{12}(f+g)=1,$

$$u_{xxxxxxx}+30u_xu_{xxxxx}+30u_{xx}u_{xxx}+180u_x^2u_{xx}-\frac{1}{5}u_{tt}+u_{xxxt}+6u_xu_{xt}+6u_tu_{xx}=0, \tag{1.4}$$

Wazwaz obtained multiple soliton solutions to (1.1) and multiple singular soliton solutions in [11]. Gómez and Salas studied some exact solutions to (1.2) by using the Cole-Hopf transformation and one improved tanh-coth method in [12]. A much nicer form of (1.1), stressing the relation to the KdV equation was also given by Gómez [12] and Ramani [13] etc. Moreover, in our recent work (see [14]), one of the authors has obtained a new bilinear form of the KdV6 equation and the multi-soliton solutions have been derived.

In this paper, we will investigate exact traveling wave solutions to the integrable equations (1.2) and (1.3) from the point of view of geometric theory of the dynamical systems (see [15]).

Setting

$$u(x,t)=u(x-vt)=u(\xi),$$

where v is the wave speed, integrating equation (1.1) with respect to ξ once and setting $\phi=u_\xi$, we have

$$\phi''''+a\phi\phi''+\frac{b-a}{2}(\phi')^2-ev\phi''+\frac{c}{3}\phi^3-\frac{v(f+g)}{2}\phi^2+dv^2\phi+\beta_0=0, \tag{1.5}$$

where β_0 is an integral constant and "$'$" stands for the derivative with respect to ξ. By making the transformation

$$y=-\Big(\phi-\frac{ev}{a}\Big),$$

(1.5) becomes the following fourth-order ordinary differential equation:

$$y''''=ayy''+\frac{b-a}{2}(y')^2-\frac{c}{3}y^3-\gamma y^2+\alpha y+\beta, \tag{1.6}$$

where

$$\begin{aligned} \gamma&=-\frac{cev}{a}+\frac{v(f+g)}{2},\\ \alpha&=-\Big(\frac{ce^2v^2}{a^2}-\frac{ev^2(f+g)}{a}+dv^2\Big),\\ \beta&=\frac{cv^3e^3}{3a^3}-\frac{e^2v^3(f+g)}{2a^2}+\frac{dev^3}{a}+\beta_0. \end{aligned}$$

Under the above transformations, the corresponding traveling wave equations of (1.2)–(1.4) become respectively as follows:

$$y'''' = 20yy'' + 10(y')^2 - 40y^3 + \frac{3}{10}v^2y - \frac{1}{100}v^3, \tag{1.7}$$

$$y'''' = 18yy'' + 9(y')^2 - 24y^3 + 4vy^2 + \frac{16}{9}v^2y - \frac{26}{243}v^3, \tag{1.8}$$

$$y'''' = 30yy'' - 60y^3 + \frac{2}{5}v^2y - \frac{1}{90}v^3. \tag{1.9}$$

Thus, if we know the parametric representations of $y(\xi)$ for the above equations, then we obtain the exact traveling wave solutions

$$u(x,t) = u(\xi) = \int \Big(\frac{ev}{a} - y(\xi)\Big)\mathrm{d}\xi$$

to (1.2)–(1.4).

We notice that the first integrals and some solution formulas to (1.7)–(1.9) have been studied by Cosgrove in [16], where (1.7) corresponds to the equation F-V with

$$\alpha = \frac{3}{10}v^2, \quad \beta = -\frac{1}{100}v^3, \quad \kappa = 0;$$

(1.8) corresponds to the equation F-VI with

$$\alpha = 4v, \quad \beta = -\frac{26}{243}v^3, \quad \kappa = 0;$$

in addition, (1.9) corresponds to the equation F-IV with

$$\alpha = \frac{2}{5}v^2, \quad \beta = -\frac{1}{90}v^3$$

in [16].

We will investigate the exact explicit traveling wave solutions to (1.7) and (1.8) in the next two sections. We will show that for two equations of (1.7) and (1.8), their solitary wave solution and "singular solitary wave solution" correspond to some orbits in a 4-dimensional phase space of two 4-dimensional dynamical systems. These orbits lie in the intersection of two level sets passing through the same equilibrium point.

2 The Exact Traveling Wave Solutions to (1.7) and Their Geometric Property

Let

$$x_1 = y, \quad x_2 = x_1' = y', \quad x_3 = x_2' = y'', \quad x_4 = x_3' = y'''.$$

Then (1.6) is equivalent to the 4-dimensional system

$$\begin{aligned} &x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4,\\ &x_4' = ax_1x_3 + \frac{(b-a)}{2}x_2^2 - \gamma x_1^3 + \alpha x_1 + \beta. \end{aligned} \tag{2.1}$$

In the following, we will study some dynamical behavior of (2.1) in the (x_1, x_2, x_3, x_4)—4-dimensional phase space for the aforementioned two special parameter cases.

Let $M(x_{1_j}, 0, 0, 0)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point E_j $(j = 1, 2, 3)$. Then we have

$$M(x_{1_j}, 0, 0, 0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \alpha - 3\gamma x_{1_j}^2 & 0 & a x_{1_j} & 0 \end{pmatrix}. \tag{2.2}$$

Corresponding to (1.7), we have the 4-dimensional system

$$\begin{aligned} & x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4, \\ & x_4' = 20x_1x_3 + 10x_2^2 - 40x_1^3 + \frac{3}{10}v^2x_1 - \frac{1}{100}v^3. \end{aligned} \tag{2.3}$$

It is easy to see that (2.2) has an elementary equilibrium point $E_1(-\frac{1}{10}v, 0, 0, 0)$ and a two-order equilibrium point $E_2(\frac{1}{20}v, 0, 0, 0)$.

The eigenvalues of $M(\frac{1}{20}v, 0, 0, 0)$ are $0, 0, \pm\sqrt{v}$ and the eigenvalues of $M(-\frac{1}{10}v, 0, 0, 0)$ are two purely imaginary pairs

$$\pm\sqrt{\Big(1 - \frac{1}{\sqrt{10}}\Big)v}\,\mathrm{i}, \quad \pm\sqrt{\Big(1 + \frac{1}{\sqrt{10}}\Big)v}\,\mathrm{i}.$$

System (2.3) has the following two first integrals (see [16]):

$$\Phi_1(x_1, x_2, x_3, x_4) = x_2x_4 - 10x_1x_2^2 + 10x_1^4 + \frac{v^3}{100}x_1 - \frac{1}{2}x_3^2 - \frac{3v^2}{20}x_1^2 + \frac{9}{3200}v^4, \tag{2.4}$$

$$\begin{aligned} \Phi_2(x_1, x_2, x_3, x_4) = {} & x_4^2 - 24x_4x_1x_2 + 120x_1^2x_2^2 - 8x_1x_3^2 + 4x_3x_2^2 + 80x_3x_1^3 + \frac{v^3}{50}x_3 \\ & - 192x_1^5 - \frac{3v^3}{25}x_1^2 - \frac{3v^2}{5}x_1x_3 + \frac{3v^2}{10}x_2^2 + \frac{12v^2}{5}x_1^3 + \frac{3}{2000}v^5. \end{aligned} \tag{2.5}$$

For two constants K_1 and K_2, the two level sets defined by

$$\Phi_1(x_1, x_2, x_3, x_4) = K_1 \quad \text{and} \quad \Phi_2(x_1, x_2, x_3, x_4) = K_2$$

determine two families of the invariant manifolds of system (2.3). Especially, we see from (2.4) and (2.5) that

$$\begin{aligned} K_{11} &= \Phi_1\Big(\frac{1}{20}v, 0, 0, 0\Big) = \frac{3}{1000}v^4, \\ K_{21} &= \Phi_2\Big(\frac{1}{20}v, 0, 0, 0\Big) = \frac{9}{6250}v^5, \\ K_{12} &= \Phi_1\Big(-\frac{1}{10}v, 0, 0, 0\Big) = \frac{21}{1600}v^4, \\ K_{22} &= \Phi_2\Big(-\frac{1}{10}v, 0, 0, 0\Big) = \frac{9}{50000}v^5. \end{aligned}$$

Thus, the two level sets (I_1) and (I_2) defined by

$$\Phi_1(x_1,x_2,x_3,x_4)=K_{11} \quad \text{and} \quad \Phi_2(x_1,x_2,x_3,x_4)=K_{21}$$

pass through the equilibrium point E_2, while the two level sets (II_1) and (II_2) defined by

$$\Phi_1(x_1,x_2,x_3,x_4)=K_{12} \quad \text{and} \quad \Phi_2(x_1,x_2,x_3,x_4)=K_{22}$$

pass through the equilibrium point E_1. By using the method given by Cosgrove in [16, Section 5], we see that in the intersection of (I_1) and (I_2), there exist the solutions to (1.7) as follows:

$$Y(\xi)=\frac{1}{4}(U(\xi)+V(\xi)), \tag{2.6}$$

where $U(\xi)$ and $V(\xi)$ are defined by

$$\begin{aligned} &\int_\infty^U \frac{\mathrm{d}\tau}{\sqrt{P(\tau)}}+\int_\infty^V \frac{\mathrm{d}\tau}{\sqrt{P(\tau)}}=C_1,\\ &\int_\infty^U \frac{\tau\mathrm{d}\tau}{\sqrt{P(\tau)}}+\int_\infty^V \frac{\tau\mathrm{d}\tau}{\sqrt{P(\tau)}}=C_2+\xi, \end{aligned} \tag{2.7}$$

and

$$P(t)=t^5-2\alpha t^3+8\beta t^2+32K_{11}t+16K_{21}=\Big(t-\frac{3}{5}v\Big)^2\Big(t+\frac{2}{5}v\Big)^3. \tag{2.8}$$

(2.7) and (2.8) imply that

$$\begin{aligned} \frac{2}{X}+\frac{2}{Y}+\frac{1}{\sqrt{v}}\ln\frac{(X-\sqrt{v})(Y-\sqrt{v})}{(X+\sqrt{v})(Y+\sqrt{v})}&=C_1,\\ -\frac{4}{5X}-\frac{4}{5Y}+\frac{3}{5\sqrt{v}}\ln\frac{(X-\sqrt{v})(Y-\sqrt{v})}{(X+\sqrt{v})(Y+\sqrt{v})}&=C_2+\xi, \end{aligned} \tag{2.9}$$

where

$$X=\sqrt{U+\frac{2}{5}v}, \quad Y=\sqrt{V+\frac{2}{5}v}$$

and $C_1,\ C_2$ are two integration constants. Thus, we have

$$\frac{(X-\sqrt{v})(Y-\sqrt{v})}{(X+\sqrt{v})(Y+\sqrt{v})}=C_3\exp(\sqrt{v}\xi), \quad \frac{1}{X}+\frac{1}{Y}=C_4-\frac{\xi}{2},$$

where

$$C_3=\exp\Big(\frac{2}{5}C_1+C_2\Big)\neq 0, \quad C_4=\frac{3}{10}C_1-\frac{1}{2}C_2.$$

Solving X and Y, (2.6) follows that

$$y(\xi)=\frac{1}{20}v+\frac{v[1+v(C_4-\frac{1}{2}\xi)^2(1-h^2(\xi))]}{4[1-\sqrt{v}(C_4-\frac{1}{2}\xi)h(\xi)]^2}, \quad \text{where } h(\xi)=\frac{1+C_3\mathrm{e}^{\sqrt{v}\xi}}{1-C_3\mathrm{e}^{\sqrt{v}\xi}}, \tag{2.10}$$

holds for all $(C_3,C_4)\in\mathbb{R}^2$. (2.10) gives rise to the very general exact parametric representations of the solutions to (1.7). Notice that

$$\text{when } \xi\to\pm\infty, \quad h(\xi)\to\mp 1$$

and

$$x_1(\xi) = Y(\xi) \to \frac{1}{20}v, \quad x_2(\xi), x_3(\xi), x_4(\xi) \to 0.$$

This means that every solution defined by (2.10) tends to the equilibrium point E_2 in the (x_1, x_2, x_3, x_4)-phase space.

Especially, for

$$C_3 = -1, \quad h(\xi) = \tanh\Big(\frac{1}{2}\sqrt{v}\xi\Big),$$

we obtain

$$y(\xi) = \frac{1}{20}v + \frac{v[1 + v(C_4 - \frac{1}{2}\xi)^2 \mathrm{sech}^2(\frac{1}{2}\sqrt{v}\xi)]}{4[1 - \sqrt{v}(C_4 - \frac{1}{2}\xi)\tanh^2(\frac{1}{2}\sqrt{v}\xi)]^2}. \tag{2.11}$$

In addition, for $C_3 = 1$, we get

$$y(\xi) = \frac{1}{20}v + \frac{v[1 - v(C_4 - \frac{1}{2}\xi)^2 \mathrm{csch}^2(\frac{1}{2}\sqrt{v}\xi)]}{4[1 - \sqrt{v}(C_4 - \frac{1}{2}\xi)\coth^2(\frac{1}{2}\sqrt{v}\xi)]^2}. \tag{2.12}$$

For a fixed ξ, letting $C_4 \to \infty$ in (2.10), we have

$$y(\xi) = \frac{1}{20}v + \frac{v[1 - h^2(\xi)]}{4h^2(\xi)}. \tag{2.13}$$

When

$$C_3 = 1 \quad \text{and} \quad C_3 = -1,$$

respectively, (2.13) gives

$$y(\xi) = \frac{1}{20}v - \frac{v}{4}\mathrm{sech}^2\Big(\frac{\sqrt{v}}{2}\xi\Big) \quad \text{and} \quad y(\xi) = \frac{1}{20}v + \frac{v}{4}\mathrm{csch}^2\Big(\frac{\sqrt{v}}{2}\xi\Big). \tag{2.14}$$

We see from

$$u(\xi) = \int \Big(\frac{1}{20}v - y(\xi)\Big)\mathrm{d}\xi$$

that (2.14) just gives rise to the Wazwaz's result in [11, Section 3], the so-called "the single soliton solution" and "the single singular soliton solution" to (1.2) as follows:

$$u(\xi) = \frac{p\mathrm{e}^{p\xi}}{1 + \mathrm{e}^{p\xi}} \quad \text{and} \quad u(\xi) = -\frac{p\mathrm{e}^{p\xi}}{1 - \mathrm{e}^{p\xi}}, \tag{2.15}$$

where $p = \frac{\sqrt{v}}{2}$. Clearly, these two solutions are only special cases of the solutions given by (2.10).

On the other hand, we consider the intersection of the two level sets (II_1) and (II_2) defined by

$$\Phi_1(x_1, x_2, x_3, x_4) = K_{12} \quad \text{and} \quad \Phi_2(x_1, x_2, x_3, x_4) = K_{22}$$

passing through the equilibrium point E_1. In this case, we know from the first formula of (2.8) that

$$P(t) = \Big(t^2 + \frac{2}{5}t - \frac{3}{50}v^2\Big)^2\Big(t - \frac{4}{5}v\Big). \tag{2.16}$$

Therefore, (2.7) leads on to

$$-\frac{\tan^{-1} X_1}{\lambda_1} - \frac{\tan^{-1} Y_1}{\lambda_1} + \frac{\tan^{-1}(\frac{\lambda_1}{\lambda_2} X_1)}{\lambda_2} + \frac{\tan^{-1}(\frac{\lambda_1}{\lambda_2} Y_1)}{\lambda_2} = C_1, \tag{2.17}$$

$$\frac{\lambda_1}{\sqrt{v}}(\tan^{-1} X_1 + \tan^{-1} Y_1) + \frac{\lambda_2}{\sqrt{v}}\Big(\tan^{-1}\Big(\frac{\lambda_1}{\lambda_2} X_1\Big) + \tan^{-1}\Big(\frac{\lambda_1}{\lambda_2} Y_1\Big)\Big) = C_2 + \sqrt{v}\xi, \tag{2.18}$$

where

$$\lambda_1 = \sqrt{\Big(1 + \frac{1}{\sqrt{10}}\Big)v}, \quad \lambda_2 = \sqrt{\Big(1 - \frac{1}{\sqrt{10}}\Big)v},$$

$$X_1 = \frac{\sqrt{U - \frac{4}{5}v}}{\lambda_1}, \qquad Y_1 = \frac{\sqrt{V - \frac{4}{5}v}}{\lambda_1}.$$

It implies that

$$\begin{aligned} \tan^{-1} X_1 + \tan^{-1} Y_1 &= C_3 + \frac{1}{2}\lambda_1 \xi, \\ \tan^{-1}\Big(\frac{\lambda_1}{\lambda_2} X_1\Big) + \tan^{-1}\Big(\frac{\lambda_1}{\lambda_2} Y_1\Big) &= C_4 + \frac{1}{2}\lambda_2 \xi, \end{aligned} \tag{2.19}$$

where C_3 and C_4 are two linear combinations of the integration constants C_1 and C_2.

Let

$$g(\xi) = \frac{3v \tan(\omega_1(\xi + \xi_1)) - \sqrt{10}\lambda_2^2 \tan(\omega_2(\xi + \xi_2))}{3v \tan(\omega_1(\xi + \xi_1)) - \sqrt{10}\lambda_1^2 \tan(\omega_2(\xi + \xi_2))},$$

where

$$\omega_1 = \frac{\lambda_1}{2}, \quad \omega_2 = \frac{\lambda_2}{2}, \quad \xi_1 = \frac{C_3}{\omega_1}, \quad \xi_2 = \frac{C_4}{\omega_2}.$$

Then (2.6) and (2.19) follow that

$$y(\xi) = \frac{2}{5}v + \frac{\lambda_1^2}{4}[(1 - g(\xi))^2 \tan^2(\omega_1(\xi + \xi_1)) - 2g(\xi)]. \tag{2.20}$$

Obviously, ω_1 and ω_2 do not have commensurability. Hence, the solution defined by (2.20) is a quasi-periodic wave solution.

To sum up, we have proved the following conclusion.

Theorem 2.1 (i) *For any real pair* $(C_3, C_4) \in \mathbb{R}^2$*, the traveling wave equation* (1.7) *of KdV*6 *equation* (1.2) *has the exact explicit solution given by* (2.10). *Geometrically, in the* 4*-dimensional phase space, the solution curves defined by*

$$(x_1(\xi) = y(\xi),\ x_2(\xi) = y'(\xi),\ x_3(\xi) = y''(\xi),\ x_4(\xi) = y'''(\xi))$$

lie in the intersection of two level manifolds

$$\Phi_1(x_1, x_2, x_3, x_4) = K_{11} \quad \text{and} \quad \Phi_2(x_1, x_2, x_3, x_4) = K_{21}$$

of system (2.3).

(ii) *There exists a quasi-periodic wave solution to* (2.3) *with the parametric representation given by* (2.20)*. Geometrically, it corresponds to the intersection curve of two level manifolds*

$$\Phi_1(x_1,x_2,x_3,x_4)=K_{12} \quad \textit{and} \quad \Phi_2(x_1,x_2,x_3,x_4)=K_{22}$$

of system (2.3)*.*

3 The Exact Traveling Wave Solutions to (1.8) and Their Geometric Property

Corresponding to (1.8), we have the 4-dimensional system

$$\begin{aligned} &x_1'=x_2, \quad x_2'=x_3, \quad x_3'=x_4,\\ &x_4'=18x_1x_3+9x_2^2-24x_1^3+4vx_1^2+\frac{16}{9}v^2x_1-\frac{26}{243}c^3. \end{aligned} \tag{3.1}$$

Now, (3.1) has three elementary equilibrium points at

$$Q_1\Big(-\frac{1}{18}(3\sqrt{3}-1)v,0,0,0\Big), \quad Q_2\Big(\frac{1}{18}v,0,0,0\Big), \quad Q_3\Big(\frac{1}{18}(3\sqrt{3}+1)v,0,0,0\Big).$$

The eigenvalues of $M(-\frac{1}{18}(3\sqrt{3}-1)v,0,0,0)$ are two purely imaginary pairs:

$$\pm(2v(\sqrt{3}-1))^{\frac{1}{2}}\mathrm{i}, \quad \pm(v(\sqrt{3}+1))^{\frac{1}{2}}\mathrm{i}.$$

The eigenvalues of $M(\frac{1}{18}v,0,0,0)$ are a real pair and a purely imaginary pair: $\pm\sqrt{2v}$, $\pm\sqrt{v}\mathrm{i}$. The eigenvalues of $M(\frac{1}{18}(3\sqrt{3}+1)v,0,0,0)$ are two real pairs

$$\pm((\sqrt{3}-1)v)^{\frac{1}{2}}, \quad \pm(2(\sqrt{3}+1)v)^{\frac{1}{2}}.$$

We know from [16] that the first two integrals of (2.11) are as follows:

$$\Psi_1(x_1,x_2,x_3,x_4)=x_2x_4-9x_1x_2^2-\frac{1}{2}x_3^2+6x_1^4-\frac{4v}{3}x_1^3-\frac{8v^2}{9}x_1^2+\frac{26v^3}{243}x_1-\frac{280}{2187}v^4, \tag{3.2}$$

$$\begin{aligned} \Psi_2(x_1,x_2,x_3,x_4)=\ &\frac{1}{3}x_3^3+4cx_3^2x_1-\frac{104}{3}vx_3x_1^3+\frac{8}{9}v^2x_3x_1^2+\frac{532}{243}v^3x_3x_1+18x_3x_1x_2^2\\ &-2vx_3x_2^2-x_2x_3x_4+12vx_1x_4x_2-18x_4x_1^2x_2-54vx_1^2x_2^2+\frac{8}{9}v^2x_4x_2\\ &-8v^2x_1x_2^2-\frac{9}{4}x_2^4-72x_1^6+x_1x_4^2+54x_1^3x_2^2-\frac{5}{9}vx_4^2-\frac{31}{27}v^3x_2^2\\ &-9x_3^2x_1^2+48x_3x_1^4-\frac{4}{9}v^2x_3^2-\frac{260}{2187}v^4x_3+72vx_1^5+\frac{4}{3}v^2x_1^4\\ &-\frac{724}{81}v^3x_1^3-\frac{4}{27}v^4x_1^2+\frac{208}{2187}v^5x_1+\frac{2263}{59049}v^6. \end{aligned} \tag{3.3}$$

For a given real pair $(K_1,K_2)\in\mathbb{R}^2$, two functions

$$\Psi_1(x_1,x_2,x_3,x_4)=K_1 \quad \text{and} \quad \Psi_2(x_1,x_2,x_3,x_4)=K_2$$

define two families of the invariant set of system (3.1). Especially, we see from (3.2) and (3.3) that

$$K_{11} = \Psi_1\Big(\frac{1}{18}v, 0, 0, 0\Big) = -\frac{1}{8}v^4, \quad K_{21} = \Psi_2\Big(\frac{1}{18}v, 0, 0, 0\Big) = \frac{1}{24}v^6.$$

Thus, the two level set (III_1) and (III_2) defined by

$$\Psi_1(x_1, x_2, x_3, x_4) = K_{11} \quad \text{and} \quad \Psi_2(x_1, x_2, x_3, x_4) = K_{21}$$

pass through the equilibrium point Q_2.

We next use the method given by Cosgrove in [16, Section 6] to study the solutions lying in the intersection of (III_1) and (III_2). We rename the parameters α, β and the integration constants K_1 and K_2 as follows:

$$\begin{aligned} \alpha &= \mu^2 A, \quad \beta = \frac{1}{4}\mu^4\alpha F - \frac{8}{243}\alpha^3, \\ K_{11} &= \mu^8\Big(24K^2 - \frac{1}{8}F^2\Big), \\ K_{21} &= \mu^{12}F\Big(12K^2 - \frac{1}{24}F^2\Big). \end{aligned}$$

For system (3.1), we have

$$\mu = 2, \quad A = v, \quad F = \frac{v^2}{8}, \quad K = \frac{1}{32}v^2.$$

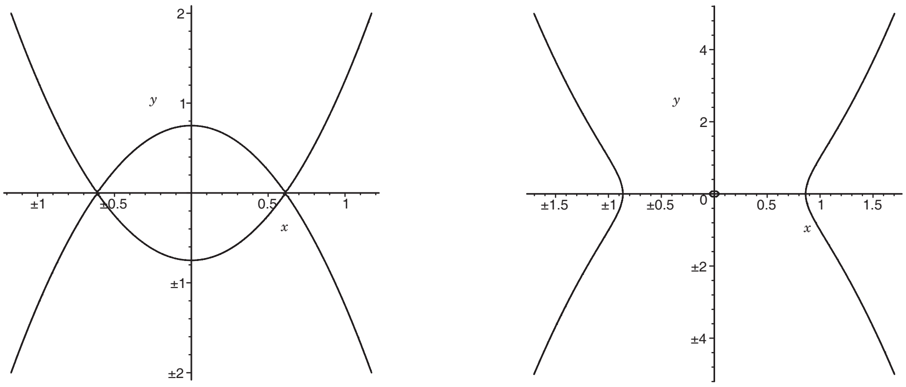
Thus, there exist the solutions to (1.8) as follows:

$$y(\xi) = \frac{(U' - V')^2}{(U - V)^2} - 4(U + V)^2 + \frac{5}{9}v, \tag{3.4}$$

where $U(\xi)$ and $V(\xi)$ are defined by

$$\begin{aligned} (U')^2 &= 4U^4 - vU^2 + \frac{1}{16}v^2 = \Big(2U^2 - \frac{1}{4}v\Big)^2, \\ (V')^2 &= 4V^4 - vV^2 = V^2(4V^2 - v). \end{aligned} \tag{3.5}$$

Notice that in the $(U, \dot{U})$-phase plane and $(V, \dot{V})$-phase plane, the two equations defined by (3.5) determine two fourth algebraic curves shown in Figures 1(a) and (b), respectively.



(a) (b)

Figure 1 The phase curves defined by (3.5).

Clearly, the first equation of (3.5) gives rise to two heteroclinic orbits connecting two critical points

$$(U,\dot{U})=\Big(\pm\sqrt{\frac{v}{8}},0\Big)$$

and 4 open orbits (the stable manifold and unstable manifold of two critical points, respectively, see Figure 1(a)), while the second equation of (3.5) gives rise to two open curves passing through two points

$$(V,\dot{V})=\Big(\pm\frac{\sqrt{v}}{2},0\Big)$$

and the origin

$$(V,\dot{V})=(0,0).$$

Thus, by using (3.5) to do integration, we obtain the following results:

(1) Corresponding to two heteroclinic orbits in Figure 1(a), we have the parametric representations

$$U(\xi)=U_1(\xi)=\pm\sqrt{\frac{v}{8}}\tanh\Big(\sqrt{\frac{v}{2}}\xi\Big).$$

(2) Corresponding to the stable manifold and the unstable manifold in both the right side and the left side of the saddle points $(\pm\sqrt{\frac{v}{8}},0)$ in Figure 1(a), we have the parametric representations

$$U(\xi)=U_2(\xi)=\pm\sqrt{\frac{v}{8}}\coth\Big(\sqrt{\frac{v}{2}}\xi\Big).$$

(3) Corresponding to the two equilibrium points $(\pm\sqrt{\frac{v}{8}},0)$ in Figure 1(a), we have the parametric representations

$$U(\xi)=U_3(\xi)=\pm\sqrt{\frac{v}{8}}.$$

(4) Corresponding to the origin $(0,0)$ in Figure 1(b), we have its parametric representation

$$V(\xi)=V_1(\xi)=0.$$

(5) Corresponding to two open curves passing through the point $(\pm\frac{\sqrt{v}}{2},0)$ in Figure 1(b), we have the parametric representations

$$V(\xi)=V_2(\xi)=\pm\frac{\sqrt{v}}{2}\sec(\sqrt{v}\xi).$$

Therefore, as some intersection curves of two level manifolds

$$\Phi_1(x_1,x_2,x_3,x_4)=-\frac{1}{8}v^4 \quad\text{and}\quad \Phi_2(x_1,x_2,x_3,x_4)=\frac{1}{24}v^6,$$

we obtain the following exact explicit non-trivial parametric representations of the solutions to (1.8) or (3.1):

$$y=x_1(\xi)=\frac{(U_1'-V_1')^2}{(U_1-V_1)^2}-4(U_1+V_1)^2+\frac{5}{9}v=\frac{v}{18}+\frac{v}{2}\mathrm{csch}^2\Big(\sqrt{\frac{v}{2}}\xi\Big), \tag{3.6}$$

$$y=x_1(\xi)=\frac{(U_2'-V_1')^2}{(U_2-V_1)^2}-4(U_2+V_1)^2+\frac{5}{9}v=\frac{v}{18}-\frac{v}{2}\mathrm{sech}^2\Big(\sqrt{\frac{v}{2}}\xi\Big), \tag{3.7}$$

$$\begin{aligned}y=x_1(\xi)&=\frac{(U_3'-V_2')^2}{(U_3-V_2)^2}-4(U_3+V_2)^2+\frac{5}{9}v=\frac{v\tan^2(\sqrt{v}\xi)\sec^2(\sqrt{v}\xi)}{(\frac{1}{\sqrt{2}}\mp\sec(\sqrt{v}\xi))^2}\\&\quad-v\Big(\frac{1}{\sqrt{2}}\pm\sec(\sqrt{v}\xi)\Big)^2+\frac{5}{9}v,\end{aligned} \tag{3.8}$$

$$\begin{aligned}y=x_1(\xi)&=\frac{(U_1'-V_2')^2}{(U_1-V_2)^2}-4(U_1+V_2)^2+\frac{5}{9}v\\&=\frac{v(\frac{1}{2}\mathrm{sech}^2(\sqrt{\frac{v}{2}}\xi)\mp\tan(\sqrt{v}\xi)\sec(\sqrt{v}\xi))^2}{(\frac{1}{\sqrt{2}}\tanh(\sqrt{\frac{v}{2}}\xi)\mp\sec(\sqrt{v}\xi))^2}\\&\quad-v\Big(\frac{1}{\sqrt{2}}\tanh\Big(\sqrt{\frac{v}{2}}\xi\Big)\pm\sec(\sqrt{v}\xi)\Big)^2+\frac{5}{9}v,\end{aligned} \tag{3.9}$$

$$\begin{aligned}y=x_1(\xi)&=\frac{(U_2'-V_2')^2}{(U_2-V_2)^2}-4(U_2+V_2)^2+\frac{5}{9}v\\&=\frac{v(\frac{1}{2}\mathrm{csch}^2(\sqrt{\frac{v}{2}}\xi)\mp\tan(\sqrt{v}\xi)\sec(\sqrt{v}\xi))^2}{(\frac{1}{\sqrt{2}}\coth(\sqrt{\frac{v}{2}}\xi)\mp\sec(\sqrt{v}\xi))^2}\\&\quad-v\Big(\frac{1}{\sqrt{2}}\coth\Big(\sqrt{\frac{v}{2}}\xi\Big)\pm\sec(\sqrt{v}\xi)\Big)^2+\frac{5}{9}v.\end{aligned} \tag{3.10}$$

We see from

$$u(\xi)=\int\Big(\frac{1}{18}v-y(\xi)\Big)\mathrm{d}\xi$$

that (3.6) and (3.7) just give rise to the Wazwaz's result in [11, Section 5], the so-called "the single soliton solution" and "the single singular soliton solution" to (1.3) as follows:

$$u(\xi)=\frac{p\mathrm{e}^{p\xi}}{1+\mathrm{e}^{p\xi}} \quad\text{and}\quad u(\xi)=-\frac{p\mathrm{e}^{p\xi}}{1-\mathrm{e}^{p\xi}}, \tag{3.11}$$

where $p=\sqrt{v}$.

To sum up, we have the following conclusion.

Theorem 3.1 The traveling wave equation (1.8) of KdV6 equation (1.3) has the exact explicit solutions given by (3.6)–(3.10). Geometrically, in the 4-dimensional phase space, these solution curves defined by $(x_1(\xi) = y(\xi),\ x_2(\xi) = y'(\xi),\ x_3(\xi) = y''(\xi),\ x_4(\xi) = y'''(\xi))$ lie in the intersection of two level manifolds $\Psi_1(x_1, x_2, x_3, x_4) = K_{11}$ and $\Psi_2(x_1, x_2, x_3, x_4) = K_{21}$ of system (3.1).

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