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Relationship Between the Restricted AKNS Flows and the Restricted KdV Flows*

Ruguang ZHOU[1]

Abstract It is well-known that every member of the KdV hierarchy of equations can be obtained from the AKNS hierarchy of equations by imposing a simple reduction. The author finds that the reduction conditions of the potentials in the spectral problem can be replaced by adding additional eigenfunction equations to the spectral problem, and then shows that the restricted KdV flows, such as the Neumann system, the Garnier system and the generalized multicomponent Hénon-Hieles system, are a kind of special reductions of the restricted AKNS flows.

Keywords The restricted AKNS flow, The restricted KdV flow, Nonlinearization of spectral problem, Reduction condition, Integrable Hamiltonian system
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1 Introduction

Over the past two decades, a large number of finite dimensional integrable Hamiltonian systems called the restricted flows were obtained from (1+1)-dimensional soliton equations (see [1–21]). In particular, the restricted AKNS flows and the restricted KdV flows including the Neumann system, the Garnier system and the generalized multicomponent Hénon-Hieles system were obtained or reobtained from the AKNS hierarchy and the KdV hierarchy, respectively (see [2–12]). These restricted flows were applied to performing numerical analysis and graphic presentations of solutions of soliton equations (see [22–24]).

On the other hand, it is well-known that most of physically interesting soliton equations such as the KdV equation, the sine-Gordon equation, the mKdV equation, the NLS equation and the DNLS equation are the reduced systems of large systems, and the reduction problem was one of the central problems in the theory of integrable systems since its early days (see [25–32]). Many systematic studies from various points of view were made. It is natural to ask what the relations between the restricted flows of reduced systems and that of large systems are.

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[1]School of Mathematical Sciences, Xuzhou Normal University, Xuzhou 221116, Jiangsu, China.
E-mail: rgzhou@public.xz.js.cn
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In the present work, we study the relationship between the restricted KdV flows and the restricted AKNS flows. We first show that the reduction condition of potential in the spectral problem of the KdV equation may be replaced by adding a pair of additional eigenfunction equations. Then, based on this, we deduce that the nonlinearization of spectral problems (NSPs) of the KdV hierarchy can be simply transplanted from that of the AKNS hierarchy. Finally, we prove that the restricted KdV flows including the celebrated C. Neumann system (G_0-constraint KdV flow), the Garnier system (G_2-constraint KdV flow) and the generalized multicomponent Hénon-Hieles system (G_4-constraint KdV flow) are a kind of special reductions of the restricted AKNS flows.

2 The AKNS Hierarchy and the KdV Hierarchy

In this section, we review the construction of the AKNS hierarchy of equations and the KdV hierarchy of equations following [21]. The AKNS hierarchy of equations consists of infinitely many evolution equations, and the n-th equation reads

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = JG_n, \quad J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad G_n = \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix}, \quad n \geq 1, \tag{2.1}$$

where b_j's, c_j's are determined recursively from

$$\begin{pmatrix} c_{j+1} \\ b_{j+1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \partial - 2v\partial^{-1}u & 2v\partial^{-1}v \\ -2u\partial^{-1}u & -\partial + 2u\partial^{-1}v \end{pmatrix} \begin{pmatrix} c_j \\ b_j \end{pmatrix}, \quad j \geq 2 \tag{2.2}$$

with initial conditions

$$b_0 = c_0 = 0, \quad b_1 = 2u, \quad c_1 = 2v,$$

where

$$\partial = \frac{\partial}{\partial x}$$

is the differential operator with respect to variable x, and

$$\partial^{-1} = \int \cdot \, \mathrm{d}x$$

is the integration operator with zero constant of integration.

G_k is called as the Lenard gradient and will play an important role in nonlinearization of spectral problems of the AKNS hierarchy. The first few Lenard gradients are as follows:

$$\begin{aligned}
G_0 &= 2(v,u)^{\mathrm{T}}, \quad G_1 = (v_x, -u_x)^{\mathrm{T}}, \\
G_2 &= \frac{1}{2}(v_{xx} - 2uv^2, u_{xx} - 2u^2v)^{\mathrm{T}}, \\
G_3 &= \frac{1}{4}(v_{xxx} - 6uvv_x, -u_{xxx} + 6uu_xv)^{\mathrm{T}}, \\
G_4 &= \frac{1}{8} \begin{pmatrix} v_{xxxx} - 4u_xvv_x - 6uv_x^2 - 8uvv_{xx} - u_{xx}v^2 + 6u^2v^3 \\ u_{xxxx} - 4uu_xv_x - 6u_x^2v - 8uvu_{xx} - 2u^2v_{xx} + 6u^3v^2 \end{pmatrix}.
\end{aligned}$$

It is not difficult to check that (2.1) is equivalent to

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0,$$

which is the compatible condition of the spectral problem

$$\phi_x = U(u,v,\lambda)\phi, \quad U(u,v,\lambda) = \begin{pmatrix} -\lambda & u \\ v & \lambda \end{pmatrix} \tag{2.3}$$

and the auxiliary spectral problem

$$\phi_{t_n} = V^{(n)}(u,v,\lambda)\phi, \quad V^{(n)}(u,v,\lambda) = \sum_{j=0}^{n} \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \lambda^{n-j}, \quad n \geq 0, \tag{2.4}$$

where λ is a spectral parameter and

$$a_0 = -2, \quad a_1 = 0, \quad a_j = \partial^{-1}(uc_j - vb_j), \quad j \geq 2.$$

The first equation in the AKNS hierarchy is a trivial one,

$$u_{t_1} = 2u_x, \quad v_{t_1} = 2v_x.$$

The second equation is

$$\begin{cases} u_{t_2} + u_{xx} - 2u^2 v = 0, \\ v_{t_2} - v_{xx} + 2uv^2 = 0. \end{cases} \tag{2.5}$$

The third equation reads

$$\begin{cases} u_{t_3} - \dfrac{1}{2}u_{xxx} + 3uu_x v = 0, \\ v_{t_3} - \dfrac{1}{2}v_{xxx} + 3uvv_x = 0, \end{cases} \tag{2.6}$$

which associates with spectral problem (2.3) and

$$\phi_t = V^{(3)}(u,v,\lambda)\phi, \tag{2.7}$$

where

$$V^{(3)}(u,v,\lambda) = \begin{pmatrix} -2\lambda^3 + uv\lambda - \frac{1}{2}(u_x v - uv_x) & 2u\lambda^2 - u_x\lambda + \frac{1}{2}(u_{xx} - 2u^2 v) \\ 2v\lambda^2 + v_x\lambda + \frac{1}{2}(v_{xx} - 2uv^2) & 2\lambda^3 - uv\lambda + \frac{1}{2}(u_x v - uv_x) \end{pmatrix}.$$

Making use of the recursion relation (2.2), we can show that all the $(2n+1)$-th $(n \geq 0)$ AKNS equations allow the reduction $v = 1$, which gives rise to the KdV hierarchy (see [33]). In particular, imposing reduction $v = 1$ to the third member yields the KdV equation

$$u_{t_3} - \frac{1}{2}u_{xxx} + 3uu_x = 0. \tag{2.8}$$

It is an already known fact that the associated spectral problems of the KdV hierarchy are just that of the corresponding AKNS system imposing $v = 1$. For example, the KdV equation (2.8) connects

$$\phi_x = U(u,1,\lambda)\phi, \quad U(u,1,\lambda) = \begin{pmatrix} -\lambda & u \\ 1 & \lambda \end{pmatrix} \tag{2.9}$$

and

$$\phi_{t_3} = V^{(3)}(u,1,\lambda)\phi,$$

where

$$V^{(3)}(u,1,\lambda) = \begin{pmatrix} -2\lambda^3 + u\lambda - \frac{1}{2}u_x & 2u\lambda^2 - u_x\lambda + \frac{1}{2}(u_{xx} - 2u^2) \\ 2\lambda^2 - u & 2\lambda^3 - u\lambda + \frac{1}{2}u_x \end{pmatrix}.$$

3 The Restricted AKNS Flows

In this section, we collect and give some materials on the nonlinearizations of spectral problems of the AKNS hierarchy (see [3, 17, 21]).

Take m distinct parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$, which are called the eigenvalue parameters, and consider m copies of the AKNS spectral problem (2.3) as follows:

$$\begin{cases} \phi_{1j,x} = -\lambda_j \phi_{1j} + u\phi_{2j}, \\ \phi_{2j,x} = v\phi_{1j} + \lambda_j \phi_{2j}, \end{cases} \quad 1 \le j \le m, \tag{3.1}$$

or in a compact form

$$\begin{cases} \Phi_{1,x} = -A\Phi_1 + u\Phi_2, \\ \Phi_{2,x} = v\Phi_1 + A\Phi_2, \end{cases} \tag{3.2}$$

where

$$\Phi_1 = (\phi_{11}, \cdots, \phi_{1m})^{\mathrm{T}},$$
$$\Phi_2 = (\phi_{21}, \cdots, \phi_{2m})^{\mathrm{T}},$$
$$A = \mathrm{diag}(\lambda_1, \cdots, \lambda_m).$$

The nonlinearization of spectral problem (3.2) is to couple a constraint between the eigenfunctions and the potentials to spectral problem (3.2), such that the resulting systems are completely integrable Hamiltonian systems. A well-known constraint due to Cao [1, 3] is

$$G_k = \sum_{j=1}^{m} \mu_j \Big(\frac{\delta\lambda_j}{\delta u}, \frac{\delta\lambda_j}{\delta v}\Big)^{\mathrm{T}}, \quad k \ge 0, \tag{3.3}$$

where G_k is the Lenard gradient defined by (2.1), $\mu_1, \mu_2, \cdots, \mu_m$ are m arbitrary nonzero constants, and $(\frac{\delta\lambda_j}{\delta u}, \frac{\delta\lambda_j}{\delta v})^{\mathrm{T}}$ is the variational derivative of λ_j with respect to potential $(u,v)^{\mathrm{T}}$.

For spectral problem (3.2), it is easy to know, up to a constant,

$$\frac{\delta\lambda_j}{\delta u} = -\phi_{2j}^2, \quad \frac{\delta\lambda_j}{\delta v} = \phi_{1j}^2, \quad 1 \le j \le m. \tag{3.4}$$

Hence, the k-th restricted AKNS flow or the G_k-constraint AKNS flow reads

$$\begin{cases} \Phi_{1,x} = -A\Phi_1 + u\Phi_2, \\ \Phi_{2,x} = v\Phi_1 + A\Phi_2, \\ G_k = (-\langle B\Phi_2, \Phi_2\rangle, \langle B\Phi_1, \Phi_1\rangle)^{\mathrm{T}}, \end{cases} \tag{3.5}$$

where

$$B = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_m)$$

and the diamond bracket denotes the Euclidean scalar product. In the following, we give some restricted AKNS flows.

3.1 The G_0-constraint AKNS flow

Noting $G_0 = (2v, 2u)^{\mathrm{T}}$ and (3.4), we have a G_0-constaint

$$u = \langle B\Phi_1, \Phi_1\rangle, \quad v = -\langle B\Phi_2, \Phi_2\rangle. \tag{3.6}$$

Here we have absorbed a factor $\frac{1}{2}$ by taking advantage of the arbitrariness of μ_j's.

Substituting (3.6) into (3.2), we then get a G_0-constraint AKNS flow,

$$\begin{cases} \Phi_{1,x} = -A\Phi_1 + \langle B\Phi_1, \Phi_1\rangle\Phi_2, \\ \Phi_{2,x} = -\langle B\Phi_2, \Phi_2\rangle\Phi_1 + A\Phi_2, \end{cases} \tag{3.7}$$

which can be written as a Hamiltonian form

$$\Phi_{1,x} = -B^{-1}\frac{\partial H_{G_0}}{\partial \Phi_2}, \quad \Phi_{2,x} = B^{-1}\frac{\partial H_{G_0}}{\partial \Phi_1}, \tag{3.8}$$

where

$$H_{G_0} = \langle A\Phi_1, B\Phi_2\rangle - \frac{1}{2}\langle B\Phi_1, \Phi_1\rangle\langle B\Phi_2, \Phi_2\rangle.$$

Furthermore, it allows Lax representation

$$\frac{\mathrm{d}}{\mathrm{d}x}L_{G_0}(\lambda) = [\widetilde{U}_{G_0}, L_{G_0}(\lambda)]$$

with

$$L_{G_0}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sum_{j=1}^{m} \frac{\mu_j}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix} \tag{3.9}$$

and

$$\widetilde{U}_{G_0} = \begin{pmatrix} -\lambda & \langle B\Phi_1, \Phi_1\rangle \\ -\langle B\Phi_2, \Phi_2\rangle & \lambda \end{pmatrix}.$$

Moreover, it can be easily checked that Lax matrix (3.9) satisfies an r-matrix relation. Hence,

$$\det L_{G_0}(\lambda) = -1 + \sum_{j=1}^{m} \frac{F_j}{\lambda - \lambda_j} \tag{3.10}$$

gives rise to m functionally independent and conserved integrals in involution

$$F_j = -2\mu_j \phi_{1j}\phi_{2j} + \sum_{\substack{k=1\\k\neq j}}^{m} \frac{\mu_j\mu_k(\phi_{1j}\phi_{2k} - \phi_{1k}\phi_{2j})^2}{\lambda_j - \lambda_k}, \quad 1 \leq j \leq m.$$

Therefore, (3.7) is a completely integrable Hamiltonian system.

3.2 The G_2-constraint AKNS flow

A G_2-constraint is

$$v_{xx} - 2uv^2 = -\langle B\Phi_2, \Phi_2\rangle, \quad u_{xx} - 2u^2 v = \langle B\Phi_1, \Phi_1\rangle. \tag{3.11}$$

Coupling it with system (3.2) yields a G_2-constraint AKNS flow

$$\begin{cases} \Phi_{1,x} = -A\Phi_1 + u\Phi_2, \\ \Phi_{2,x} = v\Phi_1 + A\Phi_2, \\ v_{xx} - 2uv^2 = -\langle B\Phi_2, \Phi_2\rangle, \\ u_{xx} - 2u^2 v = \langle B\Phi_1, \Phi_1\rangle. \end{cases} \tag{3.12}$$

Introduce four new variables

$$\phi_{N+1} = u, \quad \phi_{N+2} = v, \quad \psi_{N+1} = v_x, \quad \psi_{N+2} = u_x$$

and define a symplectic form by

$$\sum_{j=1}^{N} \mu_j \mathrm{d}\phi_{2j} \wedge \mathrm{d}\phi_{1j} + \frac{1}{2}\mathrm{d}\psi_{N+1} \wedge \mathrm{d}\phi_{N+1} + \frac{1}{2}\mathrm{d}\psi_{N+2} \wedge \mathrm{d}\phi_{N+2}$$

over $\mathbb{R}^{2m+4}$. Then (3.12) can be written in the following Hamiltonian form:

$$\begin{aligned} \Phi_{1,x} &= -B^{-1}\frac{\partial H_{G_2}}{\partial \Phi_2}, \qquad \Phi_{2,x} = B^{-1}\frac{\partial H_{G_2}}{\partial \Phi_1}, \\ \phi_{N+1,x} &= -2\frac{\partial H_{G_2}}{\partial \psi_{N+1}}, \quad \psi_{N+1,x} = 2\frac{\partial H_{G_2}}{\partial \phi_{N+1}}, \\ \phi_{N+2,x} &= -2\frac{\partial H_{G_2}}{\partial \psi_{N+2}}, \quad \psi_{N+2,x} = 2\frac{\partial H_{G_2}}{\partial \phi_{N+2}}, \end{aligned}$$

where

$$\begin{aligned} H_{G_2} = \langle B\Phi_1, A\Phi_2\rangle &- \frac{1}{2}\phi_{N+1}\langle B\Phi_2, \Phi_2\rangle + \frac{1}{2}\phi_{N+2}\langle B\Phi_1, \Phi_1\rangle \\ &+ \frac{1}{2}\phi_{N+1}^2\phi_{N+2}^2 - \frac{1}{2}\psi_{N+1}\psi_{N+2}. \end{aligned}$$

A straightforward verification shows that (3.12) allows the following Lax representation:

$$\frac{\mathrm{d}}{\mathrm{d}x}L_{G_2}(\lambda)=[U(u,v,\lambda),L_{G_2}(\lambda)],$$

where

$$L_{G_2}(\lambda)=2\begin{pmatrix}-2\lambda^2+uv & 2\lambda u-u_x\\ 2\lambda v+v_x & 2\lambda^2-uv\end{pmatrix}+\sum_{j=1}^{m}\frac{\mu_j}{\lambda-\lambda_j}\begin{pmatrix}\phi_{1j}\phi_{2j} & -\phi_{1j}^2\\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j}\end{pmatrix}$$

and

$$U(u,v,\lambda)=\begin{pmatrix}-\lambda & u\\ v & \lambda\end{pmatrix}.$$

Again Lax matrix $L_{G_2}(\lambda)$ satisfies an r-matrix relation, the required $m+2$ conserved integrals to support the integrability of (3.12) can be generated from $\det L_{G_2}(\lambda)$, and (3.12) is a completely integrable system.

3.3 The G_4-constraint AKNS flow

The G_4-constraint reads

$$\begin{cases}\frac{1}{4}\Big(\frac{1}{2}v_{xxxx}-2u_xvv_x-3uv_x^2-4uvv_{xx}-u_{xx}v^2+3u^2v^3\Big)=-\langle B\Phi_2,\Phi_2\rangle,\\ \frac{1}{4}\Big(\frac{1}{2}u_{xxxx}-2uu_xv_x-3u_x^2v-4uvu_{xx}-u^2v_{xx}+3u^3v^2\Big)=\langle B\Phi_1,\Phi_1\rangle.\end{cases}\tag{3.13}$$

Absorbing a factor $-\frac{1}{8}$, we get a G_4-constraint

$$\begin{cases}\frac{1}{2}v_{xxxx}-2u_xvv_x-3uv_x^2-4uvv_{xx}-u_{xx}v^2+3u^2v^3=\frac{1}{2}\langle B\Phi_2,\Phi_2\rangle,\\ \frac{1}{2}u_{xxxx}-2uu_xv_x-3u_x^2v-4uvu_{xx}-u^2v_{xx}+3u^3v^2=-\frac{1}{2}\langle B\Phi_1,\Phi_1\rangle.\end{cases}\tag{3.14}$$

Coupling it to system (3.2) yields a G_4-constraint AKNS flow

$$\begin{cases}\Phi_{1,x}=-A\Phi_1+u\Phi_2,\\ \Phi_{2,x}=v\Phi_1+A\Phi_2,\\ \frac{1}{2}v_{xxxx}-2u_xvv_x-3uv_x^2-4uvv_{xx}-u_{xx}v^2+3u^2v^3=\frac{1}{2}\langle B\Phi_2,\Phi_2\rangle,\\ \frac{1}{2}u_{xxxx}-2uu_xv_x-3u_x^2v-4uvu_{xx}-u^2v_{xx}+3u^3v^2=-\frac{1}{2}\langle B\Phi_1,\Phi_1\rangle.\end{cases}\tag{3.15}$$

This is a Hamiltonian system. To end this, we introduce eight new variables

$$q_{11}=u,\quad q_{12}=v,\quad q_{21}=u_x,\quad q_{22}=v_x,\quad p_{11}=-\frac{1}{2}v_{xxx}+u_xv^2+4uvv_x,$$
$$p_{12}=-\frac{1}{2}u_{xxx}+u^2v_x+4uu_xv,\quad p_{21}=\frac{1}{2}v_{xx},\quad p_{22}=\frac{1}{2}u_{xx},$$

and define a symplectic form by

$$\sum_{j=1}^{N}\mu_j\mathrm{d}\phi_{2j}\wedge\mathrm{d}\phi_{1j}+\sum_{k=1}^{2}\sum_{j=1}^{2}\mathrm{d}p_{kj}\wedge\mathrm{d}q_{kj}$$

over $\mathbb{R}^{2m+8}$. Then (3.15) can be written in the following Hamiltonian form:

$$\begin{aligned}
\Phi_{1,x} &= -B^{-1}\frac{\partial H_{G_4}}{\partial \Phi_2}, & \Phi_{2,x} &= B^{-1}\frac{\partial H_{G_4}}{\partial \Phi_1},\\
q_{11,x} &= -\frac{\partial H_{G_4}}{\partial p_{11}}, & p_{11,x} &= \frac{\partial H_{G_4}}{\partial q_{11}},\\
q_{12,x} &= -\frac{\partial H_{G_4}}{\partial p_{12}}, & p_{12,x} &= \frac{\partial H_{G_4}}{\partial q_{12}},\\
q_{21,x} &= -\frac{\partial H_{G_4}}{\partial p_{21}}, & p_{21,x} &= \frac{\partial H_{G_4}}{\partial q_{21}},\\
q_{22,x} &= -\frac{\partial H_{G_4}}{\partial p_{22}}, & p_{22,x} &= \frac{\partial H_{G_4}}{\partial q_{22}},
\end{aligned}$$

where

$$\begin{aligned}
H_{G_4} &= \langle A\Phi_1, B\Phi_2\rangle - \frac{1}{2}q_{11}\langle B\Phi_2, \Phi_2\rangle + \frac{1}{2}q_{12}\langle B\Phi_1, \Phi_1\rangle\\
&\quad - (q_{21}p_{11} + q_{22}p_{12}) + 2p_{22}p_{21} + \frac{1}{2}q_{21}^2 q_{12}^2 - q_{11}^3 q_{12}^3 + \frac{1}{2}q_{11}^2 q_{22}^2 + 4q_{11}q_{12}q_{21}q_{22}.
\end{aligned}$$

A straightforward verification shows that (3.15) allows the following Lax representation:

$$\frac{\mathrm{d}}{\mathrm{d}x}L_{G_4}(\lambda) = [U(u, v, \lambda), L_{G_4}(\lambda)],$$

where

$$L_{G_4}(\lambda) = \begin{pmatrix} A_4 & B_4 \\ C_4 & -A_4 \end{pmatrix} + \sum_{j=1}^{m} \frac{\mu_j}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}$$

and

$$U(u, v, \lambda) = \begin{pmatrix} -\lambda & u \\ v & \lambda \end{pmatrix},$$

where

$$\begin{aligned}
A_4 &= -2\lambda^4 + uv\lambda^3 - \frac{1}{2}(uv_x - u_x v)\lambda + \frac{1}{4}(uv_{xx} + u_{xx}v - u_x v_x - 3u^2 v^2),\\
B_4 &= 2u\lambda^3 - u_x\lambda^2 + \frac{1}{2}(u_{xx} - 2u^2 v)\lambda - \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x v,\\
C_4 &= 2v\lambda^3 + \lambda^2 v_x + \frac{1}{2}(v_{xx} - 2uv^2)\lambda + \frac{1}{4}v_{xxx} - \frac{3}{2}uvv_x.
\end{aligned}$$

Again Lax matrix $L_{G_4}(\lambda)$ satisfies an r-matrix relation, the required $m + 4$ conserved integrals to support the integrability of (3.15) can be generated from $\det L_{G_4}(\lambda)$, and (3.15) is a completely integrable system.

4 From the Restricted AKNS Flows to the Restricted KdV Flows

In this section, we want to establish relations between the restricted KdV flows and the restricted AKNS flows. For this purpose, we adapt the usual procedure of NSP to directly construct the restricted KdV flows from the restricted AKNS flows.

As mentioned before, the KdV hierarchy can be obtained from the odd members in the AKNS hierarchy by imposing the reduction condition $v = 1$. A simple calculation shows that: if $v = 1$ and $(\phi_1, \phi_2)^{\mathrm{T}}$ solves (2.3) with eigenvalue λ, then $(\phi_1 + 2\lambda\phi_2, \phi_2)^{\mathrm{T}}$ solves (2.3) with eigenvalue $-\lambda$. This implies that the reduction condition $v = 1$ leads to that the eigenvalues come in pair: λ and $-\lambda$.

We remark that, for N eigenparameters: $\lambda_1, \lambda_2, \cdots, \lambda_N$ ($\lambda_i \neq \pm\lambda_j, 1 \leq i, j \leq N$), N copies of the spectral problems of the KdV hierarchy

$$\begin{cases} \Phi_{1,x} = -A\Phi_1 + u\Phi_2, \\ \Phi_{2,x} = v\Phi_1 + A\Phi_2, \\ v = 1 \end{cases} \tag{4.1}$$

is equivalent to the following system:

$$\begin{cases} \Phi_{1,x} = -A\Phi_1 + u\Phi_2, \\ \Phi_{2,x} = v\Phi_1 + A\Phi_2, \\ (\Phi_1 + 2A\Phi_2)_x = A(\Phi_1 + 2A\Phi_2) + u\Phi_2, \\ \Phi_{2,x} = v(\Phi_1 + 2A\Phi_2) - A\Phi_2. \end{cases} \tag{4.2}$$

Here and after,

$$\Phi_1 = (\phi_{11}, \cdots, \phi_{1N})^{\mathrm{T}},$$

$$\Phi_2 = (\phi_{21}, \cdots, \phi_{2N})^{\mathrm{T}},$$

$$A = \mathrm{diag}(\lambda_1, \cdots, \lambda_N).$$

Actually, the last two equations in (4.2) are equivalent to $v = 1$ by applying a straightforward computation to the first two equations.

Therefore, to perform the NSP of the KdV hierarchy, we only need to consider system (4.2) instead of (4.1). This is very important for us to get the restricted KdV flows from the restricted AKNS flows because (4.2) is just a special system of $2N$ copies of the AKNS spectral problems in terms of eigenfunctions

$$(\widetilde{\phi}_{1j}, \widetilde{\phi}_{2j})^{\mathrm{T}} = (\phi_{1j}, \phi_{2j})^{\mathrm{T}}, \quad (\widetilde{\phi}_{1,N+j}, \widetilde{\phi}_{2,N+j})^{\mathrm{T}} = (\phi_{1j} + 2\lambda_j\phi_{2j}, \phi_{2j})^{\mathrm{T}}, \quad 1 \leq j \leq N$$

and corresponding eigenvalues

$$\lambda_j, -\lambda_j, \quad 1 \leq j \leq N.$$

To make this precise, we introduce the notion

$$\widetilde{A} = \mathrm{diag}(\lambda_1, \cdots, \lambda_N, -\lambda_1, \cdots, -\lambda_N), \tag{4.3}$$

$$\widetilde{\Phi}_1 = \begin{pmatrix} \Phi_1 \\ \Phi_1 + 2A\Phi_2 \end{pmatrix}, \quad \widetilde{\Phi}_2 = \begin{pmatrix} \Phi_2 \\ \Phi_2 \end{pmatrix}. \tag{4.4}$$

Then (4.2) can be written in the following form of $2N$ copies of the AKNS spectral problems:

$$\begin{cases}\widetilde{\Phi}_{1,x} = -\widetilde{A}\widetilde{\Phi}_1 + u\widetilde{\Phi}_2,\\ \widetilde{\Phi}_{2,x} = v\widetilde{\Phi}_1 + \widetilde{A}\widetilde{\Phi}_2.\end{cases} \tag{4.5}$$

This enables us to get the restricted KdV flows from the restricted AKNS flows just through a simple substitution: m by $2N$, Φ_1 by $\widetilde{\Phi}_1$ and Φ_2 by $\widetilde{\Phi}_2$. This implies that the restricted KdV flows are a kind of special reductions of the restricted AKNS flows. In the following, we exhibit the first three restricted KdV flows.

4.1 The G_0-constraint KdV flow

In contrast with (3.6), the G_0 constraint for the KdV hierarchy is

$$u = \langle \widetilde{B}\widetilde{\Phi}_1, \widetilde{\Phi}_1\rangle, \quad v = -\langle \widetilde{B}\widetilde{\Phi}_2, \widetilde{\Phi}_2\rangle, \tag{4.6}$$

where

$$\widetilde{B} = \mathrm{diag}(\mu_1, \cdots, \mu_{2N})$$

and $\mu_1, \cdots, \mu_{2N}$ are $2N$ arbitrary nonzero constants. For the sake of simplicity, we choose

$$\widetilde{B} = -\frac{1}{2}I_{2N},$$

and finally arrive at an explicit expression of the G_0-constraint

$$\begin{cases} u = -\frac{1}{2}\langle \widetilde{\Phi}_1, \widetilde{\Phi}_1\rangle \equiv -\frac{1}{2}\langle \Phi_1, \Phi_1\rangle - \frac{1}{2}\langle \Phi_1 + 2A\Phi_2, \Phi_1 + 2A\Phi_2\rangle,\\ v = \frac{1}{2}\langle \widetilde{\Phi}_2, \widetilde{\Phi}_2\rangle \equiv \langle \Phi_2, \Phi_2\rangle.\end{cases} \tag{4.7}$$

Substituting (4.7) into (4.5), we then get the following finite-dimensional system:

$$\begin{cases}\widetilde{\Phi}_{1,x} = -\widetilde{A}\widetilde{\Phi}_1 + u\widetilde{\Phi}_2,\\ \widetilde{\Phi}_{2,x} = v\widetilde{\Phi}_1 + \widetilde{A}\widetilde{\Phi}_2,\\ u = -\frac{1}{2}\langle \widetilde{\Phi}_1, \widetilde{\Phi}_1\rangle,\\ v = \frac{1}{2}\langle \widetilde{\Phi}_2, \widetilde{\Phi}_2\rangle\end{cases} \tag{4.8}$$

or equivalently

$$\begin{cases}\Phi_{1,x} = -A\Phi_1 - \frac{1}{2}\langle \Phi_1, \Phi_1\rangle\Phi_2 - \frac{1}{2}\langle \Phi_1 + 2A\Phi_2, \Phi_1 + 2A\Phi_2\rangle\Phi_2,\\ \Phi_{2,x} = \Phi_1 + A\Phi_2,\\ \langle \Phi_2, \Phi_2\rangle = 1.\end{cases} \tag{4.9}$$

Differentiating the last equation, we obtain

$$\langle \Phi_1, \Phi_2\rangle + \langle A\Phi_2, \Phi_2\rangle = 0.$$

Finally the G_0-constraint KdV flow is written as

$$\begin{cases} \Phi_{1,x} = -A\Phi_1 - \frac{1}{2}\langle\Phi_1, \Phi_1\rangle\Phi_2 - \frac{1}{2}\langle\Phi_1 + 2A\Phi_2, \Phi_1 + 2A\Phi_2\rangle\Phi_2, \\ \Phi_{2,x} = \Phi_1 + A\Phi_2, \\ \langle\Phi_2, \Phi_2\rangle = 1, \\ \langle\Phi_1, \Phi_2\rangle + \langle A\Phi_2, \Phi_2\rangle = 0. \end{cases} \tag{4.10}$$

To identify system (4.10), we make a transformation

$$\begin{cases} p_j = \phi_{1j} + \lambda_j\phi_{2j}, \\ q_j = \phi_{2j}, \end{cases} \qquad 1 \le j \le N \tag{4.11}$$

for (4.10) and get

$$\begin{cases} \mathbf{p}_x = -A^2\mathbf{q} + (\langle A^2\mathbf{q}, \mathbf{q}\rangle - \langle\mathbf{p}, \mathbf{p}\rangle)\mathbf{q}, \\ \mathbf{q}_x = \mathbf{p}, \\ \langle\mathbf{q}, \mathbf{q}\rangle = 1, \\ \langle\mathbf{q}, \mathbf{p}\rangle = 0, \end{cases} \tag{4.12}$$

where

$$\mathbf{q} = (q_1, q_2, \cdots, q_N)^{\mathrm{T}}, \quad \mathbf{p} = (p_1, p_2, \cdots, p_N)^{\mathrm{T}}.$$

Obviously, this is nothing but the well-known C. Neumann system. It is a remarkable fact that the first restricted KdV flow is the C. Neumann system (see [3]). Here we show that the C. Neumann system is a special G_0-constraint AKNS flow. In addition, we would like to point out that the integrable properties of the restricted KdV flows such as the Lax representation and the conversed integrals can be directly obtained from those of the restricted AKNS flows. For example, we can get the Lax representation of (4.10) directly from that of the G_0-constraint AKNS flow as follows:

$$L_{K_0,x} = [\widetilde{U}_{K_0}, L_{K_0}],$$

where

$$L_{K_0}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}$$
$$- \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda + \lambda_j}\begin{pmatrix} \phi_{1j}\phi_{2j} + 2\lambda_j\phi_{2j}^2 & -(\phi_{1j} + 2\lambda_j\phi_{2j})^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} - 2\lambda_j\phi_{2j}^2 \end{pmatrix},$$
$$\widetilde{U}_{K_0}(\lambda) = \begin{pmatrix} -\lambda & -\frac{1}{2}\langle\Phi_1, \Phi_1\rangle - \frac{1}{2}\langle\Phi_1 + 2A\Phi_2, \Phi_1 + 2A\Phi_2\rangle \\ 1 & \lambda \end{pmatrix}.$$

From $\det L_{K_0}(\lambda)$, we can generate $N-1$ functionally independent and conserved integrals in involution of (4.10), and (4.10) is completely integrable.

4.2 The G_2-constraint KdV flow

Taking $\widetilde{B}=\frac{1}{2}I_{2N}$, we obtain a G_2-constraint of the KdV hierarchy as follows:

$$\begin{cases} u_{xx}-2u^2v=-\dfrac{1}{2}\langle\widetilde{\Phi}_1,\widetilde{\Phi}_1\rangle\equiv-\langle\Phi_1,\Phi_1\rangle-2\langle\Phi_1,\Phi_2\rangle-2\langle A^2\Phi_2,\Phi_2\rangle,\\ v_{xx}-2v^2u=\dfrac{1}{2}\langle\widetilde{\Phi}_2,\widetilde{\Phi}_2\rangle\equiv\langle\Phi_2,\Phi_2\rangle. \end{cases} \tag{4.13}$$

Coupling (4.13) to (4.5), we get the following system:

$$\begin{cases} \widetilde{\Phi}_{1,x}=-\widetilde{A}\widetilde{\Phi}_1+u\widetilde{\Phi}_2,\\ \widetilde{\Phi}_{2,x}=v\widetilde{\Phi}_1+\widetilde{A}\widetilde{\Phi}_2,\\ u_{xx}-2u^2v=-\dfrac{1}{2}\langle\widetilde{\Phi}_1,\widetilde{\Phi}_1\rangle,\\ v_{xx}-2v^2u=\dfrac{1}{2}\langle\widetilde{\Phi}_2,\widetilde{\Phi}_2\rangle, \end{cases} \tag{4.14}$$

namely,

$$\begin{cases} \Phi_{1,x}=-A\Phi_1+u\Phi_2,\\ \Phi_{2,x}=v\Phi_1+A\Phi_2,\\ v=1,\\ u_{xx}-2u^2v=-\langle\Phi_1,\Phi_1\rangle-2\langle\Phi_1,\Phi_2\rangle-2\langle A^2\Phi_2,\Phi_2\rangle,\\ v_{xx}-2v^2u=\langle\Phi_2,\Phi_2\rangle. \end{cases} \tag{4.15}$$

Substituting $v=1$ into the last equation in (4.15) yields

$$u=\frac{1}{2}\langle\Phi_2,\Phi_2\rangle. \tag{4.16}$$

Furthermore, we find that the fourth equation of (4.15) is an identity. Hence, the G_2-constraint KdV flow (4.15) is simplified to

$$\begin{cases} \Phi_{1,x}=-A\Phi_1-\dfrac{1}{2}\langle\Phi_2,\Phi_2\rangle\Phi_2,\\ \Phi_{2,x}=\Phi_1+A\Phi_2. \end{cases} \tag{4.17}$$

It is easy to verify that, under transform (4.11), (4.17) becomes

$$\begin{cases} q_x=p,\\ p_x=A^2q-\frac{1}{2}\langle q,q\rangle q, \end{cases} \tag{4.18}$$

which is nothing but the Garnier system. Therefore, the Garnier is also a special reduction of the G_2-constraint AKNS flow.

Moreover, applying the previously established result of the G_2-constraint AKNS flow, we know that (4.17) is a completely integrable system and has Lax pair

$$
\begin{aligned}
L_{K_2}(\lambda) = & \begin{pmatrix} -\lambda^2 + \frac{1}{4}\langle\Phi_2, \Phi_2\rangle & \frac{1}{2}\lambda\langle\Phi_2, \Phi_2\rangle - \frac{1}{2}\langle\Phi_1, \Phi_2\rangle - \frac{1}{2}\langle A\Phi_2, \Phi_2\rangle \\ 1 & \lambda^2 + \frac{1}{4}\langle\Phi_2, \Phi_2\rangle \end{pmatrix} \\
& + \frac{1}{8}\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix} \\
& + \frac{1}{8}\sum_{j=1}^{N} \frac{1}{\lambda + \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} + 2\lambda_j\phi_{2j}^2 & -(\phi_{1j} + 2\lambda_j\phi_{2j})^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} - 2\lambda_j\phi_{2j}^2 \end{pmatrix}.
\end{aligned}
$$

4.3 The G_4-constraint KdV flow

We take $\widetilde{B} = \frac{1}{4}I_{2N}$ and thus obtain a G_4-constraint of the KdV hierarchy as follows:

$$
\begin{cases}
\frac{1}{2}v_{xxxx} - 2u_x v v_x - 3uv_x^2 - 4uvv_{xx} - u_{xx}v^2 + 3u^2v^3 = \frac{1}{4}\langle\widetilde{\Phi}_2, \widetilde{\Phi}_2\rangle \equiv \frac{1}{2}\langle\Phi_2, \Phi_2\rangle, \\
\frac{1}{2}u_{xxxx} - 2uu_x v_x - 3u_x^2 v - 4uvu_{xx} - u^2 v_{xx} + 3u^3v^2 \\
\quad = -\frac{1}{4}\langle\widetilde{\Phi}_1, \widetilde{\Phi}_1\rangle \equiv -\frac{1}{2}(\langle\Phi_1, \Phi_1\rangle + 2\langle A\Phi_1, \Phi_2\rangle + 2\langle A^2\Phi_2, \Phi_2\rangle).
\end{cases} \tag{4.19}
$$

Coupling it to (4.5), we get the following system:

$$
\begin{cases}
\widetilde{\Phi}_{1,x} = -\widetilde{A}\widetilde{\Phi}_1 + u\widetilde{\Phi}_2, \\
\widetilde{\Phi}_{2,x} = v\widetilde{\Phi}_1 + \widetilde{A}\widetilde{\Phi}_2, \\
\frac{1}{2}v_{xxxx} - 2u_x v v_x - 3uv_x^2 - 4uvv_{xx} - u_{xx}v^2 + 3u^2v^3 = \frac{1}{4}\langle\widetilde{\Phi}_2, \widetilde{\Phi}_2\rangle, \\
\frac{1}{2}u_{xxxx} - 2uu_x v_x - 3u_x^2 v - 4uvu_{xx} - u^2 v_{xx} + 3u^3v^2 = -\frac{1}{4}\langle\widetilde{\Phi}_1, \widetilde{\Phi}_1\rangle
\end{cases} \tag{4.20}
$$

or equivalently

$$
\begin{cases}
\Phi_{1,x} = -A\Phi_1 + u\Phi_2, \\
\Phi_{2,x} = \Phi_1 + A\Phi_2, \\
-u_{xx} + 3u^2 = \frac{1}{2}\langle\Phi_2, \Phi_2\rangle, \\
\frac{1}{2}u_{xxxx} - 3u_x^2 - 4uu_{xx} + 3u^3 = -\frac{1}{2}(\langle\Phi_1, \Phi_1\rangle + 2\langle A\Phi_1, \Phi_2\rangle + 2\langle A^2\Phi_2, \Phi_2\rangle).
\end{cases} \tag{4.21}
$$

Making use of the first three equations in (4.21), we find that the last equation is an identity. Hence, (4.21) can be simplified to

$$
\begin{cases}
\Phi_{1,x} = -A\Phi_1 + u\Phi_2, \\
\Phi_{2,x} = \Phi_1 + A\Phi_2, \\
u_{xx} - 3u^2 = -\frac{1}{2}\langle\Phi_2, \Phi_2\rangle.
\end{cases} \tag{4.22}
$$

Introducing new variables

$$
Q = u, \quad P = u_x,
$$

we can write (4.22) as a Hamiltonian form

$$\Phi_{1,x} = \frac{\partial H_{K_4}}{\partial \Phi_2}, \quad \Phi_{2,x} = -\frac{\partial H_{K_4}}{\partial \Phi_1}, \quad Q_x = \frac{\partial H_{K_4}}{\partial P}, \quad P_x = -\frac{\partial H_{K_4}}{\partial Q},$$

where

$$H_{K_4} = -\langle A\Phi_1, \Phi_2\rangle + \frac{1}{2}Q\langle \Phi_2, \Phi_2\rangle - \frac{1}{2}\langle \Phi_1, \Phi_1\rangle + \frac{1}{2}P^2 - Q^3. \tag{4.23}$$

Under transform (4.11), (4.23) becomes

$$H_{K_4} = \frac{1}{2}P^2 - Q^3 + \frac{1}{2}Q\langle q, q\rangle - \frac{1}{2}\langle p, p\rangle + \frac{1}{2}\langle A^2 q, q\rangle. \tag{4.24}$$

In comparison with the Hénon-Hieles system $(\mathbb{R}^2, \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2, H_{HH})$ with

$$H_{HH} = \frac{1}{2}p_1^2 + \frac{1}{2}w_1 q_1^2 + a q_1 q_2^2 - \frac{1}{3}b q_1^3 + \frac{1}{2}p_2^2 + \frac{1}{2}w_2 q_2^2,$$

where w_1, w_2, a, b are constants (see [34–35]), we may regard (4.24) as a generalized multicomponent Hénon-Hieles system and we show that it is a reduction of the G_4-constraint AKNS flow. Moreover, applying the previous result on the G_4-constraint AKNS flow, we know that (4.22) is a completely integrable system and has Lax pair

$$\begin{aligned} L_{K_4}(\lambda) = &\begin{pmatrix} A_{K_4} & 2u\lambda^3 - u_x\lambda^2 + \frac{1}{2}(u_{xx} - 2u^2)\lambda - \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \\ 2\lambda^3 - u\lambda & -A_{K_4} \end{pmatrix} \\ &+ \frac{1}{8}\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix} \\ &+ \frac{1}{8}\sum_{j=1}^{N} \frac{1}{\lambda + \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} + 2\lambda_j\phi_{2j}^2 & -(\phi_{1j} + 2\lambda_j\phi_{2j})^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} - 2\lambda_j\phi_{2j}^2 \end{pmatrix}, \end{aligned}$$

where

$$A_{K_4} = -2\lambda^4 + u\lambda^3 - \frac{1}{2}u_x\lambda + \frac{1}{4}u_{xx} - \frac{3}{4}u^2.$$

5 Conclusions

We have shown that the restricted KdV flows including the Neumann system, the Garnier system and the generalized multicomponent Hénon-Hieles system are a kind of special reductions of the restricted AKNS flows. We have already shown that the reduction conditions of potentials in the spectral problem may be replaced by adding additional eigenfunction equations to the spectral problem, and the NSPs of reduced systems can be simply transplanted from those of the large system. We believe that the idea is rather general and could be applied to other soliton equations.

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