Chin. Ann. Math.
33B(2), 2012, 259–270
DOI: 10.1007/s11401-012-0699-y

Chinese Annals of Mathematics, Series B

Generalized Cauchy Matrix Approach for Lattice Boussinesq-Type Equations*

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Abstract The authors generalize the Cauchy matrix approach to get exact solutions to the lattice Boussinesq-type equations: lattice Boussinesq equation, lattice modified Boussinesq equation and lattice Schwarzian Boussinesq equation. Some kinds of solutions including soliton solutions, Jordan block solutions and mixed solutions are obtained.

Keywords Lattice Boussinesq-type equations, Generalized Cauchy matrix approach, Exact solutions
2000 MR Subject Classification 39A14

1 Introduction

Partial difference equation (P△E) has been a major research target for many decades, and substantial progress has been made (see [1–9]). The lattice Boussinesq (BSQ)-type equations, which are the second member of the so-called lattice Gelfand-Dikii type hierarchies (see [10]), have attracted many researchers' attention (see [11–13]). These equations can be presented in the form of the 9-point equations on the two-dimensional lattice. For example, the lattice BSQ equation (see [10]) is

$$\begin{aligned}&\frac{p^3-q^3}{p-q+s_{n+1,m+1}-s_{n,m+2}}-\frac{p^3-q^3}{p-q+s_{n+2,m}-s_{n+1,m+1}}\\&=(p-q+s_{n+1,m}-s_{n,m+1})(2p+q-s_{n+2,m+1}+s_{n,m})\\&\quad-(p-q-s_{n+2,m+1}+s_{n+1,m+2})(2p+q-s_{n+2,m+2}+s_{n,m+1}),\end{aligned}\tag{1.1}$$

in which $s=s_{n,m}$ denotes the dependent variable of the lattice points labeled by $(n,m)\in\mathbb{Z}^2$. In (1.1), the p, q are continuous lattice parameters associated with the grid size in the directions of the lattice given by the independent variables n and m, respectively. For the sake of clarity,

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*Project supported by the National Natural Science Foundation of China (No. 11071157), the Shanghai Leading Academic Discipline Project (No. J50101) and the Postgraduate Innovation Foundation of Shanghai University (No. SHUCX111027).

we prefer to use a notation with lattice shifts denoted by

$$s = s_{n,m} \mapsto \widetilde{s} = s_{n+1,m},$$
$$s = s_{n,m} \mapsto \widehat{s} = s_{n,m+1},$$

in terms of which we also have

$$\widehat{\widetilde{s}} = s_{n+1,m+1}, \quad \widehat{\widetilde{\widetilde{s}}} = s_{n+2,m+1}, \quad \widehat{\widehat{\widetilde{s}}} = s_{n+1,m+2}, \quad \widehat{\widehat{\widetilde{\widetilde{s}}}} = s_{n+2,m+2}.$$

Besides the one-component forms, the lattice BSQ-type equations can also be described as three-component and defined on an elementary square of the 2D lattice, having 3D consistency (see [14–15]). Recently, some extended cases are also considered (see [16–18]).

So far, there are many methods which have been developed to solve the lattice equations, such as direct linearization method (see [8–9, 14, 18]), Bäcklund transformation (see [19–20]), Inverse Scattering Transformation (see [21]), Hirota bilinear method (see [15, 17, 22–24]), and so on. Recently, by using Cauchy matrix approach, Nijhoff et al. researched soliton solutions and elliptic soliton solutions to the ABS lattice equations (see [25–26]). Based on [25], some kinds of solutions to the ABS lattice equations are obtained by the generalized Cauchy matrix approach (see [27]).

In the present paper, we will generalize Cauchy matrix approach to obtain some kinds of solutions to lattice BSQ-type equations: lattice BSQ equation, lattice modified BSQ equation and lattice Schwarzian BSQ equation. Firstly, we show a system of equations which is called condition equations set (CES). From the CES, one can get the lattice BSQ-type equations by introducing some objects and considering their dynamic relationships. Furthermore, the equivalent form of the CES is derived. Finally, by solving the new CES, some exact solutions including soliton solutions, Jordan block solutions and mixed solutions are obtained.

The paper is organized as follows. In Section 2, we give the CES, starting from which one can derive the lattice BSQ-type equations. In Section 3, a new CES is obtained by taking the canonical form of the coefficient matrix K in the original CES. Through solving the new CES, some kinds of solutions besides solitons are obtained. Some remarks are listed in the final section of the paper.

2 Generalized Cauchy Matrix Approach of Lattice Boussinesq-Type Equations

2.1 Condition equation set and recurrence structure

Consider the following CES:

$$-\omega \boldsymbol{M}\boldsymbol{K} + \boldsymbol{K}\boldsymbol{M} = \boldsymbol{r}\,{}^{t}\boldsymbol{c}, \tag{2.1a}$$
$$(p\boldsymbol{I} - \boldsymbol{K})\widetilde{\boldsymbol{r}} = (p\boldsymbol{I} - \omega\boldsymbol{K})\boldsymbol{r}, \quad (q\boldsymbol{I} - \boldsymbol{K})\widehat{\boldsymbol{r}} = (q\boldsymbol{I} - \omega\boldsymbol{K})\boldsymbol{r}, \tag{2.1b}$$
$$(p\boldsymbol{I} - \boldsymbol{K})\widetilde{\boldsymbol{M}} = (p\boldsymbol{I} - \omega\boldsymbol{K})\boldsymbol{M}, \quad (q\boldsymbol{I} - \boldsymbol{K})\widehat{\boldsymbol{M}} = (q\boldsymbol{I} - \omega\boldsymbol{K})\boldsymbol{M}, \tag{2.1c}$$

where

$$\omega^2+\omega+1=0, \quad p,q\in\mathbb{C}$$

are lattice parameters, $\boldsymbol{I}$ is the $N\times N$ unit matrix, $\boldsymbol{K}$ is an $N\times N$ complex matrix with $\mathrm{Det}(\boldsymbol{K})\neq 0$ and two arbitrary eigenvalues $k_i,\ k_j$ satisfy

$$k_i-\omega k_j\neq 0, \quad i,j=1,2,\cdots,N.$$

$\boldsymbol{M}$ and $\boldsymbol{r}$ are an undetermined matrix and a column vector, respectively, and both of them are dependent on $n,\ m$, while ${}^t\boldsymbol{c}$ is a row vector independent of $n,\ m$. Here ${}^t\boldsymbol{c}$ does not mean transposition of $\boldsymbol{c}$ but just a notation, transposition is represented by T.

Firstly, we discuss the dynamics of $\boldsymbol{M}$. Taking $\widetilde{\ }$-shift and $\underset{\sim}{\ }$-shift of (2.1a), respectively, and then by using (2.1c), we have

$$\widetilde{\boldsymbol{M}}(p\boldsymbol{I}-\omega\boldsymbol{K})-(p\boldsymbol{I}-\omega\boldsymbol{K})\boldsymbol{M}=\widetilde{\boldsymbol{r}}\,{}^t\boldsymbol{c}, \tag{2.2a}$$

$$\boldsymbol{M}(p\boldsymbol{I}-\omega^2\boldsymbol{K})-(p\boldsymbol{I}-\boldsymbol{K})\underset{\sim}{\boldsymbol{M}}=\omega\boldsymbol{r}\,{}^t\boldsymbol{c}. \tag{2.2b}$$

Replacing p by q, and $\widetilde{\ }$-shift by $\widehat{\ }$-shift in (2.2), we obtain

$$\widehat{\boldsymbol{M}}(q\boldsymbol{I}-\omega\boldsymbol{K})-(q\boldsymbol{I}-\omega\boldsymbol{K})\boldsymbol{M}=\widehat{\boldsymbol{r}}\,{}^t\boldsymbol{c}, \tag{2.3a}$$

$$\boldsymbol{M}(q\boldsymbol{I}-\omega^2\boldsymbol{K})-(q\boldsymbol{I}-\boldsymbol{K})\underset{\frown}{\boldsymbol{M}}=\omega\boldsymbol{r}\,{}^t\boldsymbol{c}. \tag{2.3b}$$

(2.2) and (2.3) encode all the information on the dynamics of the matrix $\boldsymbol{M}$, w.r.t. the discrete variables $n,\ m$, in addition to (2.1a) which can be thought as the defining property of $\boldsymbol{M}$.

Now we introduce the following objects involving the matrix $\boldsymbol{M}$, from which we can get solutions to the lattice BSQ-type equations:

$$\boldsymbol{u}^{(i)}=(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r}, \tag{2.4a}$$

$${}^t\boldsymbol{u}^{(j)}={}^t\boldsymbol{c}\boldsymbol{K}^j(\boldsymbol{I}+\boldsymbol{M})^{-1}, \tag{2.4b}$$

$$S^{(i,j)}={}^t\boldsymbol{c}\boldsymbol{K}^j(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r} \tag{2.4c}$$

for $i,j\in\mathbb{Z}$. Obviously, the latter objects can also be written as

$$S^{(i,j)}={}^t\boldsymbol{c}\boldsymbol{K}^j\boldsymbol{u}^{(i)}={}^t\boldsymbol{u}^{(j)}\boldsymbol{K}^i\boldsymbol{r}. \tag{2.5}$$

However, they are not symmetric here, w.r.t. the interchange of i and j, i.e., $S^{(i,j)}\neq S^{(j,i)}$, which is different from the lattice KdV-type equations (see [25]).

Taking $\underset{\sim}{\ }$-shift and $\widetilde{\ }$-shift of (2.4a), respectively, and noting (2.2) and (2.5), we obtain

$$(p\boldsymbol{I}-\omega\boldsymbol{K})\boldsymbol{u}^{(i)}=p\widetilde{\boldsymbol{u}}^{(i)}-\widetilde{\boldsymbol{u}}^{(i+1)}+S^{(i,0)}\widetilde{\boldsymbol{u}}^{(0)}, \tag{2.6a}$$

$$\begin{aligned}\Big[\prod_{j=2}^{3}(p\boldsymbol{I}-\omega^j\boldsymbol{K})\widetilde{\boldsymbol{u}}^{(i)}\Big]=&\Big[\prod_{j=1}^{2}(p\boldsymbol{I}-\omega^jE_2)\Big]\boldsymbol{u}^{(i)}\\&+\sum_{l=1}^{2}\omega^l\Big[\prod_{j=2}^{l}(p-\omega^jE_1)\widetilde{S}^{(i,0)}\Big]\cdot\Big[\prod_{j=l+1}^{2}(p-\omega^jE_2)\Big]\boldsymbol{u}^{(0)},\end{aligned} \tag{2.6b}$$

where we have used equation (2.1a) and expression (2.5), and $E_1,\ E_2$ are two operators defined as

$$E_1 S^{(i,j)} = S^{(i,j+1)}, \quad E_2 \boldsymbol{u}^{(i)} = \boldsymbol{u}^{(i+1)}.$$

Similarly, we get

$$(q\boldsymbol{I} - \omega \boldsymbol{K})\boldsymbol{u}^{(i)} = q\widehat{\boldsymbol{u}}^{(i)} - \widehat{\boldsymbol{u}}^{(i+1)} + S^{(i,0)}\widehat{\boldsymbol{u}}^{(0)}, \tag{2.7a}$$

$$\begin{aligned}\Big[\prod_{j=2}^{3}(q\boldsymbol{I} - \omega^j \boldsymbol{K})\Big]\widehat{\boldsymbol{u}}^{(i)} &= \Big[\prod_{j=1}^{2}(q\boldsymbol{I} - \omega^j E_2)\Big]\boldsymbol{u}^{(i)} \\ &\quad + \sum_{l=1}^{2}\omega^l \Big[\prod_{j=2}^{l}(q - \omega^j E_1)\widehat{S}^{(i,0)}\Big] \cdot \Big[\prod_{j=l+1}^{2}(q - \omega^j E_2)\Big]\boldsymbol{u}^{(0)}. \end{aligned}\tag{2.7b}$$

These equations constitute the dynamics of $\boldsymbol{u}^{(i)}$. Furthermore, by using (2.1a) and (2.4a), we have the following algebraic relation:

$$\boldsymbol{K}^3 \boldsymbol{u}^{(i)} = \boldsymbol{u}^{(i+3)} - \sum_{j=0}^{2}\omega^j S^{(i,j)}\boldsymbol{u}^{(2-j)}. \tag{2.8}$$

The recurrence relations of $S(i,j)$ listed below can be derived by multiplying (2.6) and (2.7) from the left by the row vector ${}^t\boldsymbol{c}\boldsymbol{K}^j$ and using expression (2.5),

$$pS^{(i,j)} - \omega S^{(i,j+1)} = p\widetilde{S}^{(i,j)} - \widetilde{S}^{(i+1,j)} + S^{(i,0)}\widetilde{S}^{(0,j)}, \tag{2.9a}$$

$$\begin{aligned}\Big[\prod_{k=2}^{3}(p - \omega^k E_1)\widetilde{S}^{(i,j)}\Big] &= \Big[\prod_{k=1}^{2}(p - \omega^k E_3)S^{(i,j)}\Big] \\ &\quad + \sum_{l=1}^{2}\omega^l \Big[\prod_{k=2}^{l}(p - \omega^k E_1)\widetilde{S}^{(i,0)}\Big] \cdot \Big[\prod_{k=l+1}^{2}(p - \omega^k E_3)S^{(0,j)}\Big], \end{aligned}\tag{2.9b}$$

$$qS^{(i,j)} - \omega S^{(i,j+1)} = q\widehat{S}^{(i,j)} - \widehat{S}^{(i+1,j)} + S^{(i,0)}\widehat{S}^{(0,j)}, \tag{2.9c}$$

$$\begin{aligned}\Big[\prod_{k=2}^{3}(q - \omega^k E_1)\widehat{S}^{(i,j)}\Big] &= \Big[\prod_{k=1}^{2}(q - \omega^k E_3)S^{(i,j)}\Big] \\ &\quad + \sum_{l=1}^{2}\omega^l \Big[\prod_{k=2}^{l}(q - \omega^k E_1)\widehat{S}^{(i,0)}\Big] \cdot \Big[\prod_{k=l+1}^{2}(q - \omega^k E_3)S^{(0,j)}\Big], \end{aligned}\tag{2.9d}$$

where we have introduced operator E_3 defined as

$$E_3 S^{(i,j)} = S^{(i+1,j)}.$$

2.2 Three-component lattice BSQ-type equations

In the following, we choose the values of $i,\ j$ to obtain the three-component lattice BSQ-type equations. Let

$$i = j = 0.$$

Then (2.9) becomes

$$
\begin{aligned}
&ps - \omega S^{(0,1)} = p\widetilde{s} - \widetilde{S}^{(1,0)} + s\widetilde{s}, && (2.10a)\\
&p^2\widetilde{s} + p\omega\widetilde{S}^{(0,1)} + \omega^2\widetilde{S}^{(0,2)} = p^2 s + pS^{(1,0)} + S^{(2,0)} - p\widetilde{s}s - \widetilde{s}S^{(1,0)} - \omega\widetilde{S}^{(0,1)}s, && (2.10b)\\
&qs - \omega S^{(0,1)} = q\widehat{s} - \widehat{S}^{(1,0)} + s\widehat{s}, && (2.10c)\\
&q^2\widehat{s} + q\omega\widehat{S}^{(0,1)} + \omega^2\widehat{S}^{(0,2)} = q^2 s + qS^{(1,0)} + S^{(2,0)} - q\widehat{s}s - \widehat{s}S^{(1,0)} - \omega\widehat{S}^{(0,1)}s, && (2.10d)
\end{aligned}
$$

with $s = S^{(0,0)}$. From (2.10a) and (2.10c), it is easy to get the following two equations:

$$
\begin{aligned}
\widehat{S}^{(1,0)} - \widetilde{S}^{(1,0)} &= (p - q + \widehat{s} - \widetilde{s})s - p\widetilde{s} + q\widehat{s}, && (2.11a)\\
\omega(\widetilde{S}^{(0,1)} - \widehat{S}^{(0,1)}) &= (p - q + \widehat{s} - \widetilde{s})\widehat{\widetilde{s}} - p\widehat{s} + q\widetilde{s}. && (2.11b)
\end{aligned}
$$

Let us focus on (2.10b) and (2.10d). By substraction, one can delete $S^{(2,0)}$; and to delete $S^{(0,2)}$ from the remains, one can make use of (2.9a) and (2.9c), where taking

$$i = 1 \quad \text{and} \quad j = 0.$$

Then after some algebra handling, we can reach

$$\omega\widehat{\widetilde{S}}^{(0,1)} - S^{(1,0)} = -pq + (p + q - \widehat{\widetilde{s}})(p + q + s) - \frac{p^3 - q^3}{p - q + \widehat{s} - \widetilde{s}}. \tag{2.12}$$

Equations (2.10a), (2.10c) and (2.12) are referred to as three-component lattice BSQ equations.

In a similar way to [12], we can consider the dynamic relations of the objects $S^{(1,-1)}$, $S^{(-1,1)}$, $S^{(-1,-1)}$, $S^{(2,-1)}$, $S^{(-1,2)}$, and so on, from which the three-component lattice modified BSQ/Schwarzian BSQ equation can be obtained as

$$
\begin{aligned}
&\widetilde{s}_2 s_3 = p(\widetilde{x} - x), \quad \widehat{s}_2 s_3 = q(\widehat{x} - x), && (2.13a)\\
&s_2\widehat{\widetilde{s}}_3 = \frac{p^3(\widetilde{x} - x)(\widehat{\widetilde{x}} - \widehat{x}) - q^3(\widehat{x} - x)(\widehat{\widetilde{x}} - \widetilde{x})}{pq(\widehat{x} - \widetilde{x})}, && (2.13b)
\end{aligned}
$$

with

$$x = -\omega^2 S^{(-1,-1)} + \frac{n}{p} + \frac{m}{q}, \quad s_2 = \omega + S^{(0,-1)}, \quad s_3 = -\omega^2(S^{(-1,0)} - 1). \tag{2.14}$$

2.3 One-component lattice BSQ-type equations

From the above three-component lattice BSQ-type equations, one can easily get the one-component forms. The one-component lattice BSQ equation (1.1) can be derived from equations (2.11), (2.12) and the identity

$$\widehat{(\omega\widehat{\widetilde{S}}^{(0,1)} - S^{(1,0)})} - \widetilde{(\omega\widehat{\widetilde{S}}^{(0,1)} - S^{(1,0)})} = \omega\widehat{\widetilde{(\widehat{S}^{(0,1)} - \widetilde{S}^{(0,1)})}} - (\widehat{S}^{(1,0)} - \widetilde{S}^{(1,0)}). \tag{2.15}$$

In a similar way to [11] (see [12]), by removing x, s_3 from (2.13), we get the one-component lattice modified BSQ equation

$$p\Big(\frac{s_2}{\widetilde{s}_2}-\frac{\widehat{\widehat{\widetilde{s}}}_2}{\widehat{\widehat{\widetilde{\widetilde{s}}}}_2}\Big)-q\Big(\frac{s_2}{\widehat{s}_2}-\frac{\widehat{\widetilde{\widetilde{s}}}_2}{\widehat{\widehat{\widetilde{\widetilde{s}}}}_2}\Big)=\frac{p^2\widehat{\widetilde{s}}_2-q^2\widehat{\widehat{s}}_2}{p\widehat{\widehat{s}}_2-q\widehat{\widetilde{s}}_2}\cdot\frac{\widehat{\widehat{\widetilde{s}}}_2}{\widehat{s}_2}-\frac{p^2\widetilde{\widetilde{s}}_2-q^2\widehat{\widetilde{s}}_2}{p\widehat{\widetilde{s}}_2-q\widetilde{\widetilde{s}}_2}\cdot\frac{\widehat{\widetilde{\widetilde{s}}}_2}{\widetilde{s}_2}. \tag{2.16}$$

By removing s_2, s_3 from (2.13), we get the one-component lattice Schwarzian BSQ equation

$$\frac{p^3(\widehat{\widetilde{x}}-\widehat{x})(\widehat{\widehat{\widetilde{x}}}-\widehat{\widehat{x}})-q^3(\widehat{\widehat{x}}-\widehat{x})(\widehat{\widehat{\widetilde{x}}}-\widehat{\widetilde{x}})}{p^3(\widetilde{\widetilde{x}}-\widetilde{x})(\widehat{\widetilde{\widetilde{x}}}-\widehat{\widetilde{x}})-q^3(\widehat{\widetilde{x}}-\widetilde{x})(\widehat{\widetilde{\widetilde{x}}}-\widetilde{\widetilde{x}})}=\frac{(\widehat{x}-x)(\widehat{\widehat{\widetilde{\widetilde{x}}}}-\widehat{\widehat{\widetilde{x}}})(\widehat{\widetilde{x}}-\widetilde{\widetilde{x}})}{(\widetilde{x}-x)(\widehat{\widehat{\widetilde{\widetilde{x}}}}-\widehat{\widetilde{\widetilde{x}}})(\widehat{\widetilde{x}}-\widehat{\widehat{x}})}. \tag{2.17}$$

3 Explicit Solutions of CES

In a similar way to [27], we just need to discuss general solutions to the CES (2.1) according to the coefficient matrix $\boldsymbol{K}$ which takes canonical forms. We replace $\boldsymbol{K}$ by matrix $\boldsymbol{\Gamma}$ which is similar to $\boldsymbol{K}$, i.e.,

$$\boldsymbol{K}=\boldsymbol{T}^{-1}\boldsymbol{\Gamma}\boldsymbol{T},$$

and then (2.1) becomes

$$-\omega\boldsymbol{M}_1\boldsymbol{\Gamma}+\boldsymbol{\Gamma}\boldsymbol{M}_1=\boldsymbol{r}_1{}^t\boldsymbol{c}_1, \tag{3.1a}$$

$$(p\boldsymbol{I}-\boldsymbol{\Gamma})\widetilde{\boldsymbol{r}}_1=(p\boldsymbol{I}-\omega\boldsymbol{\Gamma})\boldsymbol{r}_1,\quad (q\boldsymbol{I}-\boldsymbol{\Gamma})\widehat{\boldsymbol{r}}_1=(q\boldsymbol{I}-\omega\boldsymbol{\Gamma})\boldsymbol{r}_1, \tag{3.1b}$$

$$(p\boldsymbol{I}-\boldsymbol{\Gamma})\widetilde{\boldsymbol{M}}_1=(p\boldsymbol{I}-\omega\boldsymbol{\Gamma})\boldsymbol{M}_1,\quad (q\boldsymbol{I}-\boldsymbol{\Gamma})\widehat{\boldsymbol{M}}_1=(q\boldsymbol{I}-\omega\boldsymbol{\Gamma})\boldsymbol{M}_1, \tag{3.1c}$$

with

$$\boldsymbol{M}_1=\boldsymbol{T}\boldsymbol{M}\boldsymbol{T}^{-1},\quad \boldsymbol{r}_1=\boldsymbol{T}\boldsymbol{r},\quad {}^t\boldsymbol{c}_1={}^t\boldsymbol{c}\boldsymbol{T}^{-1}, \tag{3.2}$$

where we denote

$$\boldsymbol{r}_1=(\rho_1,\rho_2,\cdots,\rho_N)^{\mathrm{T}}\quad\text{and}\quad {}^t\boldsymbol{c}_1=(c_1,c_2,\cdots,c_N)\ \text{with}\ \{c_i\neq 0\}.$$

Now, we turn to (3.1). According to the new CES, we can define the following new objects:

$$\boldsymbol{u}_1^{(i)}=(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}\boldsymbol{\Gamma}^i\boldsymbol{r}_1, \tag{3.3a}$$

$${}^t\boldsymbol{u}_1^{(j)}={}^t\boldsymbol{c}_1\boldsymbol{\Gamma}^j(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}, \tag{3.3b}$$

$$S_1^{(i,j)}={}^t\boldsymbol{c}_1\boldsymbol{\Gamma}^j(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}\boldsymbol{\Gamma}^i\boldsymbol{r}_1, \tag{3.3c}$$

which are related to (2.4) through

$$\boldsymbol{u}^{(i)}=\boldsymbol{T}^{-1}\boldsymbol{u}_1^{(i)},\quad {}^t\boldsymbol{u}^{(j)}={}^t\boldsymbol{u}_1^{(j)}\boldsymbol{T},\quad S^{(i,j)}=S_1^{(i,j)}. \tag{3.4}$$

Obviously,

$$S_1^{(i,j)}={}^t\boldsymbol{c}_1\boldsymbol{\Gamma}^j\boldsymbol{u}_1^{(i)}={}^t\boldsymbol{u}_1^{(j)}\boldsymbol{\Gamma}^i\boldsymbol{r}_1$$

and objects $S^{(i,j)}$ and $S_1^{(i,j)}$ lead to the same solutions to the lattice BSQ-type equations.

In the following we show the solutions related to $\boldsymbol{\Gamma}$.

Since the equation (3.1c) can be deduced from (3.1a) and (3.1b) (see [27]), in the following, we just give explicit expressions of $\boldsymbol{r}$ and $\boldsymbol{M}$ in the equations (3.1a) and (3.1b), where $\boldsymbol{\Gamma}$ takes different canonical forms of the $N \times N$ constant matrix $\boldsymbol{K}$, which corresponds to $\boldsymbol{K}$ having different kinds of eigenvalues.

Case 1

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{[D;N]} = \mathrm{Diag}(k_1, k_2, \cdots, k_N), \quad k_i - \omega k_j \neq 0 \quad (i, j = 1, 2, \cdots, N), \tag{3.5}$$

where $\{k_j\}$ are distinct complex numbers and $\{k_i - \omega k_j \neq 0\}$. In this case, $\boldsymbol{r}_1$ in (3.1b) is given by

$$\boldsymbol{r}_1 = \boldsymbol{\rho}_{[D;N]} \boldsymbol{I}_N^{(1)}, \quad \boldsymbol{I}_N^{(1)} = (1, 1, \cdots, 1)_N^{\mathrm{T}}, \tag{3.6a}$$

$$\boldsymbol{\rho}_{[D;N]} = \mathrm{Diag}(\rho_1, \rho_2, \cdots, \rho_N), \quad \rho_i = \Big(\frac{p - \omega k_i}{p - k_i}\Big)^n \Big(\frac{q - \omega k_i}{q - k_i}\Big)^m \rho_i^0, \tag{3.6b}$$

where $\{\rho_i\}$ are the plane-wave factors and $\{\rho_i^0\}$ are complex constants. Note that

$${}^t\boldsymbol{c}_1 = \boldsymbol{I}_N^{(1)\mathrm{T}} \boldsymbol{C}_{[D;N]}$$

with

$$\boldsymbol{C}_{[D;N]} = \mathrm{Diag}(c_1, c_2, \cdots, c_N),$$

and then $\boldsymbol{M}_1$ in (3.1a) can be expressed as

$$\boldsymbol{M}_1 = \boldsymbol{\rho}_{[D;N]} \boldsymbol{G}_{[D;N]} \boldsymbol{C}_{[D;N]}, \quad \boldsymbol{G}_{[D;N]} = \Big(\frac{1}{k_i - \omega k_j}\Big)_{i,j=1,2,\cdots,N}. \tag{3.7}$$

Obviously, $\boldsymbol{G}_{[D;N]}$ is a Cauchy matrix. In this case, the corresponding solutions $S(i,j)$ generate soliton solutions.

Case 2 Jordan-block solutions can be got by taking

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{[J;N]}(k_1) = \begin{pmatrix} k_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & k_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & k_1 \end{pmatrix}_N, \tag{3.8}$$

where $k_1 \neq 0$ is a complex number.

A special solution of $\boldsymbol{r}_1$ in (3.1b) is easily given by

$$\boldsymbol{r}_1 = \boldsymbol{\rho}_{[J;N]} \boldsymbol{I}_N^{(2)}, \quad \boldsymbol{I}_N^{(2)} = (1, 0, \cdots, 0)_N^{\mathrm{T}}, \tag{3.9a}$$

$$\boldsymbol{\rho}_{[J;N]} = \begin{pmatrix} \rho_1 & 0 & 0 & \cdots & 0 \\ \rho_1^{(1)} & \rho_1 & 0 & \cdots & 0 \\ \frac{\rho_1^{(2)}}{2!} & \rho_1^{(1)} & \rho_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\rho_1^{(N-1)}}{(N-1)!} & \frac{\rho_1^{(N-2)}}{(N-2)!} & \frac{\rho_1^{(N-3)}}{(N-3)!} & \cdots & \rho_1 \end{pmatrix}, \quad \rho_1^{(j)} = \partial_{k_1}^j \rho_1, \tag{3.9b}$$

with ρ_1 defined by (3.6b). In order to obtain the explicit expression of $\boldsymbol{M}_1$, we rewrite

$$^t\boldsymbol{c}_1 = \boldsymbol{I}_N^{(2)\mathrm{T}}\boldsymbol{C}_{[J;N]}$$

with

$$\boldsymbol{C}_{[J;N]} = \begin{pmatrix} c_1 & \cdots & c_{N-2} & c_{N-1} & c_N \\ c_2 & \cdots & c_{N-1} & c_N & 0 \\ c_3 & \cdots & c_N & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ c_N & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

and let

$$\boldsymbol{M}_1 = \boldsymbol{\rho}_{[J;N]}\boldsymbol{G}_{[J;N]}\boldsymbol{C}_{[J;N]}.$$

Then (3.1a) becomes

$$-\omega\boldsymbol{G}_{[J;N]}\boldsymbol{\Gamma}^{\mathrm{T}} + \boldsymbol{\Gamma}\boldsymbol{G}_{[J;N]} = \boldsymbol{I}_N^{(2)}\boldsymbol{I}_N^{(2)\mathrm{T}}. \tag{3.11}$$

Denote

$$\boldsymbol{G}_{[J;N]} = (\boldsymbol{G}_1^{\mathrm{T}},\ \boldsymbol{G}_2^{\mathrm{T}}, \cdots, \boldsymbol{G}_N^{\mathrm{T}})^{\mathrm{T}},$$

i.e., $\boldsymbol{G}_j$ is the j-th row vector of $\boldsymbol{G}_{[J;N]}$. Then from (3.11), we obtain the following system of equations:

$$-\omega\boldsymbol{G}_1\boldsymbol{\Gamma}^{\mathrm{T}} + k_1\boldsymbol{G}_1 = \boldsymbol{I}_N^{(2)\mathrm{T}}, \tag{3.12a}$$

$$-\omega\boldsymbol{G}_j\boldsymbol{\Gamma}^{\mathrm{T}} + \boldsymbol{G}_{j-1} + k_1\boldsymbol{G}_j = (0,0,0,\cdots,0), \quad j = 2,3,\cdots,N. \tag{3.12b}$$

Through iteration calculation, we have

$$\boldsymbol{G}_j = (-1)^{j-1}\boldsymbol{I}_N^{(2)\mathrm{T}}(k_1\boldsymbol{I}_N - \omega\boldsymbol{\Gamma}^{\mathrm{T}})^{-j}, \quad j = 1,2,3,\cdots,N. \tag{3.13}$$

Furthermore, taking the j-th derivative of (3.12a) w.r.t. k_1 and comparing it with (3.12b), we obtain

$$\boldsymbol{G}_{j+1} = \frac{\partial_{k_1}^j\boldsymbol{G}_1}{(2j)!!}.$$

Now we define

$$\boldsymbol{r}_N^{[J]}(k_1) = \boldsymbol{\mathcal{A}}\boldsymbol{r}_1, \quad \boldsymbol{M}_N^{[J]}(k_1) = \boldsymbol{\mathcal{A}}\boldsymbol{M}_1, \tag{3.14}$$

where $\boldsymbol{\mathcal{A}}$ is an arbitrary N-order lower triangular Toeplitz matrix (commuting with a Jordan block (see [28–29]). In a similar way to [29], one can prove that $\boldsymbol{r}_N^{[J]}(k_1)$ and $\boldsymbol{M}_N^{[J]}(k_1)$ give explicit expressions for all solutions to the CES (3.1) when $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{[J;N]}(k_1)$.

Remark 3.1 If we define

$$^t\boldsymbol{c}_1^{[J]} = {^t\boldsymbol{c}_1}\boldsymbol{\mathcal{A}}, \quad \boldsymbol{r}_N^{[J]}(k_1) = \boldsymbol{\mathcal{A}}\boldsymbol{r}_1, \quad \boldsymbol{M}_N^{[J]}(k_1) = \boldsymbol{\mathcal{A}}\boldsymbol{M}_1\boldsymbol{\mathcal{A}}, \tag{3.15}$$

then it leads to the same solution as (3.14) for the arbitrariness of $\boldsymbol{\mathcal{A}}$.

Case 3 Here we consider $\boldsymbol{\Gamma}$ in two mixed cases: (i) Diagonal-Jordan block; (ii) Jordan block-Jordan block. The corresponding solutions are called mixed solutions.

Subcase 3.1

$$\boldsymbol{\Gamma}=\boldsymbol{\Gamma}^{(1)}_{[m]}=\mathrm{Diag}(\boldsymbol{\Gamma}_{[D;N_1]},\boldsymbol{\Gamma}_{[J;N_2]}(k_{N_1+1}))_{N\times N}, \tag{3.16}$$

which corresponds to the N-order matrix $\boldsymbol{K}$ have N_1 distinct complex eigenvalues $k_1,k_2,\cdots,k_{N_1}$ and N_2 complex eigenvalues k_{N_1+1} ($\{k_i-\omega k_j\neq 0\}$).

Special solutions of $\boldsymbol{r}_1$ and $\boldsymbol{M}_1$ in this case are easily given by

$$\boldsymbol{r}_1=\mathrm{Diag}(\boldsymbol{\rho}_{[D;N_1]},\boldsymbol{\rho}_{[J;N_2]})(\boldsymbol{I}^{(1)\mathrm{T}}_{N_1},\boldsymbol{I}^{(2)\mathrm{T}}_{N_2})^{\mathrm{T}}, \tag{3.17a}$$

$${}^{t}\boldsymbol{c}_1=(\boldsymbol{I}^{(1)\mathrm{T}}_{N_1},\boldsymbol{I}^{(2)\mathrm{T}}_{N_2})\cdot\mathrm{Diag}(\boldsymbol{C}_{[D;N_1]},\boldsymbol{C}_{[J;N_2]}), \tag{3.17b}$$

$$\boldsymbol{M}_1=\mathrm{Diag}(\boldsymbol{\rho}_{[D;N_1]},\boldsymbol{\rho}_{[J;N_2]})\cdot\boldsymbol{G}^{(1)}_{[m]}\cdot\mathrm{Diag}(\boldsymbol{C}_{[D;N_1]},\boldsymbol{C}_{[J;N_2]}). \tag{3.17c}$$

Decomposing $\boldsymbol{G}^{(1)}_{[m]}$ to

$$\boldsymbol{G}^{(1)}_{[m]}=\begin{pmatrix}\boldsymbol{G}^{(1)}_{1,1} & \boldsymbol{G}^{(1)}_{1,2}\\ \boldsymbol{G}^{(1)}_{2,1} & \boldsymbol{G}^{(1)}_{2,2}\end{pmatrix}, \tag{3.18}$$

and taking it into (3.1a), we get

$$-\omega\boldsymbol{G}^{(1)}_{1,1}\boldsymbol{\Gamma}_{[D;N_1]}+\boldsymbol{\Gamma}_{[D;N_1]}\boldsymbol{G}^{(1)}_{1,1}=\boldsymbol{I}^{(1)}_{N_1}\boldsymbol{I}^{(1)\mathrm{T}}_{N_1}, \tag{3.19a}$$

$$-\omega\boldsymbol{G}^{(1)}_{1,2}\boldsymbol{\Gamma}^{\mathrm{T}}_{[J;N_2]}+\boldsymbol{\Gamma}_{[D;N_1]}\boldsymbol{G}^{(1)}_{1,2}=\boldsymbol{I}^{(1)}_{N_1}\boldsymbol{I}^{(2)\mathrm{T}}_{N_2}, \tag{3.19b}$$

$$-\omega\boldsymbol{G}^{(1)}_{2,1}\boldsymbol{\Gamma}_{[D;N_1]}+\boldsymbol{\Gamma}_{[J;N_2]}\boldsymbol{G}^{(1)}_{2,1}=\boldsymbol{I}^{(2)}_{N_2}\boldsymbol{I}^{(1)\mathrm{T}}_{N_1}, \tag{3.19c}$$

$$-\omega\boldsymbol{G}^{(1)}_{2,2}\boldsymbol{\Gamma}^{\mathrm{T}}_{[J;N_2]}+\boldsymbol{\Gamma}_{[J;N_2]}\boldsymbol{G}^{(1)}_{2,2}=\boldsymbol{I}^{(2)}_{N_2}\boldsymbol{I}^{(2)\mathrm{T}}_{N_2}. \tag{3.19d}$$

The solutions $\boldsymbol{G}^{(1)}_{1,1}$, $\boldsymbol{G}^{(1)}_{1,2}$, $\boldsymbol{G}^{(1)}_{2,1}$ and $\boldsymbol{G}^{(1)}_{2,2}$ can be expressed as

$$\boldsymbol{G}^{(1)}_{1,1}=\Big(\frac{1}{k_i-\omega k_j}\Big)_{i,j=1,2,\cdots,N_1}, \tag{3.20a}$$

$$\boldsymbol{G}^{(1)}_{2,2}=(\boldsymbol{G}^{(1)\mathrm{T}}_{2,2;1},\boldsymbol{G}^{(1)\mathrm{T}}_{2,2;2},\cdots,\boldsymbol{G}^{(1)\mathrm{T}}_{2,2;N_2})^{\mathrm{T}}, \tag{3.20b}$$

$$\boldsymbol{G}^{(1)}_{2,2;j}=(-1)^{j-1}\boldsymbol{I}^{(2)\mathrm{T}}_{N_2}(k_{N_1+1}\boldsymbol{I}_{N_2}-\omega\boldsymbol{\Gamma}^{\mathrm{T}}_{[J;N_2]})^{-j},\quad j=1,2,\cdots,N_2, \tag{3.20c}$$

$$\boldsymbol{G}^{(1)}_{1,2}=(\boldsymbol{G}^{(1)\mathrm{T}}_{1,2;1},\boldsymbol{G}^{(1)\mathrm{T}}_{1,2;2},\cdots,\boldsymbol{G}^{(1)\mathrm{T}}_{1,2;N_1})^{\mathrm{T}}, \tag{3.20d}$$

$$\boldsymbol{G}^{(1)}_{1,2;j}=\boldsymbol{I}^{(2)\mathrm{T}}_{N_2}(k_j\boldsymbol{I}_{N_2}-\omega\boldsymbol{\Gamma}^{\mathrm{T}}_{[J;N_2]})^{-1},\quad j=1,2,\cdots,N_1, \tag{3.20e}$$

$$\boldsymbol{G}^{(1)}_{2,1}=(\boldsymbol{G}^{(1)}_{2,1;1},\boldsymbol{G}^{(1)}_{2,1;2},\cdots,\boldsymbol{G}^{(1)}_{2,1;N_1}), \tag{3.20f}$$

$$\boldsymbol{G}^{(1)}_{2,1;j}=\boldsymbol{I}^{(2)\mathrm{T}}_{N_2}(-\omega k_j\boldsymbol{I}_{N_2}+\boldsymbol{\Gamma}^{\mathrm{T}}_{[J;N_2]})^{-1},\quad j=1,2,\cdots,N_1. \tag{3.20g}$$

Explicit expressions for general solutions to the CES (3.1) when $\boldsymbol{\Gamma}^{(1)}_{[m]}$ is (3.16) can be given by

$$\boldsymbol{r}^{[D;J]}_N=\mathrm{Diag}(\mathscr{A},\boldsymbol{\mathcal{A}})\boldsymbol{r}_1,\quad \boldsymbol{M}^{[D;J]}_N=\mathrm{Diag}(\mathscr{A},\boldsymbol{\mathcal{A}})\boldsymbol{M}_1, \tag{3.21}$$

where $\mathscr{A}$ is an arbitrary N_1-order constant diagonal matrix and $\boldsymbol{\mathcal{A}}$ is an arbitrary N_2-order lower triangular Toeplitz matrix.

Subcase 3.2

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(2)}_{[m]} = \mathrm{Diag}(\boldsymbol{\Gamma}_{[J;N_1]}(k_1), \boldsymbol{\Gamma}_{[J;N_2]}(k_2))_{N\times N}, \tag{3.22}$$

which corresponds to the matrix $\boldsymbol{K}$ having N_1 complex eigenvalues k_1 and N_2 complex eigenvalues k_2.

Special solutions of $\boldsymbol{r}_1$ and $\boldsymbol{M}_1$ in this case are easily given by

$$\boldsymbol{r}_1 = \mathrm{Diag}(\boldsymbol{\rho}_{[J;N_1]}, \boldsymbol{\rho}_{[J;N_2]})(\boldsymbol{I}^{(2)\mathrm{T}}_{N_1}, \boldsymbol{I}^{(2)\mathrm{T}}_{N_2})^{\mathrm{T}}, \tag{3.23a}$$

$${}^{t}\boldsymbol{c}_1 = (\boldsymbol{I}^{(2)\mathrm{T}}_{N_1}, \boldsymbol{I}^{(2)\mathrm{T}}_{N_2})\mathrm{Diag}(\boldsymbol{C}_{[J;N_1]}, \boldsymbol{C}_{[J;N_2]}), \tag{3.23b}$$

$$\boldsymbol{M}_1 = \mathrm{Diag}(\boldsymbol{\rho}_{[J;N_1]}, \boldsymbol{\rho}_{[J;N_2]}) \cdot \boldsymbol{G}^{(2)}_{[m]} \cdot \mathrm{Diag}(\boldsymbol{C}_{[J;N_1]}, \boldsymbol{C}_{[J;N_2]}). \tag{3.23c}$$

Decomposing $\boldsymbol{G}^{(2)}_{[m]}$ to

$$\boldsymbol{G}^{(2)}_{[m]} = \begin{pmatrix} \boldsymbol{G}^{(2)}_{1,1} & \boldsymbol{G}^{(2)}_{1,2} \\ \boldsymbol{G}^{(2)}_{2,1} & \boldsymbol{G}^{(2)}_{2,2} \end{pmatrix}, \tag{3.24}$$

and taking it into (3.1a), we get a system of equations similar to (3.19). The solutions $\boldsymbol{G}^{(2)}_{1,1}$, $\boldsymbol{G}^{(2)}_{1,2}$, $\boldsymbol{G}^{(2)}_{2,1}$ and $\boldsymbol{G}^{(2)}_{2,2}$ can be expressed as

$$\boldsymbol{G}^{(2)}_{1,1} = (\boldsymbol{G}^{(2)\mathrm{T}}_{1,1;1}, \boldsymbol{G}^{(2)\mathrm{T}}_{1,1;2}, \cdots, \boldsymbol{G}^{(2)\mathrm{T}}_{1,1;N_1})^{\mathrm{T}}, \tag{3.25a}$$

$$\boldsymbol{G}^{(2)}_{1,1;j} = (-1)^{j-1}\boldsymbol{I}^{(2)\mathrm{T}}_{N_1}(k_1\boldsymbol{I}_{N_1} - \omega\boldsymbol{\Gamma}^{\mathrm{T}}_{[J;N_1]})^{-j}, \quad j = 1, 2, \cdots, N_1, \tag{3.25b}$$

$$\boldsymbol{G}^{(2)}_{1,2} = (\boldsymbol{G}^{(2)\mathrm{T}}_{1,2;1}, \boldsymbol{G}^{(2)\mathrm{T}}_{1,2;2}, \cdots, \boldsymbol{G}^{(2)\mathrm{T}}_{1,2;N_1})^{\mathrm{T}}, \tag{3.25c}$$

$$\boldsymbol{G}^{(2)}_{1,2;j} = (-1)^{j-1}\boldsymbol{I}^{(2)\mathrm{T}}_{N_2}(k_1\boldsymbol{I}_{N_2} - \omega\boldsymbol{\Gamma}_{[J;N_2]\mathrm{T}})^{-j}, \quad j = 1, 2, \cdots, N_1, \tag{3.25d}$$

$$\boldsymbol{G}^{(2)}_{2,1} = (\boldsymbol{G}^{(2)\mathrm{T}}_{2,1;1}, \boldsymbol{G}^{(2)\mathrm{T}}_{2,1;2}, \cdots, \boldsymbol{G}^{(2)\mathrm{T}}_{2,1;N_2})^{\mathrm{T}}, \tag{3.25e}$$

$$\boldsymbol{G}^{(2)}_{2,1;j} = (-1)^{j-1}\boldsymbol{I}^{(2)\mathrm{T}}_{N_1}(k_2\boldsymbol{I}_{N_1} - \omega\boldsymbol{\Gamma}_{[J;N_1]\mathrm{T}})^{-j}, \quad j = 1, 2, \cdots, N_2, \tag{3.25f}$$

$$\boldsymbol{G}^{(2)}_{2,2} = (\boldsymbol{G}^{(2)\mathrm{T}}_{2,2;1}, \boldsymbol{G}^{(2)\mathrm{T}}_{2,2;2}, \cdots, \boldsymbol{G}^{(2)\mathrm{T}}_{2,2;N_2})^{\mathrm{T}}, \tag{3.25g}$$

$$\boldsymbol{G}^{(2)}_{2,2;j} = (-1)^{j-1}\boldsymbol{I}^{(2)^{\mathrm{T}}}_{N_2}(k_2\boldsymbol{I}_{N_2} - \omega\boldsymbol{\Gamma}_{[J;N_2]\mathrm{T}})^{-j}, \quad j = 1, 2, \cdots, N_2. \tag{3.25h}$$

Furthermore, it is easy to prove

$$\boldsymbol{G}^{(2)}_{1,2;j} = \frac{\partial^{j-1}_{k_1}\boldsymbol{G}^{(2)}_{1,2;1}}{(j-1)!}, \quad j = 1, 2, \cdots, N_1$$

and

$$\boldsymbol{G}^{(2)}_{2,1;j} = \frac{\partial^{j-1}_{k_2}\boldsymbol{G}^{(2)}_{1,2;1}}{(j-1)!}, \quad j = 1, 2, 3, \cdots, N_2.$$

Explicit expressions for general solutions to the CES (3.1) when $\boldsymbol{\Gamma}^{(2)}_{[m]}$ is (3.22) can be given by defining

$$\boldsymbol{r}^{[J;J]}_{N} = \mathrm{Diag}(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathfrak{B}})\boldsymbol{r}_1, \quad \boldsymbol{M}^{[J;J]}_{N} = \mathrm{Diag}(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathfrak{B}})\boldsymbol{M}_1, \tag{3.26}$$

where $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ are arbitrary N_1 and N_2-order lower triangular Toeplitz matrices.

We end this section with some remarks.

Remark 3.2 For convenience, in Case 3 we just discuss two kinds of mixed forms of the matrix $\mathbf{\Gamma}$: diagonal-Jordan block and Jordan block-Jordan block. The calculation process can be applied to more mixed forms.

Remark 3.3 We compare our work with [15, 18], in which the authors studied the soliton solutions to the lattice BSQ-type equations through the bilinear method and the direct linearization method. In these two references, the soliton solutions to the lattice BSQ-type equations are consist of two kinds of the plane-wave factors (see one soliton solution). But here, we can only use one plane-wave factor (see Case 1).

Remark 3.4 As

$$\mathbf{\Gamma} = \mathrm{Diag}(\mathbf{\Gamma}_{[J;N_1]}(k_1), \mathbf{\Gamma}_{[J;N_2]}(k_2), \cdots, \mathbf{\Gamma}_{[J;N_s]}(k_s)),$$

we can consider its degeneration. If it degenerates to $\mathbf{\Gamma} = \mathrm{Diag}(k_1, k_2, \cdots, k_s)$, then the corresponding solutions degenerate to solitons. If it degenerates to

$$\mathbf{\Gamma} = \mathrm{Diag}(k_1, k_2, \cdots, k_{s_1}, \mathbf{\Gamma}_{[J;N_{s_1+1}]}(k_{s_1+1}), \cdots, \mathbf{\Gamma}_{[J;N_s]}(k_s)), \quad s_1 < s,$$

then the corresponding solutions degenerate to the first kind of mixed solutions (see Subcase 3.1).

Remark 3.5 An interesting question is when $\boldsymbol{K}$ only has zero eigenvalue (see [29]), how can one get the corresponding solution (a rational solution) to the lattice BSQ-type equation by using the generalized Cauchy matrix approach?

Acknowledgment The first author thanks Prof. F. W. Nijhoff for providing [11–12].

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