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Fermionization of Sharma-Tasso-Olver System*

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Abstract By applying the fermionization approach, the inverse version of the bosonization approach, to the Sharma-Tasso-Olver (STO) equation, three simple supersymmetric extensions of the STO equation are obtained from the Painlevé analysis. Furthermore, some types of special exact solutions to the supersymmetric extensions are obtained.

Keywords STO equation, Supersymmetric integrable systems, Fermionization approach
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1 Introduction

In the field of mathematical physics, the research of supersymmetric integrable systems (see [1–5]) is a very important direction. In nonlinear science, some supersymmetric integrable systems of the soliton equations have been given, such as the Korteweg-de Vries (KdV) equation (see [6–9]), the modified Korteweg-de Vries (mKdV) equation (see [9]), the Sawada-Kotera (SK) equation (see [10]), the sine-Gordon (sG) equation (see [11]) and the two-boson equation (see [12]). Furthermore, many effective methods have been proposed and developed to solve the supersymmetric equations.

On the other hand, because the existence of anticommutative fiermionic fields brought some difficulties in dealing with supersymmetric equations, the bosonization approach (see [13]) of supersymmetric systems was proposed, which changes a supersymmetric model to a system of equations with respect to bosonic fields. That means the fermion systems can be solved via boson systems. Therefore, a natural interesting question arises:

Can the inverse procedure of the bosonization (which may be called fermionization) be developed to find integrable fermion systems (integrable coupled fermion systems and/or supersymmetric systems)?

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In the next section, we propose a simple method to find possible integrable fermion extensions from a known boson system. In Section 3, taking the Sharma-Tasso-Olver (STO) equation (see [14–19])

$$u_t + 3\alpha u_x^2 + 3\alpha u^2 u_x + 3\alpha u u_{xx} + \alpha u_{xxx} = 0 \tag{1.1}$$

with the arbitrary constant α as a simple example, we find three types of supersymmetric STO extensions. The last section is a short summary and discussion.

2 Fermionization Approach: General

In [13], the authors solved the supersymmetric KdV (SKdV) equation

$$\Phi_t + \Phi_{xxx} + 3(\mathcal{D}\Phi_x)\Phi + 3(\mathcal{D}\Phi)\Phi_x = 0, \tag{2.1}$$

where

$$\Phi(\theta, x, t) = \xi(x, t) + \theta u(x, t),$$

θ is a Grassmann variable, and $\mathcal{D} = \partial_\theta + \theta\partial_x$ is the supersymmetric covariant derivative, via the bosonization approach such that the exact solutions to the SKdV equation with arbitrary number of fermionic parameters can be obtained by only solving some suitable bosonic systems. Especially, by looking for the exact solutions to the SKdV equation with two fermionic parameters $\{\xi_1, \xi_2\}$, it is enough to solve the following bosonic system with four bosonic fields v_1, v_2, u_0 and u_1:

$$u_{0t} + u_{0xxx} + 6u_0 u_{0x} = 0, \tag{2.2}$$

$$v_{1t} + v_{1xxx} + 3u_0 v_{1x} + 3u_{0x} v_1 = 0, \tag{2.3}$$

$$v_{2t} + v_{2xxx} + 3u_0 v_{2x} + 3u_{0x} v_2 = 0, \tag{2.4}$$

$$u_{1t} + u_{1xxx} + 6u_0 u_{1x} + 6u_{0x} u_1 = 3(v_1 v_{2xx} - v_2 v_{1xx}), \tag{2.5}$$

while the corresponding solutions to the SKdV read

$$\Phi(\theta, x, t) = v_1\xi_1 + v_2\xi_2 + \theta(u_0 + u_1\xi_1\xi_2). \tag{2.6}$$

It is interesting that the boson system (2.2)–(2.5) possesses the following properties:

(i) The first equation for the field u_0 is just the original KdV equation which is not coupled with other boson fields.

(ii) All other equations are only linear equations.

(iii) If the fields v_1 and v_2 are proportional to u_0, then the v_1 and v_2 equations are satisfied identically.

(iv) The homogeneous part of the u_1 equation (2.5) is just the symmetry equation of the usual KdV equation and the homogeneous part vanishes when v_1 and v_2 are proportional to u_0.

The above properties are similar in the exact solutions to the SKdV and other $N = 1$ supersymmetry systems with arbitrary fermionic parameters. For the higher N supersymmetric models, the conclusion should be modified.

One of the important problems in soliton theory is how to extend an integrable boson system to a supersymmetric one. The above bosonization procedure provides us with a new possible way, the fermionization approach, i.e., the inverse procedure of the bosonization approach, to find integrable supersymmetry extensions. The fermionization approach may be summarized into the following three steps:

Step 1 For a given bosonic system

$$F(u, u_x, u_t, u_{xx}, \cdots) \equiv F(u) = 0, \tag{2.7}$$

construct a four-boson system

$$F(u_0) = 0, \tag{2.8}$$

$$F_1(u_0)v_1 = 0, \tag{2.9}$$

$$F_1(u_0)v_2 = 0, \tag{2.10}$$

$$F'(u_0)u_1 = G(v_1, v_2, u_0) \tag{2.11}$$

with

$$F_1(u_0)u_0 \equiv F(u_0), \quad G(u_0, u_0, u_0) \equiv 0, \quad F'(u_0)f \equiv \frac{\mathrm{d}}{\mathrm{d}\epsilon}F(u_0 + \epsilon f)\Big|_{\epsilon=0}. \tag{2.12}$$

Step 2 For a fixed operator $F_1(u_0)$ and the function $G(v_1, v_2, u_0)$, find some possible integrable bosonic systems under some different senses.

Step 3 Use the bosonic fields u_0, v_1, v_2 and u_1 to construct a superfield $\Phi(\theta, x, t)$ via the transformation (2.6).

In the next section, we take the STO equation as a simple example to construct some possible $N = 1$ supersymmetric STO extensions.

3 Fermionization Approach: STO System

Step 1 According to the steps of the fermionization procedure described in the previous section, for the STO equation (1.1), we select the following four-boson system as a starting point:

$$u_{0t} + 3\alpha u_{0x}^2 + 3\alpha u_0^2 u_{0x} + 3\alpha u_0 u_{0xx} + \alpha u_{0xxx} = 0, \tag{3.1}$$

$$\begin{aligned} &v_{1t} + 3\alpha u_{0x} v_{1x} + \alpha[a u_0^2 v_{1x} + (3-a) u_0 u_{0x} v_1] \\ &+ \alpha[b u_{0xx} v_1 + (3-b) u_0 v_{1xx}] + \alpha v_{1xxx} = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &v_{2t} + 3\alpha u_{0x} v_{2x} + \alpha[a u_0^2 v_{2x} + (3-a) u_0 u_{0x} v_2] \\ &+ \alpha[b u_{0xx} v_2 + (3-b) u_0 v_{2xx}] + \alpha v_{2xxx} = 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &u_{1t} + 6\alpha u_{0x} u_{1x} + 3\alpha u_0^2 u_{1x} + 6\alpha u_0 u_{0x} u_1 + 3\alpha u_1 u_{0xx} + 3\alpha u_0 u_{1xx} \\ &+ \alpha u_{1xxx} + \alpha c[(v_{1x} v_2 - v_{2x} v_1) u_0]_x + \alpha d (v_{1xx} v_2 - v_{2xx} v_1)_x = 0. \end{aligned} \tag{3.4}$$

It is clear that the conditions shown in (2.12) are valid for a and b being arbitrary functions of u_0 and c and d being arbitrary functions of u_0, v_1 and v_2. However, for later simplicity, we only consider the special case with

$$a = \text{const.}, \quad b = \text{const.}, \quad c = 3 - a, \quad d = b. \tag{3.5}$$

Step 2 Now we use the well-known Painlevé analysis to select out some possible Painlevé integrable models from (3.1)–(3.4). Let $\phi(x,t) = 0$ be the singularity manifold of the system (3.1)–(3.4) and

$$\begin{aligned} u_0 &= \sum_{i=0}^{\infty} u_{0i} \phi^{i+\beta_1}, \quad v_1 = \sum_{i=0}^{\infty} v_{1i} \phi^{i+\beta_2}, \\ v_2 &= \sum_{i=0}^{\infty} v_{1i} \phi^{i+\beta_3}, \quad u_1 = \sum_{i=0}^{\infty} u_{1i} \phi^{i+\beta_4}, \end{aligned} \tag{3.6}$$

where

$$u_{00} \neq 0, \quad v_{10} \neq 0, \quad v_{20} \neq 0, \quad u_{10} \neq 0.$$

The leading singularities are determined by substituting

$$u_0 = u_{00} \phi^{\beta_1}, \quad v_1 = v_{10} \phi^{\beta_2}, \quad v_2 = v_{20} \phi^{\beta_3}, \quad u_1 = u_{10} \phi^{\beta_4}$$

into the system (3.1)–(3.4). This gives

$$\beta_1 = -1, \quad \beta_2 = -1, \quad \beta_3 = -1, \quad \beta_4 = -2$$

and two branches: (i) $u_{00} = \phi_x$, (ii) $u_{00} = 2\phi_x$.

Branch (i) $u_{00} = \phi_x$. Substituting (3.5) into system (3.1)–(3.4), we can obtain the recursion relations to determine the functions u_{0j}, u_{1j}, v_{1j} and v_{2j}

$$\begin{pmatrix} P_1(j)\phi_x^3 & 0 & 0 & 0 \\ P_4(j)\phi_x^2 v_{10} & P_2(j)\phi_x^3 & 0 & 0 \\ P_4(j)\phi_x^2 v_{20} & 0 & P_2(j)\phi_x^3 & 0 \\ P_5(j)\phi_x^2 u_{10} & P_6(j)\phi_x^3 v_{20} & P_6(j)\phi_x^3 v_{10} & P_3(j)\phi_x^3 \end{pmatrix} \begin{pmatrix} u_{0j} \\ v_{1j} \\ v_{2j} \\ u_{1j} \end{pmatrix} = \begin{pmatrix} F_{1j} \\ F_{2j} \\ F_{3j} \\ F_{4j} \end{pmatrix}, \tag{3.7}$$

where

$$
\begin{aligned}
&P_1(j) = (j+1)(j-1)(j-3),\\
&P_2(j) = j[j^2 - (b+3)j + a + 3b - 1],\\
&P_3(j) = j(j-2)(j-4),
\end{aligned}
$$

$P_4(j)$, $P_5(j)$, $P_6(j)$ are polynomials in j, and F_{1j}, F_{2j}, F_{3j}, F_{4j} are functions of

$$\{u_{01}, u_{02}, \cdots, u_{0,j-1}, v_{10}, v_{11}, \cdots, v_{1,j-1}, v_{20}, v_{21}, \cdots, v_{2,j-1}, u_{10}, u_{11}, \cdots, u_{1,j-1}\}.$$

It is clear that, the resonances, i.e., values of j for the recursion formula are not defined, occur when

$$
\begin{aligned}
&\det \begin{vmatrix} P_1(j)\phi_x^3 & 0 & 0 & 0 \\ P_4(j)\phi_x^2 v_{10} & P_2(j)\phi_x^3 & 0 & 0 \\ P_4(j)\phi_x^2 v_{20} & 0 & P_2(j)\phi_x^3 & 0 \\ P_5(j)\phi_x^2 u_{10} & P_6(j)\phi_x^3 v_{20} & P_6(j)\phi_x^3 v_{10} & P_3(j)\phi_x^3 \end{vmatrix}\\
&= P_1(j)P_2^2(j)P_3(j)\phi_x^{12} = 0.
\end{aligned} \tag{3.8}
$$

In order for the system (3.1)–(3.4) to pass the Painlevé test (possibly under some conditions on the parameters a and b), it is necessary that $P_2(j)$ should be written under the form $j(j - j_1)(j - j_2)$ for some integer values of j_1, j_2. The implicit expressions for these numbers are

$$j_1 + j_2 = b + 3, \quad j_1 j_2 = a + 3b - 1. \tag{3.9}$$

Therefore, a case by case analysis for resonance conditions yields in the following cases for j_1, $j_2 \leq 7$:

$$
\begin{aligned}
&\text{Case A} \qquad j_1 = 1,\ j_2 = 2 \to a = 3,\ b = 0,\\
&\text{Case B} \qquad j_1 = 2,\ j_2 = 4 \to a = 0,\ b = 3,\\
&\text{Case C} \qquad j_1 = 4,\ j_2 = 5 \to a = 3,\ b = 6.
\end{aligned} \tag{3.10}
$$

Branch (ii) $u_{00} = 2\phi_x$. For this auxiliary branch, the Painlevé analysis requires checking additional resonance conditions in Case A, B and C, respectively.

In Case A, the resonances occur at $j = -2, -1, -1, -1, -1, 0, 0, 0, 1, 1, 3, 4$.

In Case B, the resonances occur at $j = -2, -1, -1, 0, 0, 0, 1, 1, 3, 4, 5, 5$.

In Case C, the resonances occur at $j = -2, -1, -1, 0, 0, 0, 3, 4, 5, 5, 7, 7$.

The detailed calculation shows that all the resonance conditions are satisfied and then all the three cases possess the Painlevé property.

Step 3 Finally, it is not difficult to verify that the above three cases lead to three supersymmetric STO (SSTO) equations

$$\Phi_t + 3\alpha\Phi_x D\Phi_x + 3\alpha(D\Phi)^2\Phi_x + 3\alpha(D\Phi)\Phi_{xx} + \alpha\Psi_{xxx} = 0, \tag{3.11}$$

$$\Phi_t + 3\alpha\Phi_x D\Phi_x + 3\alpha\Phi D\Phi D\Phi_x + 3\alpha\Phi D\Phi_{xx} + \alpha\Psi_{xxx} = 0, \tag{3.12}$$

$$\Phi_t + 3\alpha\Phi_x D\Phi_x + 3\alpha(D\Phi)^2\Phi_x + 6\alpha\Phi D\Phi_{xx} - 3\alpha D\Phi\Phi_{xx} + \alpha\Psi_{xxx} = 0 \tag{3.13}$$

with Φ being given by (2.6).

The direct Painlevé tests for these three cases show that they are all Painlevé integrable.

4 Special Solutions

To obtain special solutions to these SSTO equations, we just study traveling wave solutions of the system (3.1)–(3.4).

Introducing the traveling wave variable

$$X = x - ct + X_0,$$

where c and X_0 are constants, the system (3.1)–(3.4) can be changed to the ordinary differential equation system

$$-cu_{0X} + 3\alpha u_{0X}^2 + 3\alpha u_0^2 u_{0X} + 3\alpha u_0 u_{0XX} + \alpha u_{0XXX} = 0, \tag{4.1}$$

$$\begin{aligned}&-cv_{1X} + 3\alpha u_{0X}v_{1X} + \alpha[au_0^2 v_{1X} + (3-a)u_0u_{0X}v_1]\\&+\alpha[bu_{0XX}v_1 + (3-b)u_0v_{1XX}] + \alpha v_{1XXX} = 0,\end{aligned} \tag{4.2}$$

$$\begin{aligned}&-cv_{2X} + 3\alpha u_{0X}v_{2X} + \alpha[au_0^2 v_{2X} + (3-a)u_0u_{0X}v_2]\\&+\alpha[bu_{0XX}v_2 + (3-b)u_0v_{2XX}] + \alpha v_{2XXX} = 0,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&-cu_{1X} + 6\alpha u_{0X}u_{1X} + 3\alpha u_0^2u_{1X} + 6\alpha u_0u_{0X}u_1 + 3\alpha u_1u_{0XX} + 3\alpha u_0u_{1XX}\\&+\alpha u_{1XXX} + \alpha(3-a)[(v_{1X}v_2 - v_{2X}v_1)u_0]_X + \alpha b(v_{1XX}v_2 - v_{2XX}v_1)_X = 0.\end{aligned} \tag{4.4}$$

We first solve out u_{0X} from (4.1), which can be changed to

$$\partial_X(u_0 + \partial_X)\Big(u_{0X} + u_0^2 - \frac{c}{\alpha}\Big) = 0. \tag{4.5}$$

It is clear that (4.5) can be solved by

$$u_{0X} = -u_0^2 + \frac{c}{\alpha}. \tag{4.6}$$

The linear inhomogeneous ODE (4.3) can be directly integrated once

$$-cu_1 + 3\alpha u_0^2u_1 + 3\alpha(u_0u_1)_X + \alpha u_{1XX} + r(X) = 0, \tag{4.7}$$

where the inhomogeneous term is

$$r(X) = \alpha(3-a)(v_{1X}v_2 - v_{2X}v_1)u_0 + \alpha b(v_{1XX}v_2 - v_{2XX}v_1) + C_1. \tag{4.8}$$

From (4.6), it is known that the variable transformations

$$X \to u_0 \tag{4.9}$$

can be used. Under the transformations (4.9), we have

$$u_1(X) = U_1(u_0), \quad v_1(X) = V_1(u_0), \quad v_2(X) = V_2(u_0). \tag{4.10}$$

Using the transformation and (4.6), the ODE system (4.2)–(4.4) becomes

$$(\alpha u_0^2 - c)^2 \frac{\mathrm{d}^3 V_1}{\mathrm{d}u_0^3} + \alpha(b+3)(\alpha u_0^2 - c)u_0 \frac{\mathrm{d}^2 V_1}{\mathrm{d}u_0^2} + \alpha^2 u_0 \Big[(2b+a-3)u_0 \frac{\mathrm{d}V_1}{\mathrm{d}u_0} - V_1\Big] = 0, \tag{4.11}$$

$$(\alpha u_0^2 - c)^2 \frac{\mathrm{d}^3 V_2}{\mathrm{d}u_0^3} + \alpha(b+3)(\alpha u_0^2 - c)u_0 \frac{\mathrm{d}^2 V_2}{\mathrm{d}u_0^2} + \alpha^2 u_0 \Big[(2b+a-3)u_0 \frac{\mathrm{d}V_2}{\mathrm{d}u_0} - V_2\Big] = 0, \tag{4.12}$$

$$(\alpha u_0^2 - c)^3 \frac{\mathrm{d}^3 U_1}{\mathrm{d}u_0^3} + 3\alpha u_0(\alpha u_0^2 - c)^2 \frac{\mathrm{d}^2 U_1}{\mathrm{d}u_0^2} - 3\alpha(\alpha u_0^2 - c)^2 \frac{\mathrm{d}U_1}{\mathrm{d}u_0} = \alpha^2 R(u_0), \tag{4.13}$$

where

$$\begin{aligned} R(u_0) = {} & \frac{b}{\alpha}(\alpha u_0^2 - c)^2 \Big(V_2 \frac{\mathrm{d}^2 V_1}{\mathrm{d}u_0^2} - V_1 \frac{\mathrm{d}^2 V_2}{\mathrm{d}u_0^2}\Big) + (2b+a-3)u_0 \\ & \times (\alpha u_0^2 - c)\Big(V_2 \frac{\mathrm{d}V_1}{\mathrm{d}u_0} - V_1 \frac{\mathrm{d}V_2}{\mathrm{d}u_0}\Big) + C_1. \end{aligned} \tag{4.14}$$

Solving (4.11)–(4.13) yields

$$V_1 = C_2 u_0 + u_0 \int \frac{C_3(\alpha u_0^2 - c)^{\frac{\delta}{4}} + C_4(\alpha u_0^2 - c)^{-\frac{\delta}{4}}}{u_0^2(\alpha u_0^2 - c)^{\frac{1+b}{4}}} \mathrm{d}u_0, \tag{4.15}$$

$$V_2 = C_5 u_0 + u_0 \int \frac{C_6(\alpha u_0^2 - c)^{\frac{\delta}{4}} + C_7(\alpha u_0^2 - c)^{-\frac{\delta}{4}}}{u_0^2(\alpha u_0^2 - c)^{\frac{1+b}{4}}} \mathrm{d}u_0, \tag{4.16}$$

$$\begin{aligned} U_1 = {} & C_8(\alpha u_0^2 - c) + C_9 u_0 \sqrt{\alpha u_0^2 - c} - \frac{c}{\alpha}(\alpha u_0^2 - c) \int \frac{R(u_0)u_0}{(\alpha u_0^2 - c)^2} \mathrm{d}u_0 \\ & + \frac{c}{\alpha} u_0 \sqrt{\alpha u_0^2 - c} \int \frac{R(u_0)}{(\alpha u_0^2 - c)^{\frac{3}{2}}} \mathrm{d}u_0, \end{aligned} \tag{4.17}$$

where

$$\delta = \sqrt{b^2 - 6b - 4a + 13}$$

and C_i $(i = 1, 2, \cdots, 9)$ are arbitrary constants. Thus, we have obtained a special type of two-fermionic-parameter traveling wave solution to the system (3.1)–(3.4)

$$u = u_0 + \xi_1\xi_2\Big\{C_8(\alpha u_0^2 - c) + C_9 u_0\sqrt{\alpha u_0^2 - c} - \frac{c}{\alpha}(\alpha u_0^2 - c)\int \frac{R(u_0)u_0}{(\alpha u_0^2 - c)^2}\mathrm{d}u_0 + \frac{c}{\alpha}u_0\sqrt{\alpha u_0^2 - c}\int \frac{R(u_0)}{(\alpha u_0^2 - c)^{\frac{3}{2}}}\mathrm{d}u_0\Big\}, \tag{4.18}$$

$$v = \xi_1\Big\{C_2 u_0 + u_0\int \frac{C_3(\alpha u_0^2 - c)^{\frac{\delta}{4}} + C_4(\alpha u_0^2 - c)^{-\frac{\delta}{4}}}{u_0^2(\alpha u_0^2 - c)^{\frac{1+b}{4}}}\mathrm{d}u_0\Big\} + \xi_2\Big\{C_5 u_0 + u_0\int \frac{C_6(\alpha u_0^2 - c)^{\frac{\delta}{4}} + C_7(\alpha u_0^2 - c)^{-\frac{\delta}{4}}}{u_0^2(\alpha u_0^2 - c)^{\frac{1+b}{4}}}\mathrm{d}u_0\Big\}, \tag{4.19}$$

with the known solution u_0 to (4.6).

So, in Case A, u possesses the form of (4.18), and

$$v = \xi_1\Big[C_2 + C_3 u_0 + C_4\sqrt{\alpha u_0^2 - c}\Big] + \xi_2\Big[C_5 + C_6 u_0 + C_7\sqrt{\alpha u_0^2 - c}\Big] \tag{4.20}$$

with $R(u_0) = C_1$ being a constant.

In case B, u possesses the form of (4.18), and

$$v = \xi_1\Big[C_2 u_0 + C_3\sqrt{\alpha u_0^2 - c} + \frac{C_4}{\sqrt{\alpha u_0^2 - c}}\Big] + \xi_2\Big[C_5 u_0 + C_6\sqrt{\alpha u_0^2 - c} + \frac{C_7}{\sqrt{\alpha u_0^2 - c}}\Big] \tag{4.21}$$

with

$$R(u_0) = \frac{3}{\alpha}(\alpha u_0^2 - c)^2\Big(V_2\frac{\mathrm{d}^2V_1}{\mathrm{d}u_0^2} - V_1\frac{\mathrm{d}^2V_2}{\mathrm{d}u_0^2}\Big) + 3u_0(\alpha u_0^2 - c)\Big(V_2\frac{\mathrm{d}V_1}{\mathrm{d}u_0} - V_1\frac{\mathrm{d}V_2}{\mathrm{d}u_0}\Big) + C_1.$$

In Case C, u possesses the form of (4.18), and

$$v = \xi_1\Big[C_2 u_0 + \frac{C_3(2\alpha u_0^2 - c)}{\sqrt{\alpha u_0^2 - c}} - 3C_4\alpha u_0 \mathrm{arctanh}\Big(\frac{\sqrt{\alpha c}}{c}u_0\Big) + \frac{C_4\sqrt{c\alpha}(3\alpha u_0^2 - 2c)}{\alpha u_0^2 - c}\Big] + \xi_2\Big[C_5 u_0 + \frac{C_6(2\alpha u_0^2 - c)}{\sqrt{\alpha u_0^2 - c}} - 3C_7\alpha u_0 \mathrm{arctanh}\Big(\frac{\sqrt{\alpha c}}{c}u_0\Big) + \frac{C_7\sqrt{c\alpha}(3\alpha u_0^2 - 2c)}{\alpha u_0^2 - c}\Big] \tag{4.22}$$

with

$$R(u_0) = \frac{6}{\alpha}(\alpha u_0^2 - c)^2\Big(V_2\frac{\mathrm{d}^2V_1}{\mathrm{d}u_0^2} - V_1\frac{\mathrm{d}^2V_2}{\mathrm{d}u_0^2}\Big) + 12u_0(\alpha u_0^2 - c)\Big(V_2\frac{\mathrm{d}V_1}{\mathrm{d}u_0} - V_1\frac{\mathrm{d}V_2}{\mathrm{d}u_0}\Big) + C_1.$$

In addition, in Case A, by making a dependent variable transformation

$$\Phi = D\ln f(x, t, \theta), \tag{4.23}$$

we find that (3.11) is transformed into

$$D\Big(\frac{f_t + \alpha f_{xxx}}{f}\Big) = 0. \tag{4.24}$$

Therefore, we can obtain the following N-soliton solution:

$$f = 1 + \sum_{n=1}^{N} \mathrm{e}^{k_n x - \alpha k_n^3 t + \xi_n\theta + x^0}, \tag{4.25}$$

where k_n $(n = 1, 2, \cdots, N)$ and x^0 are constants, ξ_n $(n = 1, 2, \cdots, N)$ are odd (fermionic) constants. Using the transformation (4.23), we obtain the solution to (3.11)

$$\Phi = \frac{\sum\limits_{n=1}^{N} (\theta k_n + \xi_n) \mathrm{e}^{k_n x - \alpha k_n^3 t + \xi_n \theta + x^0}}{1 + \sum\limits_{n=1}^{N} \mathrm{e}^{k_n x - \alpha k_n^3 t + \xi_n \theta + x^0}}. \tag{4.26}$$

5 Summary and Discussion

In summary, we developed a new method, the fermionization approach which is an inverse procedure of the bosonization approach, to find possible supersymmetric integrable extensions of known integrable boson systems. Especially, taking the STO equation as a simple example, three types of Painlevé integrable SSTO systems are found. The exact traveling wave solutions to the SSTO systems in the usual space with two fermionic parameters can be obtained simply by integration.

The fermionization approach shows us that arbitrary solutions to bosonic models can be extended to those of the supersymmetric models simply by taking $v_1 \sim v_2 \sim u_0$ and $u_1 = \sigma(u_0)$, where u_0 is a solution to the original bosonic model and $\sigma(u_0)$ is any symmetry of the original bosonic system.

In this paper, we study only the possible fermionic extensions of the boson systems under some constraints, for instance, the conditions (2.12) and (3.5). The possible other types of integrable systems should be studied further.

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