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Shape Analysis of Bounded Traveling Wave Solutions and Solution to the Generalized Whitham-Broer-Kaup Equation with Dissipation Terms*

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Abstract This paper deals with the problem of the bounded traveling wave solutions' shape and the solution to the generalized Whitham-Broer-Kaup equation with the dissipation terms which can be called WBK equation for short. The authors employ the theory and method of planar dynamical systems to make comprehensive qualitative analyses to the above equation satisfied by the horizontal velocity component $u(\xi)$ in the traveling wave solution $(u(\xi), H(\xi))$, and then give its global phase portraits. The authors obtain the existent conditions and the number of the solutions by using the relations between the components $u(\xi)$ and $H(\xi)$ in the solutions. The authors study the dissipation effect on the solutions, find out a critical value r^*, and prove that the traveling wave solution $(u(\xi), H(\xi))$ appears as a kink profile solitary wave if the dissipation effect is greater, i.e., $|r| \geq r^*$, while it appears as a damped oscillatory wave if the dissipation effect is smaller, i.e., $|r| < r^*$. Two solitary wave solutions to the WBK equation without dissipation effect is also obtained. Based on the above discussion and according to the evolution relations of orbits corresponding to the component $u(\xi)$ in the global phase portraits, the authors obtain all approximate damped oscillatory solutions $(\widetilde{u}(\xi), \widetilde{H}(\xi))$ under various conditions by using the undetermined coefficients method. Finally, the error between the approximate damped oscillatory solution and the exact solution is an infinitesimal decreasing exponentially.

Keywords Generalized Whitham-Broer-Kaup equation, Shape analysis, Solitary wave solution, Damped oscillatory solution, Error estimate
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1 Introduction

To describe the movement of the shallow water wave, people have proposed many mathemat-

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ical models, such as the nonlinear shallow water long-wave approximate system of equations, i.e., Whitham-Broer equation (see [1–2]),

$$\begin{cases} u_t - H_x - uu_x + \dfrac{1}{2}u_{xx} = 0, \\ H_t - (Hu)_x - \dfrac{1}{2}u_{xx} = 0, \end{cases} \tag{1.1}$$

the modified Boussinesq equation (see [3])

$$\begin{cases} u_t + H_x + uu_x = 0, \\ H_t + (Hu)_x + u_{xxx} = 0, \end{cases} \tag{1.2}$$

and so on. Many researchers have studied (1.1) and (1.2). For example, Kaup [4] and Ablowite [5] studied the inverse scattering transformation solutions to (1.1) and (1.2). Kapershmidt [6] discussed the symmetry and conservation law of (1.1) and (1.2). Fan, Zhang and Yan [7–10] investigated the solitary wave solutions and the exact solutions in other forms. Recently, many researchers have studied the following Whitham-Broer-Kaup equation (WBK equation):

$$\begin{cases} u_t + H_x + uu_x + \beta u_{xx} = 0, \\ H_t + (Hu)_x + \delta u_{xxx} - \beta H_{xx} = 0. \end{cases} \tag{1.3}$$

WBK equation was obtained in the study of the shallow wave under the Boussinesq approximation through the Hamilton theory and the variation method by Whitham [1], Broer [2] and Kaup [4]. In (1.3), $u = u(x,t)$ is the horizontal velocity field, $H = H(x,t)$ is the height deviating from the equilibrium position of the liquid level, δ and β are the coefficients of the dispersion term and the dissipation term, respectively. Fan, et al. obtained a kind of the Backlund transformation and a group of solitary wave solutions to the WBK equation (1.3) in [9]. Salah and Dogan [11] presented the blow-up solutions and approximate periodic solutions to (1.3) by the Adomian decomposition method, while Rafei and Daniali also gave some blow-up traveling wave solutions and periodic traveling wave solutions by the improved iteration method in [12].

Since damping is inevitable in the wave movement, dissipation rises. Whitham [13] pointed out that one of basic problems which need to be concerned for nonlinear evolution equations is how dissipation affects the nonlinear system, and this paper considers the following generalized WBK equation with the dissipation terms:

$$\begin{cases} u_t + bH_x + auu_x + \beta u_{xx} = 0, \\ H_t + a(Hu)_x + ru_{xx} - \beta H_{xx} + \delta u_{xxx} = 0, \end{cases} \tag{1.4}$$

where $a \neq 0,\ b \neq 0$ are the arbitrary constants, δ is the coefficient of the dispersion term, and r, β are the coefficients of the dissipation terms. Obviously, if

$$a = -1, \quad b = -1, \quad r = \delta = 0, \quad \beta = \frac{1}{2},$$

(1.4) becomes (1.1); if

$$a = b = 1, \quad \beta = r = 0, \quad \delta = 1,$$

(1.4) becomes a modified Boussinesq equation (1.2); if

$$a = b = 1 \quad \text{and} \quad r = 0,$$

(1.4) becomes the WBK equation (1.3); if

$$a = b = 1, \quad r = \delta = 0,$$

(1.4) becomes the Kapershmidt equation (see [6])

$$\begin{cases} u_t + H_x + uu_x + \beta u_{xx} = 0, \\ H_t + (Hu)_x - \beta H_{xx} = 0. \end{cases} \tag{1.5}$$

In this paper, we focus on investigating the dissipation effect (i.e., $r \neq 0$) on the evolution of traveling wave solutions' shape for (1.4) and the way to solve it. With the variation of dissipation effect from being strong to being weak, what does the shape of the traveling wave solution to (1.4) appear as? We will prove that the bounded traveling wave solution to (1.4) appears as a kink profile solitary wave solution if the dissipation coefficient $|r|$ is large, while it appears as a damped oscillatory solution if the dissipation coefficient $|r|$ is small. Then we will give a critical value which can characterize the scale of the dissipation effect. We will concentrate our efforts on obtaining the approximate damped oscillatory solutions to (1.4) when $|r|$ is small, and giving their error estimates. (It is very important to give the error estimates, or it will be unreliable to put the approximate solution into use in physics and mechanics.)

This paper is organized as follows. In Section 2, to fully grasp the existent conditions and the number of bounded traveling wave solutions, we will employ the theory and method of the planar dynamical system to make comprehensive qualitative analyses of the traveling wave solutions to (1.4). In this section, we will present five global phase portraits of the traveling wave solutions to (1.4), and give the existent conditions of the kink profile solitary wave solutions and the oscillatory traveling wave solutions. In Section 3, we mainly study the dissipation effect on the shape of the bounded traveling wave solutions. In Section 4, the kink profile solitary wave solutions to (1.4) for $r = 0$ and $r \neq 0$ are presented. Based on the evolution relation of orbits in global phase portraits, we design the structure of the damped oscillatory solutions, and by the undetermined coefficients method, we obtain approximate damped oscillatory solutions to (1.4) under various conditions. In Section 5, we study the error estimates between the exact solutions and the approximate damped oscillatory solutions. The difficulty of this problem is that we only know the approximate damped oscillatory solutions, but do not know the exact solutions. To overcome it, we use some transformations and the idea of the homogenization principle to establish the integral equations reflecting the relations between the exact solutions and the approximate damped oscillatory solutions, and then give the error estimates for the approximate solutions. The errors obtained by the method in Section 4 are infinitesimals decreasing exponentially. In Section 6, we give a simple summary of this paper.

The main significance of this paper lies in the following points: (1) We can clearly see the dissipation effect on the shape of the traveling wave solutions to (1.4) in our paper. In Section 3, we obtain a critical value r^* which can be used to characterize the scale of the dissipation effect, and give the conclusion that a bounded traveling wave solution to (1.4) appears as a kink profile solitary wave if the dissipation effect is large, i.e., r is greater than or equal to r^*, while it appears as a damped oscillatory wave if the dissipation effect is small, i.e., r is less than r^*. From Theorem 4.1 in this paper, we know that the solutions to (1.4) with no dissipation effect ($r = 0$) are solitary waves. Generally speaking, the solitary wave is the traveling wave whose energy is centralized relatively, and this wave has great damages on affected objects. A damped oscillatory wave has a bell profile head and an oscillatory tail, and its destructive force is less than a solitary wave. Its energy loses in movement and with time going by it decreases gradually. For the decreasing kink wave, since its energy loses promptly, the destructive force is smaller. Using the critical value r^* given by this paper may provide a reference for the person to control the practical model which (1.4) represents, such as adjusting the system parameter to obtain more flatter wave. Thus we can reduce the hazard to nature and loss in economy. (2) This paper gives a method which can be applied to find approximate damped oscillatory solutions for other coupled nonlinear evolution equations with the dissipation effect. Firstly, we make shape analyses for an equation. Secondly, we obtain the solutions with no dissipation effect. Finally, according to the evolution relation of orbits in global phase portraits the damped oscillatory solutions respond to, we obtain their approximate damped oscillatory solutions. And the errors obtained by the method in this paper between the approximate damped oscillatory solutions and their exact solutions are the infinitesimals decreasing exponentially.

2 Existence of Kink Profile Solitary Wave Solutions and Oscillatory Traveling Wave Solutions to (1.4)

By using the wave transformation

$$u(x,t) = u(\xi) = u(x - vt),$$
$$H(x,t) = H(\xi) = H(x - vt),$$

where v is the wave speed, (1.4) is reduced to two nonlinear ordinary differential equations

$$-vu'(\xi) + au(\xi)u'(\xi) + bH'(\xi) + \beta u''(\xi) = 0, \tag{2.1}$$
$$-vH'(\xi) + a(u(\xi)H(\xi))' + ru''(\xi) - \beta H''(\xi) + \delta u'''(\xi) = 0. \tag{2.2}$$

Let

$$C_- = \lim_{\xi \to -\infty} u(\xi), \qquad C_+ = \lim_{\xi \to +\infty} u(\xi),$$
$$C_-^H = \lim_{\xi \to -\infty} H(\xi), \quad C_+^H = \lim_{\xi \to +\infty} H(\xi).$$

Assume that the solution $u(\xi)$ we are looking for satisfies the following conditions:

$$u'(\xi), u''(\xi) \to 0, \quad |\xi| \to +\infty, \tag{2.3}$$

and the asymptotic values of $u(\xi)$ are the solutions to the following equation:

$$a^2x^3 - 3avx^2 + 2v^2x = 0. \tag{2.4}$$

At the same time, the asymptotic values of $H(\xi)$ satisfy

$$C_+^H = \frac{C_+}{b}\Big(v - \frac{a}{2}C_+\Big), \quad C_-^H = \frac{C_-}{b}\Big(v - \frac{a}{2}C_-\Big). \tag{2.5}$$

From (2.1) and (2.2), we can obtain

$$H(\xi) = \frac{v}{b}u(\xi) - \frac{a}{2b}u^2(\xi) - \frac{\beta}{b}u'(\xi), \tag{2.6}$$

while $u(\xi)$ satisfies

$$\Big(\frac{\beta^2}{b} + \delta\Big)u''(\xi) + ru'(\xi) - \frac{a^2}{2b}u^3(\xi) + \frac{3av}{2b}u^2(\xi) - \frac{v^2}{b}u(\xi) = 0. \tag{2.7}$$

Applying the translation transformation

$$u(\xi) = U(\xi) + \frac{v}{a} \tag{2.8}$$

to (2.7), we have

$$\overline{r} = \frac{rb}{\beta^2 + \delta b}, \quad \overline{b} = \frac{a^2}{2(\beta^2 + \delta b)}, \tag{2.9}$$

and we can translate (2.7) into

$$U''(\xi) + \overline{r}U'(\xi) - \overline{b}\Big(U^3(\xi) - \frac{v^2}{a^2}U(\xi)\Big) = 0. \tag{2.10}$$

Thus, under the hypotheses (2.3)–(2.5), to study the existence of bounded traveling wave solutions to (1.4) is equivalent to studying the existence of bounded solutions to (2.10).

For the sake of studying the existence of the boundary solutions to (2.10), we let

$$x = U(\xi), \quad y = U'(\xi).$$

Then (2.10) can be reformulated as a planar dynamic system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\xi} = y \triangleq P(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = -\overline{r}y + \overline{b}\Big(x^3 - \dfrac{v^2}{a^2}x\Big) \triangleq Q(x,y). \end{cases} \tag{2.11}$$

Owing to

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = -\overline{r},$$

by Bendixson-Dulac's criterion, we have the following proposition for system (2.11).

Proposition 2.1 *If $r \neq 0$, then system* (2.11) *does not have any closed orbit or singular closed orbit with a finite number of singular points on* (x, y) *phase plane. Furthermore, there exists no periodic traveling wave solution or bell profile solitary wave solution to* (2.9) *as* $r \neq 0$.

Let

$$f(x) = x^3 - \frac{v^2}{a^2}x. \tag{2.12}$$

Obviously, system (2.11) has three singular points $P_i(x_i, 0)$ $(i = 1, 2, 3)$, where

$$x_1 = -\Big|\frac{v}{a}\Big|, \quad x_2 = 0, \quad x_3 = \Big|\frac{v}{a}\Big|.$$

For clarity and non-repetitiveness, we assume that $b > 0,\ \delta \geq 0$ throughout the paper. Other cases can be discussed in a similar way.

We use

$$J(x_i, 0) = \begin{pmatrix} 0 & 1 \\ \overline{b}f'(x_i) & -\overline{r} \end{pmatrix}, \quad i = 1, 2, 3$$

to denote the Jacobian matrix of the linearized system of system (2.11) at singular points $P_i(x_i, 0)$ $(i = 1, 2, 3)$.

In the following, we study the types of singular points for system (2.11).

2.1 In the case of $r = 0$

In this case, system (2.11) has the first integral

$$H(x,y) = \frac{1}{2}y^2 - \overline{b}\Big(\frac{x^4}{4} - \frac{v^2}{2a^2}x^2\Big) = h. \tag{2.13}$$

According to the theory of planar dynamical systems (see [14]), it is easy to see that P_2 is a center, and P_1 and P_3 are saddle points.

2.2 In the case of $r < 0$

In the case of $r < 0$, the characteristic equations of the linearized system of system (2.11) at singular points $P_i(x_i, 0)$ $(i = 1, 2, 3)$ are

$$\lambda^2 + \overline{r}\lambda - \overline{b}f'(x_i) = 0, \quad i = 1, 2, 3.$$

The discriminants corresponding to these characteristic equations are denoted by

$$\triangle_i = \overline{r}^2 + 4\overline{b}f'(x_i) = \Big(\frac{rb}{\beta^2+\delta b}\Big)^2 + \frac{2a^2}{\beta^2+\delta b}\Big(3x_i^2 - \frac{v^2}{a^2}\Big), \quad i = 1, 2, 3.$$

(1) If $\Delta_2 \geq 0$, namely, $r \leq -|\frac{v}{b}|\sqrt{2(\beta^2+\delta b)}$, P_2 is an unstable nodal point.

(2) If $\Delta_2 < 0$, namely, $-|\frac{v}{b}|\sqrt{2(\beta^2+\delta b)} < r < 0$, P_2 is an unstable focus point.

2.3 In the case of $r > 0$

(1) If $\Delta_2 \geq 0$, i.e., $r \geq |\frac{v}{b}|\sqrt{2(\beta^2+\delta b)}$, P_2 is a stable nodal point.

(2) If $\Delta_2 < 0$, i.e., $0 < r < |\frac{v}{b}|\sqrt{2(\beta^2+\delta b)}$, P_2 is a stable focus point.

Applying Poincaré transformation to analyze singular points at the infinity of system (2.11), it is clear that there exist a couple of singular points at infinity A_1, A_2 on y axis. Meanwhile, the circumference of Poincaré disk is an orbit. Furthermore, we can prove that there exists an elliptical domain around A_1, A_2, respectively.

Based on the above results and the theory of planar dynamical systems, we present the global phase portraits (Figures 1–5) of system (2.11). Figures 1–5 are sequentially global phase portraits of system (2.11) in cases of $r = 0$; $r < 0,\ \Delta_2 \geq 0$; $r < 0,\ \Delta_2 \leq 0$; $r > 0,\ \Delta_2 \geq 0$; $r > 0,\ \Delta_2 \leq 0$.

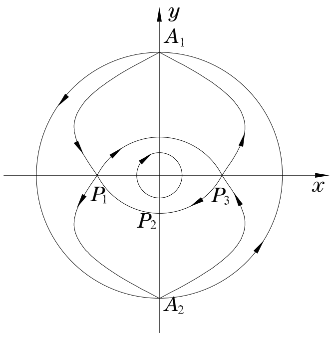

Figure 1 $\quad r = 0$

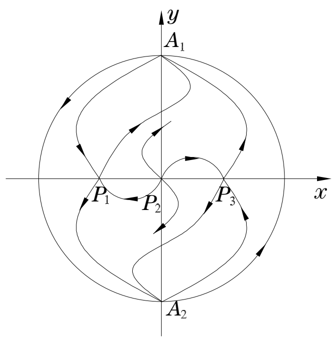

Figure 2 $\quad r < 0,\ \Delta_2 \geq 0$

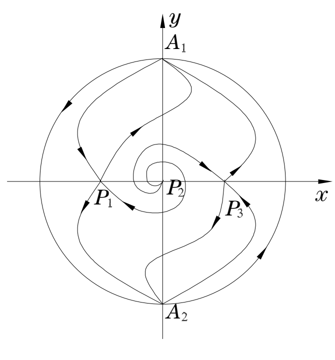

Figure 3 $\quad r < 0,\ \Delta_2 \leq 0$

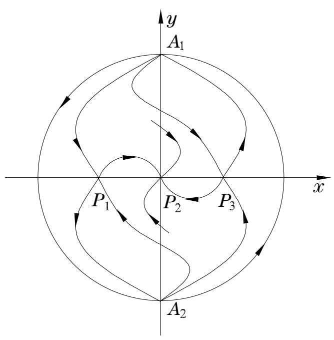

Figure 4 $\quad r > 0,\ \Delta_2 \geq 0$

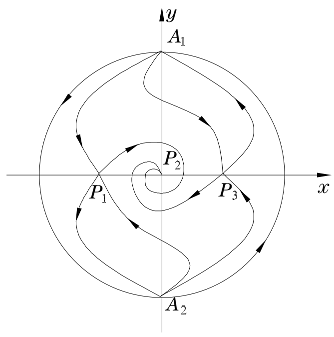

Figure 5 $\quad r > 0,\ \Delta_2 \leq 0$

From Figures 1–5, we have the following propositions.

Proposition 2.2 *If $r = 0$, except P_1, P_2, P_3 and $L(P_1, P_3), L(P_3, P_1)$, the non-periodic solutions to system* (2.11) *are unbounded. Moreover, the coordinate values of the points on these orbits tend to be infinite.*

Proposition 2.3 *If $r < 0$, except P_1, P_2, P_3 and $L(P_2, P_1), L(P_2, P_3)$, the non-periodic solutions to system* (2.11) *are unbounded. Moreover, the coordinate values of the points on these orbits tend to be infinite.*

Proposition 2.4 *If $r > 0$, except P_1, P_2, P_3 and $L(P_1, P_2), L(P_3, P_2)$, the non-periodic solutions to system* (2.11) *are unbounded. Moreover, the coordinate values of the points on these orbits tend to be infinite.*

A heteroclinic orbit of the planar dynamic system (2.11) corresponds to a kink profile solitary wave solution or an oscillatory solution to (2.10). Therefore, from Propositions 2.2–2.4 and Figures 1–5, we derive the following lemma.

Lemma 2.1 (1) *If $r = 0$,* (2.10) *has two solitary wave solutions $U_i(\xi)$ $(i = 1, 2)$, corresponding to $L(P_1, P_3)$ and $L(P_3, P_1)$ shown in Figure* 1*, respectively.*

(2) *If $r \neq 0$,* (2.10) *has two solutions $U_i(\xi)$ $(i = 1, 2)$. If $r < 0$, $U_1(\xi)$ corresponds to $L(P_2, P_1)$ shown in Figure* 2 *and Figure* 3*, $U_2(\xi)$ corresponds to $L(P_2, P_3)$ shown in Figure* 2 *and Figure* 3*; If $r > 0$, $U_1(\xi)$ corresponds to $L(P_1, P_2)$ shown in Figure* 4 *and Figure* 5*, and $U_2(\xi)$ corresponds to $L(P_3, P_2)$ shown in Figure* 4 *and Figure* 5.

Since $u(\xi)$ and $U(\xi)$ satisfy relational expression (2.8), and $H(\xi)$ and $u(\xi)$ satisfy the relational expression (2.6), we can get the following theorem.

Theorem 2.1 *Suppose $a \neq 0$, $b > 0$, $\beta^2 + b\delta > 0$ and* (2.3)–(2.5) *are established.*

(1) *If $r = 0$,* (1.4) *has two solitary wave solutions, and $u(\xi)$ described in the horizontal velocity component is kink-shaped, corresponding to $L(P_1, P_3)$ and $L(P_3, P_1)$ shown in Figure* 1;

(2) *If $r \neq 0$,* (1.4) *has two bounded traveling wave solutions $(u_i(\xi), H_i(\xi))$ $(i = 1, 2)$, where $u_i(\xi)$ $(i = 1, 2)$ are kink-shaped or oscillatory-shaped. If $r < 0$, $u_1(\xi)$ corresponds to $L(P_2, P_1)$ shown in Figure* 2 *and Figure* 3*, while $u_2(\xi)$ corresponds to $L(P_2, P_3)$ shown in Figure* 2 *and Figure* 3*; If $r > 0$, $u_1(\xi)$ corresponds to $L(P_1, P_2)$ shown in Figure* 4 *and Figure* 5*, while $u_2(\xi)$ corresponds to $L(P_3, P_2)$ shown in Figure* 4 *and Figure* 5.

3 Relations Between the Shape of Bounded Traveling Wave Solutions and the Parameter r of (1.4)

In a similar way of Theorem 2.1, in order to investigate the relations between the shape of bounded traveling wave solutions and the parameter r of (1.4), we firstly consider the relations between the shape of bounded solutions and the parameter $\overline{r}$ of (2.10).

From (2.12), we know that $f(x) = 0$ has three different real roots x_1, x_2, x_3 with $x_1 < x_2 < x_3$. Therefore, (2.10) can be written as

$$U''(\xi) + \overline{r}U'(\xi) - \overline{b}(U(\xi) - x_1)(U(\xi) - x_2)(U(\xi) - x_3) = 0. \tag{3.1}$$

In order to make use of the existent research results to derive the conclusion what we want, we make use of the transformation $V(\xi) = \frac{U(\xi)-x_1}{x_3-x_1}$ to (3.1). Then we can obtain

$$V''(\xi) + \overline{r}V'(\xi) + b'V(\xi)(1 - V(\xi))(V(\xi) - w) = 0, \tag{3.2}$$

where

$$w = \frac{x_2 - x_1}{x_3 - x_2} = \frac{1}{2}, \quad b' = \overline{b}(x_3 - x_1)^2 = \frac{2v^2}{\beta^2 + b\delta}.$$

Evidently, the singular points of (3.2) are $P_1'(0,0), P_2'(w,0), P_3'(1,0)$, which correspond to the singular points $P_1(x_1,0), P_2(x_2,0), P_3(x_3,0)$ of system (2.11), respectively. Since the linear transformation keeps the properties of singular points, the results on $P_1(x_1,0), P_2(x_2,0), P_3(x_3,0)$ given in Section 2 under the different parameter conditions also hold for $P_1'(0,0), P_2'(w,0), P_3'(1,0)$ under the corresponding conditions.

In the case of $r \neq 0$, we know from the qualitative analyses in Section 2 that (3.1) only satisfies the following alternative:

(1) $U(-\infty) = x_2,\ U(+\infty) = x_1$;

(2) $U(-\infty) = x_2,\ U(+\infty) = x_3$;

(3) $U(-\infty) = x_1,\ U(+\infty) = x_2$;

(4) $U(-\infty) = x_3,\ U(+\infty) = x_2$.

That is to say, (3.2) only has the following four cases:

(i) $V(-\infty) = w,\ V(+\infty) = 0$;

(ii) $V(-\infty) = w,\ V(+\infty) = 1$;

(iii) $V(-\infty) = 0,\ V(+\infty) = w$;

(iv) $V(-\infty) = 1,\ V(+\infty) = w$.

Let

$$\overline{r}^* = 2\sqrt{\omega(1-\omega)b'} = -|v|\sqrt{\frac{2}{\beta^2 + b\delta}}, \tag{3.3}$$

$$r^* = \left|\frac{v}{b}\right|\sqrt{2(\beta^2 + \delta b)}. \tag{3.4}$$

From the results in [15–17] and Lemma 2.1 in the previous section, we have

Lemma 3.1 *Supposing $\overline{r} \neq 0,\ \beta^2 + b\delta > 0$, we can obtain two solutions, $V_1(\xi)$ and $V_2(\xi)$, belonging to* (3.2)*,*

(1) *If* $\overline{r} \le -\overline{r}^*$, $V_1(\xi)$ *is a monotonically decreasing solution and satisfies the case* (i), *while* $V_2(\xi)$ *is a monotonically increasing solution and satisfies the case* (ii);

(2) *If* $-\overline{r}^* < \overline{r} < 0$, $V_1(\xi)$ *is the oscillatory solution and satisfies the case* (i), *and* $V_2(\xi)$ *is also the oscillatory solution and satisfies the case* (ii);

(3) *If* $0 < \overline{r} < \overline{r}^*$, $V_1(\xi)$ *is the oscillatory solution and satisfies the case* (iii), *and* $V_2(\xi)$ *is also the oscillatory solution and satisfies the case* (iv);

(4) *If* $\overline{r} \ge \overline{r}^*$, $V_1(\xi)$ *is a monotonically increasing solution and satisfies the case* (iii), *while* $V_2(\xi)$ *is a monotonically decreasing solution and satisfies the case* (iv).

Since $u_i(\xi)$ and $V_i(\xi)$ satisfy the linear relation, and the solitary wave solution $(u(\xi), H(\xi))$ to (1.2) satisfies (2.6), we can obtain the fact that if $u(\xi)$ is an oscillatory solution, $H(\xi)$ must be an oscillatory solution. According to Lemma 3.1, we can arrive at

Theorem 3.1 *Supposing the conditions of Theorem* 2.1 *to be established, the relations between the shape of traveling wave solutions* $(u_i(\xi), H_i(\xi))$ $(i = 1, 2)$ *to* (1.4) *and the dissipation coefficient* r *will satisfy*

(1) *If* $r \le -r^*$, *then* $u_1(\xi)$ *of the* $(u_1(\xi), H_1(\xi))$ *monotonically decreases, satisfying*

$$\lim_{\xi\to-\infty} u_1(\xi) = \frac{v}{a}, \quad \lim_{\xi\to+\infty} u_1(\xi) = 0,$$

$H_1(\xi)$ *satisfies*

$$\lim_{\xi\to-\infty} H_1(\xi) = \frac{v^2}{2ab}, \quad \lim_{\xi\to+\infty} H_1(\xi) = 0,$$

and $U_1(\xi) = u_1(\xi) - \frac{v}{a}$ *corresponds to* $L(P_2, P_1)$ *shown in Figure* 2. *At the same time,* $u_2(\xi)$ *of* $(u_2(\xi), H_2(\xi))$ *monotonically increases, satisfying*

$$\lim_{\xi\to-\infty} u_2(\xi) = \frac{v}{a}, \quad \lim_{\xi\to+\infty} u_2(\xi) = \frac{2v}{a},$$

$H_2(\xi)$ *satisfies*

$$\lim_{\xi\to-\infty} H_2(\xi) = \frac{v^2}{2ab}, \quad \lim_{\xi\to+\infty} H_2(\xi) = 0,$$

and $U_2(\xi) = u_2(\xi) - \frac{v}{a}$ *corresponds to* $L(P_2, P_3)$ *shown in Figure* 2.

(2) *If* $-r^* < r < 0$, *then* $u_1(\xi)$ *and* $H_1(\xi)$ *of* $(u_1(\xi), H_1(\xi))$ *are both oscillatory, satisfying*

$$\lim_{\xi\to-\infty} u_1(\xi) = \frac{v}{a}, \qquad \lim_{\xi\to+\infty} u_1(\xi) = 0,$$
$$\lim_{\xi\to-\infty} H_1(\xi) = \frac{v^2}{2ab}, \qquad \lim_{\xi\to+\infty} H_1(\xi) = 0,$$

and $U_1(\xi) = u_1(\xi) - \frac{v}{a}$ *corresponds to* $L(P_2, P_1)$ *shown in Figure* 3. *At the same time,* $u_2(\xi)$ *and* $H_2(\xi)$ *of* $(u_2(\xi), H_2(\xi))$ *are also oscillatory, satisfying*

$$\lim_{\xi\to-\infty} u_2(\xi) = \frac{v}{a}, \qquad \lim_{\xi\to+\infty} u_2(\xi) = \frac{2v}{a},$$
$$\lim_{\xi\to-\infty} H_2(\xi) = \frac{v^2}{2ab}, \qquad \lim_{\xi\to+\infty} H_2(\xi) = 0,$$

and $U_2(\xi) = u_2(\xi) - \frac{v}{a}$ *corresponds to* $L(P_2, P_3)$ *shown in Figure* 3.

(3) *If* $0 < r < r^*$, *then* $u_1(\xi)$ *and* $H_1(\xi)$ *of* $(u_1(\xi), H_1(\xi))$ *are both oscillatory, satisfying*

$$\lim_{\xi\to-\infty} u_1(\xi) = 0, \qquad \lim_{\xi\to+\infty} u_1(\xi) = \frac{v}{a},$$
$$\lim_{\xi\to-\infty} H_1(\xi) = 0, \qquad \lim_{\xi\to+\infty} H_1(\xi) = \frac{v^2}{2ab},$$

and $U_1(\xi) = u_1(\xi) - \frac{v}{a}$ *corresponds to* $L(P_1, P_2)$ *shown in Figure* 5. *At the same time,* $u_2(\xi)$ *and* $H_2(\xi)$ *of* $(u_2(\xi), H_2(\xi))$ *are also oscillatory, satisfying*

$$\lim_{\xi\to-\infty} u_2(\xi) = \frac{2v}{a}, \qquad \lim_{\xi\to+\infty} u_2(\xi) = \frac{v}{a},$$
$$\lim_{\xi\to-\infty} H_2(\xi) = 0, \qquad \lim_{\xi\to+\infty} H_2(\xi) = \frac{v^2}{2ab},$$

and $U_2(\xi) = u_2(\xi) - \frac{v}{a}$ *corresponds to* $L(P_3, P_2)$ *shown in Figure* 5.

(4) *If* $r \geq r^*$, *then* $u_1(\xi)$ *of* $(u_1(\xi), H_1(\xi))$ *monotonically increases, satisfying*

$$\lim_{\xi\to-\infty} u_1(\xi) = 0, \quad \lim_{\xi\to+\infty} u_1(\xi) = \frac{v}{a},$$

$H_1(\xi)$ *satisfies*

$$\lim_{\xi\to-\infty} H_1(\xi) = 0, \quad \lim_{\xi\to+\infty} H_1(\xi) = \frac{v^2}{2ab},$$

and $U_1(\xi) = u_1(\xi) - \frac{v}{a}$ *corresponds to* $L(P_1, P_2)$ *shown in Figure* 4. *At the same time,* $u_2(\xi)$ *of* $(u_2(\xi), H_2(\xi))$ *monotonically decreases, satisfying*

$$\lim_{\xi\to-\infty} u_2(\xi) = \frac{2v}{a}, \quad \lim_{\xi\to+\infty} u_2(\xi) = \frac{v}{a}.$$

$H_2(\xi)$ *satisfies*

$$\lim_{\xi\to-\infty} H_2(\xi) = 0, \quad \lim_{\xi\to+\infty} H_2(\xi) = \frac{v^2}{2ab},$$

and $U_2(\xi) = u_2(\xi) - \frac{v}{a}$ *corresponds to* $L(P_3, P_2)$ *shown in Figure* 4.

In the following, we proceed to prove that oscillatory traveling wave solutions to (1.4) have the damped property. We take the case of $-r^* < r < 0$ as an example, and the damped oscillatory property of the case of $0 < r < r^*$ can be discussed similarly.

Theorem 3.2 (1) *If* $-r^* < r < 0$, *the oscillatory traveling wave solution* $u_1(\xi)$ *of* $(u_1(\xi), H_1(\xi))$ *has the maximum at* $\overline{\xi}_1$. *The solution has the monotonically decreasing property at the right of* $\overline{\xi}_1$ *and the damped oscillatory property at the left of* $\overline{\xi}_1$. *Namely, there exist countably infinite maximum points* $\overline{\xi}_i$ $(i = 1, 2, \cdots)$ *and minimum points* $\underline{\xi}_i$ $(i = 1, 2, \cdots)$ *on* ξ *axis, such that*

$$\begin{cases} -\infty < \cdots < \underline{\xi}_n < \overline{\xi}_n < \underline{\xi}_{n-1} < \overline{\xi}_{n-1} < \cdots < \underline{\xi}_1 < \overline{\xi}_1 < +\infty, \\ \lim\limits_{n\to\infty} \overline{\xi}_n = \lim\limits_{n\to\infty} \underline{\xi}_n = -\infty, \end{cases} \tag{3.5}$$

and moreover,

$$\begin{cases} u_1(+\infty) < u_1(\underline{\xi}_1) < u_1(\underline{\xi}_2) < \cdots < u_1(\underline{\xi}_n) < \cdots < u_1(-\infty) < \cdots \\ \quad < u_1(\overline{\xi}_n) < u_1(\overline{\xi}_{n-1}) < \cdots < u_1(\overline{\xi}_1), \\ \lim\limits_{n\to\infty} u_1(\overline{\xi}_n) = \lim\limits_{n\to\infty} u_1(\underline{\xi}_n) = \dfrac{v}{a}, \end{cases} \tag{3.6}$$

$$\lim_{n\to\infty}(\overline{\xi}_n - \overline{\xi}_{n+1}) = \lim_{n\to\infty}(\underline{\xi}_n - \underline{\xi}_{n+1}) = \frac{4\pi(\beta^2+\delta b)}{\sqrt{2v^2(\beta^2+\delta b) - r^2b^2}}. \tag{3.7}$$

Correspondingly, the oscillatory traveling wave solution $H_1(\xi)$ *of* $(u_1(\xi), H_1(\xi))$ *has the maximum at* $\overline{\eta}_1$. *The solution has the monotonically decreasing property at the right of* $\overline{\eta}_1$ *and the damped oscillatory property at the left of* $\overline{\eta}_1$. *Namely, there exist countably infinite maximum points* $\overline{\eta}_i$ $(i = 1, 2, \cdots)$ *and minimum points* $\underline{\eta}_i$ $(i = 1, 2, \cdots)$ *on* ξ *axis, such that*

$$\begin{cases} -\infty < \cdots < \underline{\eta}_n < \overline{\eta}_n < \underline{\eta}_{n-1} < \overline{\eta}_{n-1} < \cdots < \underline{\eta}_1 < \overline{\eta}_1 < +\infty, \\ \lim\limits_{n\to\infty} \overline{\eta}_n = \lim\limits_{n\to\infty} \underline{\eta}_n = -\infty, \end{cases} \tag{3.8}$$

and moreover,

$$\begin{cases} H_1(+\infty) < H_1(\underline{\eta}_1) < H_1(\underline{\eta}_2) < \cdots < H_1(\underline{\eta}_n) < \cdots < H_1(-\infty) < \cdots \\ \quad < H_1(\overline{\eta}_n) < H_1(\overline{\eta}_{n-1}) < \cdots < H_1(\overline{\eta}_1), \\ \lim\limits_{n\to\infty} H_1(\overline{\eta}_n) = \lim\limits_{n\to\infty} H_1(\underline{\eta}_n) = \dfrac{v^2}{2ab}. \end{cases} \tag{3.9}$$

(2) *If* $-r^* < r < 0$, *the oscillatory traveling wave solution* $u_2(\xi)$ *of* $(u_2(\xi), H_2(\xi))$ *has a minimum at* $\underline{\xi'}_1$. *The solution has the monotonically increasing property at the right of* $\underline{\xi'}_1$ *and the damped oscillatory property at the left of* $\underline{\xi'}_1$. *Namely, there exist countably infinite maximum points* $\overline{\xi'}_i$ $(i = 1, 2, \cdots)$ *and minimum points* $\underline{\xi'}_i$ $(i = 1, 2, \cdots)$ *on* ξ *axis, such that*

$$\begin{cases} -\infty < \cdots < \underline{\xi'}_n < \overline{\xi'}_n < \underline{\xi'}_{n-1} < \overline{\xi'}_{n-1} < \cdots < \underline{\xi'}_1 < \overline{\xi'}_1 < +\infty, \\ \lim\limits_{n\to\infty} \overline{\xi'}_n = \lim\limits_{n\to\infty} \underline{\xi'}_n = -\infty, \end{cases} \tag{3.10}$$

and moreover,

$$\begin{cases} u_2(-\infty) < u_2(\underline{\xi'}_1) < u_2(\underline{\xi'}_2) < \cdots < u_2(\underline{\xi'}_n) < \cdots < u_2(-\infty) < \cdots \\ \quad < u_2(\overline{\xi'}_n) < u_2(\overline{\xi'}_{n-1}) < \cdots < u_2(\overline{\xi'}_1), \\ \lim\limits_{n\to\infty} u_2(\overline{\xi'}_n) = \lim\limits_{n\to\infty} u_2(\underline{\xi'}_n) = u_2(-\infty) = \dfrac{v}{a}, \end{cases} \tag{3.11}$$

$$\lim_{n\to\infty}(\overline{\xi'}_n - \overline{\xi'}_{n+1}) = \lim_{n\to\infty}(\underline{\xi'}_n - \underline{\xi'}_{n+1}) = \frac{4\pi(\beta^2+\delta b)}{\sqrt{2v^2(\beta^2+\delta b) - r^2b^2}}. \tag{3.12}$$

Correspondingly, the oscillatory traveling wave solution $H_2(\xi)$ *of* $(u_2(\xi), H_2(\xi))$ *has the maximum at* $\overline{\eta'}_1$. *The solution has the monotonically decreasing property at the right of* $\overline{\eta'}_1$ *and the* *damped oscillatory property at the left of* $\overline{\eta'}_1$. *Namely, there exist countably infinite maximum points* $\overline{\eta'}_i$ $(i = 1, 2, \cdots)$ *and minimum points* $\underline{\eta'}_i$ $(i = 1, 2, \cdots)$ *on* ξ *axis, such that*

$$\begin{cases} -\infty < \cdots < \underline{\eta'}_n < \overline{\eta'}_n < \underline{\eta'}_{n-1} < \overline{\eta'}_{n-1} < \cdots < \underline{\eta'}_1 < \overline{\eta'}_1 < +\infty, \\ \lim\limits_{n\to\infty} \overline{\eta'}_n = \lim\limits_{n\to\infty} \underline{\eta'}_n = -\infty, \end{cases} \tag{3.13}$$

and moreover,

$$\begin{cases} H_2(+\infty) < H_2(\underline{\eta'}_1) < H_2(\underline{\eta'}_2) < \cdots < H_2(\underline{\eta'}_n) < \cdots \\ \quad < H_2(-\infty) < \cdots < H_2(\overline{\eta'}_n) < H_2(\overline{\eta'}_{n-1}) < \cdots < H_2(\overline{\eta'}_1), \\ \lim\limits_{n\to\infty} H_2(\overline{\eta'}_n) = \lim\limits_{n\to\infty} H_2(\underline{\eta'}_n) = \dfrac{v^2}{2ab}. \end{cases} \tag{3.14}$$

Proof (1) Since $u_1(\xi)$ and $H_1(\xi)$ of $(u_1(\xi), H_1(\xi))$ satisfy the relational expression (2.6), where $u_1(\xi)$ satisfies (2.7), $u(\xi)$ and $U(\xi)$ satisfy the relational expression (2.8). Therefore, we can obtain the properties of the traveling wave solution to (1.4) through studying those of (2.10). Under the conditions given in Theorem 3.2(1), we have $\beta^2 + b\delta > 0$ and $-r^* < r < 0$. Thus, P_1 is a saddle point and P_2 is an unstable focus point. Moreover, $L(P_1, P_2)$ tends to P_2 spirally as $\xi \to -\infty$. It is easy to see that the intersection points of $L(P_2, P_1)$ and x axis on the right hand of P_2 correspond to the maximum points of $u_1(\xi)$, while the intersection points of $L(P_2, P_1)$ and x axis on the left hand of P_2 correspond to the minimum points of $u_1(\xi)$. Hence, both (3.5) and (3.6) hold. Furthermore, when $L(P_2, P_1)$ approaches to P_2 sufficiently, its properties tend to the properties of the linear approximate solution to (2.11) at P_2. The frequency of the orbit rotating around P_2 tends to $\sqrt{\frac{-r^2b^2+2v^2(\beta^2+\delta b)}{4(\beta^2+\delta b)^2}}$. Therefore, (3.7) holds.

(2) The proof is similar to that of (1).

Summarizing the above results, we can obtain that the appearance of the kink profile solitary wave solutions and oscillatory traveling wave solutions of $u(\xi)$ are determined by the dissipation term ru_{xx}. r^* in (3.4) is just the critical value that can distinguish $u(\xi)$ into the kink profile solitary wave solutions or the oscillatory traveling wave solutions. When $|r|$ is large, namely $|r| \geq r^*$, $u(\xi)$ appears as a kink profile solitary wave solution; when $|r|$ is small, namely $0 < |r| < r^*$, $u(\xi)$ appears as an oscillatory traveling wave solution. At the same time, the whole traveling wave solution appears as oscillatory, which has the damped property (we call it damped oscillatory solution in this paper).

4 Kink Profile Solitary Wave Solutions and Approximate Damped Oscillatory Solutions to (1.4)

The traveling wave solution to (1.4) appears as a pair of solutions $(u(\xi), H(\xi))$, where $u(\xi)$ satisfies (2.7) and $H(\xi)$ satisfies (2.6). Therefore, we only need to obtain the solution to (2.7).

By Theorem 2.1 we can see that (2.7) has the kink profile solitary wave solutions as $r = 0$, while (2.7) has the kink profile solitary wave solutions and the damped oscillatory solutions as $r \neq 0$. In this section, we will derive exact solitary wave solutions and approximate damped oscillatory solutions. We consider the different cases respectively as follows.

4.1 Kink profile solitary wave solutions to (1.4)

To obtain the kink profile solitary wave solutions to (1.4), inspired by [18], we assume that (2.7) has the kink profile solution in the following form:

$$u(\xi) = \frac{A\mathrm{e}^{\alpha(\xi-\xi_0)}}{1+\mathrm{e}^{\alpha(\xi-\xi_0)}} + D, \tag{4.1}$$

where A, α, D are undetermined constants. We can obtain

$$u'(\xi) = \frac{A\alpha\mathrm{e}^{\alpha(\xi-\xi_0)}}{(1+\mathrm{e}^{\alpha(\xi-\xi_0)})^2}, \quad u''(\xi) = \frac{A\alpha^2\mathrm{e}^{\alpha(\xi-\xi_0)} - A\alpha^2\mathrm{e}^{2\alpha(\xi-\xi_0)}}{(1+\mathrm{e}^{\alpha(\xi-\xi_0)})^3}. \tag{4.2}$$

Substituting (4.1) and (4.2) into (2.7), we have

$$\begin{cases} a^2D^3 - 3avD^2 + 2v^2D = 0, \\ a^2A^2 + (3a^2D - 3av)A + (3a^2D^2 - 6aDv + 2v^2) = 0, \\ (2\beta^2 + 2b\delta)\alpha^2 + 2br\alpha + (3a^2D^2 - 6aDv + 2v^2) = 0, \\ (3a^2D - 3av)A + (2\beta^2 + 2b\delta)\alpha^2 - 2br\alpha + 2(3a^2D^2 - 6aDv + 2v^2) = 0. \end{cases} \tag{4.3}$$

By solving it, we can obtain the following solutions to (2.7):

(1) If $r = 0$, there exist two kink profile solitary wave solutions to (2.7) in the form of

$$u_+(\xi) = \frac{v}{a}\tanh\Big[\frac{1}{2}\sqrt{\frac{v^2}{\beta^2+b\delta}}(\xi-\xi_0)\Big] + \frac{v}{a}, \tag{4.4}$$

$$u_-(\xi) = -\frac{v}{a}\tanh\Big[\frac{1}{2}\sqrt{\frac{v^2}{\beta^2+b\delta}}(\xi-\xi_0)\Big] + \frac{v}{a}, \tag{4.5}$$

where $U_+(\xi) = u_+(\xi) - \frac{v}{a}$ corresponds to $L(P_3, P_1)$ shown in Figure 1, $U_-(\xi) = u_-(\xi) - \frac{v}{a}$ corresponds $L(P_1, P_3)$ shown in Figure 1.

(2) If

$$r = -\frac{3v}{2b}\sqrt{b\delta+\beta^2} \quad \text{and} \quad r = \frac{3v}{2b}\sqrt{b\delta+\beta^2},$$

(2.7) has two kink profile solitary wave solutions

$$u_1(\xi) = \frac{v}{2a}\tanh\Big[\frac{br}{6(\beta^2+b\delta)}(\xi-\xi_0)\Big] + \frac{v}{2a}$$

and

$$u_2(\xi) = -\frac{v}{2a}\tanh\Big[\frac{br}{6(\beta^2+b\delta)}(\xi-\xi_0)\Big] + \frac{3v}{2a},$$

where $U_1(\xi) = u_1(\xi) - \frac{v}{a}$ corresponds to $L(P_2, P_1)$ shown in Figure 2 or $L(P_1, P_2)$ shown in Figure 4; $U_2(\xi) = u_2(\xi) - \frac{v}{a}$ corresponds to $L(P_2, P_3)$ shown in Figure 2 or $L(P_3, P_2)$ shown in Figure 4.

Remark 4.1 From (2.3) and (2.6), we can see that if $H(-\infty) = H(+\infty)$, that is,

$$\frac{v}{b}u(-\infty) - \frac{a}{2b}u^2(-\infty) = \frac{v}{b}u(+\infty) - \frac{a}{2b}u^2(+\infty),$$

the solution $H(\xi)$ corresponds to a bell profile solitary wave solution.

So we have the following theorem.

Theorem 4.1 (1) *If* $r = 0$, (1.4) *has the following two solitary wave solutions:*

$$\begin{cases} u_+(\xi) = \dfrac{v}{a}\tanh\Big[\dfrac{1}{2}\sqrt{\dfrac{v^2}{\beta^2 + b\delta}}(\xi - \xi_0)\Big] + \dfrac{v}{a}, \\ H_+(\xi) = \dfrac{v^2}{2ab}\Big(1 - \dfrac{\beta}{v}\sqrt{\dfrac{v^2}{\beta^2 + b\delta}}\Big)\text{sech}^2\Big[\dfrac{1}{2}\sqrt{\dfrac{v^2}{\beta^2 + b\delta}}(\xi - \xi_0)\Big], \end{cases} \tag{4.6}$$

$$\begin{cases} u_-(\xi) = -\dfrac{v}{a}\tanh\Big[\dfrac{1}{2}\sqrt{\dfrac{v^2}{\beta^2 + b\delta}}(\xi - \xi_0)\Big] + \dfrac{v}{a}, \\ H_-(\xi) = \dfrac{v^2}{2ab}\Big(1 + \dfrac{\beta}{v}\sqrt{\dfrac{v^2}{\beta^2 + b\delta}}\Big)\text{sech}^2\Big[\dfrac{1}{2}\sqrt{\dfrac{v^2}{\beta^2 + b\delta}}(\xi - \xi_0)\Big], \end{cases} \tag{4.7}$$

where $u_+(\xi)$ *and* $u_-(\xi)$ *are both kink-shaped, while* $H_+(\xi)$ *and* $H_-(\xi)$ *are both bell-shaped.*

(2) *If*

$$r = -\frac{3v}{2b}\sqrt{b\delta + \beta^2} \quad \textit{and} \quad r = \frac{3v}{2b}\sqrt{b\delta + \beta^2},$$

(1.4) *has the following kink profile solitary wave solutions*:

$$\begin{cases} u_1(\xi) = \dfrac{v}{2a}\tanh\Big[\dfrac{br}{6(\beta^2 + b\delta)}(\xi - \xi_0)\Big] + \dfrac{v}{2a}, \\ H_1(\xi) = \dfrac{v}{b}u_1(\xi) - \dfrac{a}{2b}u_1^2(\xi) - \dfrac{\beta}{b}u_1'(\xi), \end{cases} \tag{4.8}$$

$$\begin{cases} u_2(\xi) = -\dfrac{v}{2a}\tanh\Big[\dfrac{br}{6(\beta^2 + b\delta)}(\xi - \xi_0)\Big] + \dfrac{3v}{2a}, \\ H_2(\xi) = \dfrac{v}{b}u_2(\xi) - \dfrac{a}{2b}u_2^2(\xi) - \dfrac{\beta}{b}u_2'(\xi), \end{cases} \tag{4.9}$$

where $u_i(\xi)$ $(i = 1, 2)$ *and* $H_j(\xi)$ $(j = 1, 2)$ *are both kink-shaped.*

Remark 4.2 (1) (4.6) and (4.7) obtained from Theorem 4.1(1) are just the solutions given in Theorem 2.1(1).

(2) We can get the absolute value

$$|r| = \frac{3v}{2b}\sqrt{b\delta + \beta^2} > \frac{v}{b}\sqrt{2(b\delta + \beta^2)} = r^*$$

from Theorem 4.1(2), and (4.8) and (4.9) that satisfy (1) and (4) in Theorem 3.1. So Theorem 4.1(2) is just the concrete presentation of (1) and (4) in Theorem 3.1.

4.2 Approximate damped oscillatory solutions to (1.4)

In this section, we study the approximate damped oscillatory solutions to (1.4) as $r \neq 0$. Note that the discriminant of the characteristic equation of (2.11) at singular points $(x_i, 0)$ $(i = 1, 2, 3)$ is

$$\triangle_i = \bar{r}^2 + 4\bar{b}f'(x_i) = \Big(\frac{rb}{\beta^2+\delta b}\Big)^2 + \frac{2a^2}{\beta^2+\delta b}\Big(3x_i^2 - \frac{v^2}{a^2}\Big), \quad i = 1, 2, 3.$$

From the analysis in the previous section, we know that the damped oscillatory solutions to (2.7) will arise when $|r|$ is small but not zero. Since these exact expressions are difficult to obtain, in the following, we will give the approximate solutions of the damped oscillatory solutions. We take the approximate damped oscillatory solutions corresponding to $L(P_2, P_1)$ shown in Figure 3 as an example to explain our method. It is easy to see that focus-saddle orbit $L(P_2, P_1)$ in Figure 3 comes from the break of the heteroclinic orbit $L(P_3, P_1)$ in Figure 1 under the effect of the dissipation term ru_{xx}. Hence, the non-oscillatory part of the damped oscillatory solution corresponding to $L(P_2, P_1)$ can be denoted by the kink profile solitary wave solution of the form $u_-(\xi)$ as in (4.5), while the oscillatory part of the damped oscillatory wave solution can be expressed approximatively by the following form:

$$\widetilde{u}(\xi) = \mathrm{e}^{\alpha(\xi-\xi_0)}(A_1\cos(B(\xi-\xi_0)) - A_2\sin(B(\xi-\xi_0))) + C, \quad \xi \in (-\infty, \xi_0], \tag{4.10}$$

where A_1, A_2, B, C, α are undetermined constants.

Substituting (4.10) into (2.7), and neglecting the terms including $\mathrm{e}^{3\alpha(\xi-\xi_0)}$ and $\mathrm{e}^{2\alpha(\xi-\xi_0)}$, we have

$$\begin{cases} B^2 = \dfrac{-r^2b^2 + 2v^2(\beta^2+\delta b)}{4(\beta^2+\delta b)^2}, \\ \alpha = -\dfrac{rb}{2(\beta^2+\delta b)}, \\ a^2C^3 - 3avC^2 + 2v^2C = 0. \end{cases} \tag{4.11}$$

In order to derive the approximate damped oscillatory solution to (1.4), it still requires some conditions to connect (4.5) and (4.10). At first, we should take

$$\lim_{\xi\to-\infty} u(\xi) = \lim_{\xi\to-\infty} \widetilde{u}(\xi) = \frac{v}{a},$$

and thus $C = \frac{v}{a}$.

In order to make the proof easy, we take ξ_0' as a connective point (as shown in Figure 6), where $\xi_0' \in (\xi_0 - \frac{\pi}{B}, \xi_0)$, and choose

$$\frac{\mathrm{d}^i}{\mathrm{d}\xi^i}\widetilde{u}(\xi_0') = \frac{\mathrm{d}^i}{\mathrm{d}\xi^i}u_-(\xi_0'), \quad i = 0, 1, \tag{4.12}$$

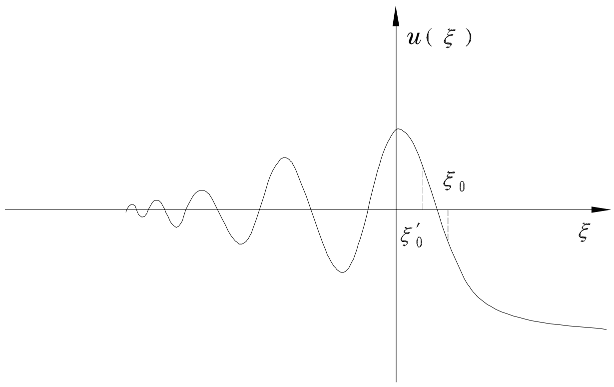

Figure 6 Location of the connective point ξ_0'

that is

$$e^{\alpha(\xi_0'-\xi_0)}(A_1\cos(B(\xi_0'-\xi_0))-A_2\sin(B(\xi_0'-\xi_0)))+\frac{v}{a}=u_-(\xi_0'), \tag{4.13}$$

$$\begin{aligned}&\alpha e^{\alpha(\xi_0'-\xi_0)}(A_1\cos(B(\xi_0'-\xi_0))-A_2\sin(B(\xi_0'-\xi_0)))\\ &-Be^{\alpha(\xi_0'-\xi_0)}(A_1\sin(B(\xi_0'-\xi_0))+A_2\cos(B(\xi_0'-\xi_0)))=u_-'(\xi_0'),\end{aligned} \tag{4.14}$$

as connective conditions. From (4.13) and (4.14), we obtain

$$\begin{cases}A_1=e^{-\alpha(\xi_0'-\xi_0)}\Big(\Big(\dfrac{\alpha}{B}\sin(B(\xi_0'-\xi_0))+\cos(B(\xi_0'-\xi_0))\Big)\Big(u_-(\xi_0')-\dfrac{v}{a}\Big)\\ \qquad -\dfrac{1}{B}\sin(B(\xi_0'-\xi_0))u_-'(\xi_0')\Big),\\ A_2=A_1\cot(B(\xi_0'-\xi_0))-\dfrac{(u_-(\xi_0')-\frac{v}{a})e^{-\alpha(\xi_0'-\xi_0)}}{\sin(B(\xi_0'-\xi_0))}.\end{cases} \tag{4.15}$$

The value of $(A_1\cos(B(\xi-\xi_0))-A_2\sin(B(\xi-\xi_0)))+C$ is independent of the value of B. Without loss of generality, let $B>0$ throughout this paper.

Theorem 4.2 *If* $-|\frac{v}{b}|\sqrt{2(\beta^2+\delta b)}<r<0$, (1.4) *has the damped oscillatory solution* $(u_1(\xi),H_1(\xi))$. *The approximate solution of* $u_1(\xi)$ *is in the form of*

$$\begin{aligned}u_1(\xi)&\approx\widetilde{u}_1(\xi)\\ &=\begin{cases}-\dfrac{v}{a}\tanh\Big[\dfrac{1}{2}\sqrt{\dfrac{v^2}{\beta^2+\delta b}}(\xi-\xi_0)\Big]+\dfrac{v}{a}, & \xi\in(\xi_0',+\infty),\\ e^{\alpha(\xi-\xi_0)}(A_1\cos(B(\xi-\xi_0))-A_2\sin(B(\xi-\xi_0)))+\dfrac{v}{a}, & \xi\in(-\infty,\xi_0'],\end{cases}\end{aligned} \tag{4.16}$$

where

$$\alpha=-\frac{rb}{2(\beta^2+\delta b)},\quad B=\sqrt{\frac{-r^2b^2+2v^2(\beta^2+\delta b)}{4(\beta^2+\delta b)^2}},$$

A_1,A_2 *are given by* (4.15), *which make* $U_1(\xi)=u_1(\xi)-\frac{v}{a}$ *correspond to the focus-saddle orbit* $L(P_2,P_1)$ *in Figure* 3. *The approximate solution of* $H_1(\xi)$ *is in the form of*

$$H_1(\xi)\approx\widetilde{H}_1(\xi)=\begin{cases}\dfrac{v}{b}\widetilde{u}_1(\xi)-\dfrac{a}{2b}\widetilde{u}_1^2(\xi)-\dfrac{\beta}{b}\widetilde{u}_1'(\xi), & \xi\in(-\infty,\xi_0'],\\ H_-(\xi), & \xi\in(\xi_0',+\infty),\end{cases} \tag{4.17}$$

where $H_-(\xi)$ *is determined by* (4.7).

The shapes of $\widetilde{u}_1(\xi)$ and $\widetilde{H}_1(\xi)$ with

$$v = 1.3, \quad \xi_0 = 0, \quad \xi_0' = 0.3, \quad \beta = 2.05, \quad \delta = 2.2, \quad r = -0.4$$

are shown in Figures 7 and 8, respectively.

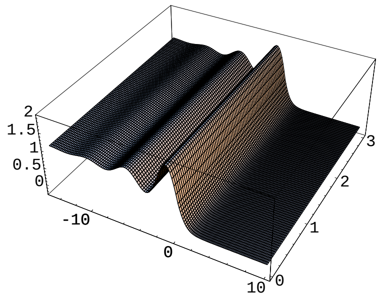

Figure 7 The graphics of $\widetilde{u}_1(\xi)$

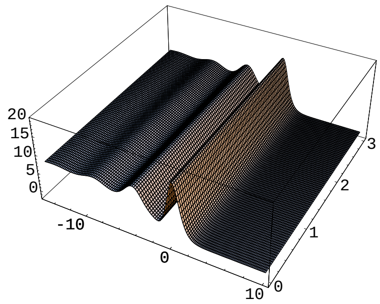

Figure 8 The graphics of $\widetilde{H}_1(\xi)$

Similarly, the focus-saddle orbit $L(P_2, P_3)$ in Figure 3 comes from the break of the heteroclinic orbit $L(P_1, P_3)$ in Figure 1 under the effect of the dissipation term ru_{xx}. Hence, the non-oscillatory part of the damped oscillatory solution to (2.7) corresponding to $L(P_2, P_3)$ can be denoted by the kink profile solitary wave solution of the form $u_+(\xi)$ as in (4.4), while the oscillatory part of the damped oscillatory wave solution can be expressed approximatively by (4.10).

Theorem 4.3 *If* $-|\frac{v}{b}|\sqrt{2(\beta^2+\delta b)} < r < 0$, (1.4) *has the damped oscillatory solution* $(u_2(\xi), H_2(\xi))$. *The approximate solution of* $u_2(\xi)$ *is in the form of*

$$u_2(\xi) \approx \widetilde{u}_2(\xi) = \begin{cases} \dfrac{v}{a}\tanh\Big[\dfrac{1}{2}\sqrt{\dfrac{v^2}{\beta^2+\delta b}}(\xi-\xi_0)\Big] + \dfrac{v}{a}, & \xi \in (\xi_0', +\infty), \\ e^{\alpha(\xi-\xi_0)}(A_1'\cos(B(\xi-\xi_0)) - A_2'\sin(B(\xi-\xi_0))) + \dfrac{v}{a}, & \xi \in (-\infty, \xi_0'], \end{cases} \tag{4.18}$$

where

$$\begin{cases} A_1' = e^{-\alpha(\xi_0'-\xi_0)}\Big(\Big(\dfrac{\alpha}{B}\sin(B(\xi_0'-\xi_0)) + \cos(B(\xi_0'-\xi_0))\Big)\Big(u_+(\xi_0') - \dfrac{v}{a}\Big) \\ \qquad -\dfrac{1}{B}\sin(B(\xi_0'-\xi_0))u_+'(\xi_0')\Big), \\ A_2' = A_1'\cot(B(\xi_0'-\xi_0)) - \dfrac{(u_+(\xi_0') - \frac{v}{a})e^{-\alpha(\xi_0'-\xi_0)}}{\sin(B(\xi_0'-\xi_0))}, \end{cases} \tag{4.19}$$

where B, α *are the same as* B, α *in Theorem* 4.2, *which make* $U_2(\xi) = u_2(\xi) - \frac{v}{a}$ *correspond to the focus-saddle orbit* $L(P_2, P_3)$ *in Figure* 3. *The approximate solution of* $H_2(\xi)$ *is in the form* *of*

$$H_2(\xi) \approx \widetilde{H}_2(\xi) = \begin{cases} \frac{v}{b}u_2(\xi) - \frac{a}{2b}u_2^2(\xi) - \frac{\beta}{b}u_2'(\xi), & \xi \in (-\infty, \xi_0'], \\ H_+(\xi), & \xi \in (\xi_0', +\infty), \end{cases} \tag{4.20}$$

where H_+ *is determined by* (4.6).

The shapes of $\widetilde{u}_2(\xi)$ and $\widetilde{H}_2(\xi)$ with

$$v = 1.3, \quad \xi_0 = 0, \quad \xi_0' = 0.3, \quad \beta = 2.05, \quad \delta = 2.2, \quad r = -0.4$$

are shown in Figures 9 and 10, respectively.

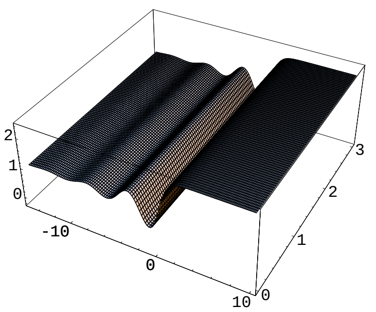

Figure 9 The graphics of $\widetilde{u}_2(\xi)$

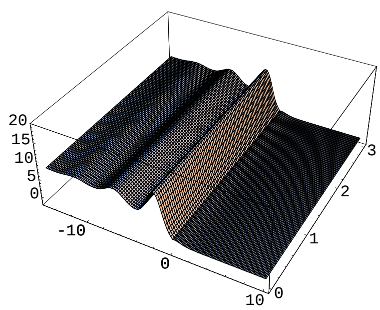

Figure 10 The graphics of $\widetilde{H}_2(\xi)$

In the same way, we can obtain the approximate expression of $(u_i(\xi), H_i(\xi))$ $(i = 1, 2)$ as

$$0 < r < \left|\frac{v}{b}\right|\sqrt{2(\beta^2 + \delta b)}.$$

5 Error Estimates of Approximate Damped Oscillatory Solutions to (1.4)

In this section, we will investigate error estimates between the exact damped oscillatory solutions and the approximate solutions given in Section 4. Therefore, we always assume the dissipation coefficient r satisfies

$$0 < r^2 < \frac{2v^2}{b^2}(\beta^2 + b\delta).$$

We still take the focus-saddle orbit $L(P_2, P_1)$ in Figure 3 as an example. Error estimates between other damped oscillatory solutions and their approximate solutions can be obtained similarly. Just for this reason and to avoid tedious marking, we denote $u_1(\xi)$ which corresponds to $L(P_2, P_1)$ shown in Figure 3 as $u(\xi)$, and $H_1(\xi)$ as $H(\xi)$.

Since the properties of the solutions are unchangeable as moving on x axis, we move the connective point ξ_0' to the origin. Then we let $\xi_1 = \xi_0 - \xi_0' > 0$, which can translate (4.16) into

$$u(\xi) \approx \widetilde{u}(\xi) = \begin{cases} -\dfrac{v}{a}\tanh\Big[\dfrac{1}{2}\sqrt{\dfrac{v^2}{\beta^2+\delta b}}(\xi-\xi_1)\Big] + \dfrac{v}{a}, & \xi \in (0,+\infty), \\ \mathrm{e}^{\alpha(\xi-\xi_1)}(A_1''\cos(B(\xi-\xi_1)) - A_2''\sin(B(\xi-\xi_1))) + \dfrac{v}{a}, & \xi \in (-\infty, 0], \end{cases} \tag{5.1}$$

where

$$\alpha = -\frac{rb}{2(\beta^2+b\delta)}, \quad B = \sqrt{\frac{-r^2b^2+2v^2(\beta^2+b\delta)}{4(\beta^2+b\delta)^2}}.$$

$$A_1'' = \mathrm{e}^{\alpha\xi_1}\Big(\Big(-\frac{\alpha}{B}\sin(B\xi_1)+\cos(B\xi_1)\Big)\Big(u_-(0)-\frac{v}{a}\Big)+\frac{1}{B}\sin(B\xi_1)u_-'(0)\Big),$$

$$A_2'' = -A_1''\cot(B\xi_1) + \frac{(u_-(0)-\frac{v}{a})\mathrm{e}^{\alpha\xi_1}}{\sin(B\xi_1)}.$$

Since

$$a^2D^3 - 3avD^2 + 2v^2D = 0$$

has three different real roots which are 0, $\frac{v}{a}$ and $\frac{2v}{a}$, (2.7) can be translated into

$$u''(\xi) + \overline{r}u'(\xi) - \overline{b}u(\xi)\Big(u(\xi)-\frac{v}{a}\Big)\Big(u(\xi)-\frac{2v}{a}\Big) = 0, \tag{5.2}$$

where

$$\overline{r} = \frac{rb}{\beta^2+b\delta}, \quad \overline{b} = \frac{a^2}{2(\beta^2+b\delta)}.$$

In order to obtain the error estimate of the approximate damped oscillatory solution corresponding to $L(P_2, P_1)$, we substitute

$$V(\xi) = \frac{au(\xi)-v}{2v} \tag{5.3}$$

and $\xi = \xi_1 - \eta$ ($\eta > \xi_1$) into (5.2). At the same time,

$$\widetilde{V}(\eta) = V(\xi_1 - \eta) = V(\xi). \tag{5.4}$$

Therefore, we transform the exact damped oscillatory solutions to the problem (2.7) which satisfy the initial condition

$$u(\xi_0') = u_-(\xi_0'), \quad u'(\xi_0') = u_-'(\xi_0') \tag{5.5}$$

into the exact solutions to the following initial value problem:

$$\begin{cases} \widetilde{V}_{\eta\eta}(\eta) - \overline{r}\widetilde{V}_\eta(\eta) + \dfrac{\overline{b}v^2}{a^2}\widetilde{V}(\eta) = \dfrac{4\overline{b}v^2}{a^2}\widetilde{V}^3(\eta), \\ \widetilde{V}(\xi_1) = \dfrac{au_-(0)-v}{2v}, \quad \widetilde{V}_\eta(\xi_1) = \dfrac{au_-'(0)}{2v}. \end{cases} \tag{5.6}$$

At first, we use the homogenization theory to study the following initial value problem:

$$\begin{cases} \overline{V}_{\eta\eta}(\eta) - \overline{r}\overline{V}_{\eta}(\eta) + \dfrac{\overline{b}v^2}{a^2}\overline{V}(\eta) = \dfrac{4\overline{b}v^2}{a^2}\widetilde{V}^3(\eta), \\ \overline{V}(\xi_1) = \dfrac{au_-(0) - v}{2v}, \quad \overline{V}_{\eta}(\xi_1) = \dfrac{au'_-(0)}{2v}, \end{cases} \tag{5.7}$$

where $\widetilde{V}(\eta)$ satisfies (5.6). We can permit the establishment of the following two lemmas.

Lemma 5.1 *Let $\overline{V}_1(\eta)$ and $\overline{V}_2(\eta)$ be the solutions to*

$$\begin{cases} \overline{V}_{\eta\eta}(\eta) - \overline{r}\overline{V}_{\eta}(\eta) + \dfrac{\overline{b}v^2}{a^2}\overline{V}(\eta) = 0, \\ \overline{V}(\xi_1) = \dfrac{au_-(0) - v}{2v}, \quad \overline{V}_{\eta}(\xi_1) = \dfrac{au'_-(0)}{2v} \end{cases} \tag{5.8}$$

and

$$\begin{cases} \overline{V}_{\eta\eta}(\eta) - \overline{r}\overline{V}_{\eta}(\eta) + \dfrac{\overline{b}v^2}{a^2}\overline{V}(\eta) = \dfrac{4\overline{b}v^2}{a^2}\widetilde{V}^3(\eta), \\ \overline{V}(\xi_1) = 0, \quad \overline{V}_{\eta}(\xi_1) = 0, \end{cases} \tag{5.9}$$

respectively. Then $\overline{V}_1(\eta) + \overline{V}_2(\eta)$ is the solution to (5.7).

Lemma 5.2 *Let $\overline{V}_3(\eta,\tau)$ be the solution to*

$$\begin{cases} \overline{V}_{\eta\eta}(\eta) - \overline{r}\overline{V}_{\eta}(\eta) + \dfrac{\overline{b}v^2}{a^2}\overline{V}(\eta) = 0, \\ \overline{V}(\tau) = 0, \quad \overline{V}_{\eta}(\tau) = \dfrac{4\overline{b}v^2}{a^2}\widetilde{V}^3(\tau) \end{cases} \quad (\eta > \tau). \tag{5.10}$$

Then $\int_{\xi_1}^{\eta} \overline{V}_3(\eta,\tau)\mathrm{d}\tau$ is the solution to (5.9).

We can obtain the solution to (5.8) as

$$\overline{V}_1(\eta) = \mathrm{e}^{-\overline{\alpha}_1\eta}(c_1\cos(\overline{\beta}_1\eta) + c_2\sin(\overline{\beta}_1\eta)), \tag{5.11}$$

where

$$\begin{aligned} \overline{\alpha}_1 &= -\frac{rb}{2(\beta^2 + b\delta)} = \alpha, \quad \overline{\beta}_1 = \sqrt{\frac{-r^2b^2 + 2v^2(\beta^2 + b\delta)}{4(\beta^2 + b\delta)^2}} = B, \\ c_1 &= \frac{a\mathrm{e}^{\overline{\alpha}_1\xi_1}}{2v}\Big(\Big(-\frac{\overline{\alpha}_1}{\overline{\beta}_1}\sin(\overline{\beta}_1\xi_1) + \cos(\overline{\beta}_1\xi_1)\Big)\Big(u_-(0) - \frac{v}{a}\Big) + \frac{1}{\overline{\beta}_1}\sin(\overline{\beta}_1\xi_1)u'_-(0)\Big), \\ c_2 &= -c_1\cot(\overline{\beta}_1\xi_1) + \frac{a(u_-(0) - \frac{v}{a})\mathrm{e}^{\overline{\alpha}_1\xi_1}}{2v\sin(\overline{\beta}_1\xi_1)}. \end{aligned}$$

To solve $\overline{V}_3(\eta,\tau)$ in the initial value problem (5.10), we let $t = \eta - \tau$. Then we obtain

$$\begin{cases} W_{tt}(t,\tau) - \overline{r}W_t(t,\tau) + \dfrac{\overline{b}v^2}{a^2}W(t,\tau) = 0, \\ W(0,\tau) = 0, \quad W_t(0,\tau) = \dfrac{4\overline{b}v^2}{a^2}\widetilde{V}^3(\tau), \end{cases} \tag{5.12}$$

where $W_t(t,\tau)=\overline{V}_3(\eta,\tau)$, and the solution to the initial value problem (5.12) is

$$W(t,\tau)=c_3\mathrm{e}^{-\overline{\alpha}_1 t}\sin(\overline{\beta}_1 t),$$

where $c_3=-\frac{4\bar{b}v^2}{\overline{\beta}_1 a^2}\widetilde{V}^3(\eta,\tau)$. Therefore, the solution to the initial value problem (5.10) is

$$\overline{V}_3(\eta,\tau)=-\frac{4\bar{b}v^2}{\overline{\beta}_1 a^2}\widetilde{V}^3(\tau)\mathrm{e}^{-\overline{\alpha}_1(\eta-\tau)}\sin(\overline{\beta}_1(\eta-\tau)).$$

Further, the solution to the initial value problem (5.9) is

$$\overline{V}_2(\eta)=\int_{\xi_1}^{\eta}\overline{V}_3(\eta,\tau)\mathrm{d}\tau=-\frac{4\bar{b}v^2}{\overline{\beta}_1 a^2}\int_{\xi_1}^{\eta}\widetilde{V}^3(\tau)\mathrm{e}^{-\overline{\alpha}_1(\eta-\tau)}\sin(\overline{\beta}_1(\eta-\tau))\mathrm{d}\tau.$$

Now, the solution to the initial value problem (5.7) is

$$\overline{V}(\eta)=\overline{V}_1(\eta)+\overline{V}_2(\eta).$$

And because the solution to the initial value problem (5.6) satisfies the initial value problem (5.7), according to the uniqueness theorem of solutions, we can arrive at $\widetilde{V}(\eta)=\overline{V}(\eta)$, namely,

$$\widetilde{V}(\eta)=\mathrm{e}^{-\overline{\alpha}_1\eta}(c_1\cos(\overline{\beta}_1\eta)+c_2\sin(\overline{\beta}_1\eta))-\frac{4\bar{b}v^2}{\overline{\beta}_1 a^2}\int_{\xi_1}^{\eta}\widetilde{V}^3(\tau)\mathrm{e}^{-\overline{\alpha}_1(\eta-\tau)}\sin(\overline{\beta}_1(\eta-\tau))\mathrm{d}\tau. \quad (5.13)$$

At first, we substitute $\eta=\xi_1-\xi$ and (5.3) into (5.13). In the following, we make $\tau'=-\tau$ in (5.13) and the transformed integral variable is still denoted by τ. In this way, we can obtain the inexplicit damped oscillatory solution to (1.4) which satisfies (5.5) such that

$$\begin{aligned} u(\xi)-\frac{v}{a}&=\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}(\bar{c}_1\cos(\overline{\beta}_1(\xi-\xi_1))+\bar{c}_2\sin(\overline{\beta}_1(\xi-\xi_1)))\\ &\quad+\frac{\bar{b}}{\overline{\beta}_1}\int_{-\xi_1}^{\xi-\xi_1}\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1-\tau)}\sin(\overline{\beta}_1(\xi-\xi_1-\tau))\Big(u(\tau)-\frac{v}{a}\Big)^3\mathrm{d}\tau, \end{aligned} \quad (5.14)$$

where

$$\begin{aligned} \bar{c}_1&=\mathrm{e}^{\alpha\xi_1}\Big(\Big(-\frac{\alpha}{B}\sin(B\xi_1)+\cos(B\xi_1)\Big)\Big(u_-(0)-\frac{v}{a}\Big)+\frac{1}{B}\sin(B\xi_1)u_-'(0)\Big),\\ \bar{c}_2&=\bar{c}_1\cot(B\xi_1)-\frac{(u_-(0)-\frac{v}{a})\mathrm{e}^{\alpha\xi_1}}{\sin(B\xi_1)}. \end{aligned}$$

Obviously, $\overline{\beta}_1,\bar{c}_1,\bar{c}_2$ in (5.14) are equal to $B,A_1,-A_2$ in (5.1), respectively.

$$\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}(\bar{c}_1\cos(\overline{\beta}_1(\xi-\xi_1))+\bar{c}_2\sin(\overline{\beta}_1(\xi-\xi_1)))+\frac{v}{a}$$

is the approximate damped oscillatory solution given by the second statement of (5.1). Therefore, (5.14) reflects the relationship between the approximate damped oscillatory solution and the corresponding exact solution as $\xi<0$.

In order to derive the error estimates between the approximate damped oscillatory solution which corresponds to $L(P_2,P_1)$ and the exact solution, we start from (5.13). Since the damped oscillatory solution $u(\xi)$ is a bounded solution, and in addition, $\widetilde{V}(\eta) = V(\xi) = \frac{au(\xi)-v}{2v}$, there exist $M > 0,\ M_1 > 0$, such that $|u(\xi)| < M,\ |\widetilde{V}(\tau)| < M_1$. Furthermore, from (5.13), we have

$$|\widetilde{V}(\eta)| \leq C_1 \mathrm{e}^{-\overline{\alpha}_1 \eta} + \frac{4\bar{b}v^2 T_1}{\overline{\beta}_1 a^2} \mathrm{e}^{-\overline{\alpha}_1 \eta} \int_{\xi_1}^{\eta} \mathrm{e}^{\overline{\alpha}_1 \tau} |\widetilde{V}(\tau)| \mathrm{d}\tau,$$

where $C_1 = |c_1| + |c_2|,\ T_1 = M_1^2$. Since $\overline{\alpha}_1 > 0$, for any η_1 satisfying $0 < \eta_1 \leq \eta$, we have

$$|\widetilde{V}(\eta)| \leq C_1 \mathrm{e}^{-\overline{\alpha}_1 \eta_1} + \frac{4\bar{b}v^2 T_1}{\overline{\beta}_1 a^2} \mathrm{e}^{-\overline{\alpha}_1 \eta_1} \int_{\xi_1}^{\eta} \mathrm{e}^{\overline{\alpha}_1 \tau} |\widetilde{V}(\tau)| \mathrm{d}\tau.$$

By Gronwall inequality, we can obtain

$$|\widetilde{V}(\eta)| \leq C_1 \mathrm{e}^{-\overline{\alpha}_1 \eta_1} \exp\Big(\frac{4\bar{b}v^2 T_1}{\overline{\beta}_1 a^2 \overline{\alpha}_1} \mathrm{e}^{-\overline{\alpha}_1 \eta_1} (\mathrm{e}^{\overline{\alpha}_1 \eta} - \mathrm{e}^{\overline{\alpha}_1 \xi_1}) \Big).$$

Since the above η_1 we have selected is arbitrary and $\eta_1 \leq \eta$, we make $\eta_1 \to \eta$ in the above statement and find

$$|\widetilde{V}(\eta)| \leq C_1 \mathrm{e}^{-\overline{\alpha}_1 \eta} \exp\Big(\frac{4\bar{b}v^2 T_1}{\overline{\beta}_1 a^2 \overline{\alpha}_1} (1 - \mathrm{e}^{\overline{\alpha}_1 (\xi_1 - \eta)}) \Big).$$

Since $\overline{\alpha}_1 > 0,\ \eta > \xi_1,\ 0 < \mathrm{e}^{\overline{\alpha}_1(\xi_1 - \eta)} < 1$. Let $T_2 = C_1 \exp\big(\frac{4\bar{b}v^2 T_1}{\overline{\beta}_1 a^2 \overline{\alpha}_1}\big)$, we obtain

$$|\widetilde{V}(\eta)| \leq T_2 \mathrm{e}^{-\overline{\alpha}_1 \eta}, \quad \eta > \xi_1 > 0. \tag{5.15}$$

On these grounds, we substitute $\eta = \xi_1 - \xi$ and (5.3) into (5.15), and then we will have

$$\Big| u(\xi) - \frac{v}{a} \Big| \leq \frac{2v}{a} T_2 \mathrm{e}^{\overline{\alpha}_1 (\xi - \xi_1)}, \quad \xi < \xi_1. \tag{5.16}$$

(5.16) shows the amplitude estimate of the damped oscillatory solution $u(\xi)$ corresponding to $L(P_2, P_1)$ in Figure 3. From (5.16), we have that $u(\xi)$ tends to $\frac{v}{a}$ as $\xi \to -\infty$.

By (5.15) and (5.13), we have

$$\begin{aligned} &|\widetilde{V}(\eta) - (\mathrm{e}^{-\overline{\alpha}_1 \eta}(c_1 \cos(\overline{\beta}_1 \eta) + c_2 \sin(\overline{\beta}_1 \eta)))| \\ \leq\ & \frac{4\bar{b}v^2}{\overline{\beta}_1 a^2} \int_0^{\eta} (\widetilde{V}(\tau))^2 |\widetilde{V}(\tau)| \mathrm{e}^{-\overline{\alpha}_1 (\eta - \tau)} \mathrm{d}\tau \\ \leq\ & \frac{2\bar{b}v^2 T_2^3}{\overline{\beta}_1 a^2 \overline{\alpha}_1} \mathrm{e}^{-\overline{\alpha}_1 \eta} (1 - \mathrm{e}^{-2\overline{\alpha}_1 \eta}), \quad \eta > 0. \end{aligned} \tag{5.17}$$

Substituting (5.3), (5.4) and $\eta = \xi_1 - \xi$ into (5.17), we will obtain

$$\Big| u(\xi) - \Big(\mathrm{e}^{\overline{\alpha}_1 (\xi - \xi_1)} \Big(\bar{c}_1 \cos(\overline{\beta}_1 (\xi - \xi_1)) + \bar{c}_2 \sin(\overline{\beta}_1 (\xi - \xi_1)) + \frac{v}{a} \Big) \Big) \Big| \leq T_3 \mathrm{e}^{\overline{\alpha}_1 (\xi - \xi_1)}, \quad \xi < \xi_1, \tag{5.18}$$

where

$$T_3 = \frac{2\bar{b}|v|^3 T_2^3}{\overline{\beta}_1 |a|^3 \overline{\alpha}_1} \mathrm{e}^{-\overline{\alpha}_1 \xi_1}.$$

(5.18) indicates the error between the exact damped oscillatory solution corresponding to $L(P_2, P_1)$ and the approximate solution (5.1). Due to

$$\varepsilon_1(\xi) = T_3 \mathrm{e}^{\overline{\alpha}_1 (\xi - \xi_1)} = o(\mathrm{e}^{\overline{\alpha}_1 (\xi - \xi_1)}), \quad \xi \to -\infty,$$

it is meaningful for (4.16) to be an approximate solution of the horizontal velocity field $u(\xi)$ in (1.4) when the conditions in Theorem 4.2 hold. In the same way, (4.18) is meaningful to be the approximate solution of the horizontal velocity field $u(\xi)$ in (1.4) when the conditions in Theorem 4.3 hold.

In the following, we come to study the error estimates between the damped oscillatory solutions and the approximate solutions of $H(\xi)$. Letting

$$\begin{aligned} &\widetilde{u}(\xi) = \mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}(\overline{c}_1\cos(\overline{\beta}_1(\xi-\xi_1)) + \overline{c}_2\sin(\overline{\beta}_1(\xi-\xi_1))) + \frac{v}{a}, \quad \xi\in(-\infty,0], \\ &\widetilde{H}(\xi) = \frac{v}{b}\widetilde{u}(\xi) - \frac{a}{2b}\widetilde{u}^2(\xi) - \frac{\beta}{b}\widetilde{u}'(\xi), \quad \xi\in(-\infty,0], \end{aligned} \tag{5.19}$$

we can arrive at

$$|H(\xi)-\widetilde{H}(\xi)| = \frac{v}{b}(u(\xi)-\widetilde{u}(\xi)) - \frac{a}{2b}(u(\xi)+\widetilde{u}(\xi))(u(\xi)-\widetilde{u}(\xi)) - \frac{\beta}{b}(u'(\xi)-\widetilde{u}'(\xi)). \tag{5.20}$$

Since

$$\begin{aligned} &u'(\xi)-\widetilde{u}'(\xi) \\ &= (u(\xi)-\widetilde{u}(\xi))' \\ &= \Big(\frac{\overline{b}}{\overline{\beta}_1}\int_{-\xi_1}^{\xi-\xi_1} \mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1-\tau)}\sin(\overline{\beta}_1(\xi-\xi_1-\tau))\Big(u(\tau)-\frac{v}{a}\Big)^3\mathrm{d}\tau\Big)' \\ &= \overline{\alpha}_1(u(\xi)-\widetilde{u}(\xi)) - \overline{b}\int_{-\xi_1}^{\xi-\xi_1} \mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1-\tau)}\cos(\overline{\beta}_1(\xi-\xi_1-\tau))\Big(u(\tau)-\frac{v}{a}\Big)^3\mathrm{d}\tau \\ &= \overline{\alpha}_1(u(\xi)-\widetilde{u}(\xi)) + \sigma(\xi), \end{aligned} \tag{5.21}$$

we can obtain

$$\begin{aligned} |\sigma(\xi)| &= |\overline{b}|\Big|\int_{-\xi_1}^{\xi-\xi_1} \mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1-\tau)}\cos(\overline{\beta}_1(\xi-\xi_1-\tau))\Big(u(\tau)-\frac{v}{a}\Big)^3\mathrm{d}\tau\Big| \\ &\le |\overline{b}|\Big(M+\Big|\frac{v}{a}\Big|\Big)\Big|\int_{-\xi_1}^{\xi-\xi_1} \mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1-\tau)}\Big(\frac{2v}{a}T_2\mathrm{e}^{\overline{\alpha}_1\tau}\Big)^2\mathrm{d}\tau\Big| \\ &= |\overline{b}|\Big(M+\Big|\frac{v}{a}\Big|\Big)\Big|\Big(\frac{2v}{a}T_2\Big)^2\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}\int_{-\xi_1}^{\xi-\xi_1}\mathrm{e}^{\overline{\alpha}_1\tau}\mathrm{d}\tau\Big| \\ &= |\overline{b}|\Big(M+\Big|\frac{v}{a}\Big|\Big)\Big(\frac{2v}{a}T_2\Big)^2\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}\Big(\frac{1}{\overline{\alpha}_1}\Big)|\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}-\mathrm{e}^{-\overline{\alpha}_1\xi_1}| \\ &\le T_4\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}, \quad \xi<\xi_1 \end{aligned} \tag{5.22}$$

from (5.16). From (5.19), (5.21) and (5.18), (5.22), we know

$$\begin{aligned} |H(\xi)-\widetilde{H}(\xi)| &= \frac{v}{b}(u(\xi)-\widetilde{u}(\xi)) - \frac{a}{2b}(u(\xi)+\widetilde{u}(\xi))(u(\xi)-\widetilde{u}(\xi)) - \frac{\beta}{b}(u'(\xi)-\widetilde{u}'(\xi)) \\ &\le \Big(\Big|\frac{v}{b}\Big| + \Big|\frac{a}{2b}\Big||u(\xi)+\widetilde{u}(\xi)| + \Big|\frac{\beta}{b}\cdot\overline{\alpha}_1\Big|\Big)|u(\xi)-\widetilde{u}(\xi)| + \Big|\frac{\beta}{b}\Big|\sigma(\xi) \\ &\le \Big(\Big|\frac{v}{b}\Big| + \Big|\frac{a}{2b}\Big|\Big|M+\Big(C_1+\frac{v}{a}\Big)\Big| + \Big|\frac{\beta}{b}\cdot\overline{\alpha}_1\Big|\Big)|u(\xi)-\widetilde{u}(\xi)| + \Big|\frac{\beta}{b}\Big|\sigma(\xi) \\ &\le T_5\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}, \quad \xi<\xi_1. \end{aligned} \tag{5.23}$$

In addition, since

$$\begin{aligned}\widetilde{u}'(\xi) &= \overline{\alpha}_1 \mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}(c_1\cos(\overline{\beta}_1(\xi-\xi_1))+c_2\sin(\overline{\beta}_1(\xi-\xi_1)))\\ &\quad+\overline{\beta}_1\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}(c_1\sin(\overline{\beta}_1(\xi-\xi_1))-c_2\cos(\overline{\beta}_1(\xi-\xi_1)))\\ &\le |\overline{\alpha}_1+\overline{\beta}_1|\cdot C_1\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)},\end{aligned}$$

we can obtain

$$\begin{aligned}&\Big|H(\xi)-\frac{v^2}{2ab}\Big|\\ &=\Big|\frac{v}{b}u(\xi)-\frac{a}{2b}u^2(\xi)-\frac{\beta}{b}u'(\xi)-\frac{v^2}{2ab}\Big|\\ &=\Big|\frac{v}{b}\Big(u(\xi)-\frac{v}{a}\Big)-\frac{a}{2b}\Big(u^2(\xi)-\frac{v^2}{a^2}\Big)-\frac{\beta}{b}(u'(\xi)-0)\Big|\\ &\le\Big|\frac{v}{b}\Big|\Big|u(\xi)-\frac{v}{a}\Big|+\Big|\frac{a}{2b}\Big|\Big|u^2(\xi)-\frac{v^2}{a^2}\Big|+\Big|\frac{\beta}{b}\Big||u'(\xi)-\widetilde{u}'(\xi)|+\Big|\frac{\beta}{b}\Big||\widetilde{u}'(\xi)-0|\\ &\le\Big|\frac{v}{b}\Big|\frac{2v}{a}T_2\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}+\Big|\frac{a}{2b}\Big|\Big(M+\Big|\frac{v}{a}\Big|\Big)\frac{2v}{a}T_2\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}\\ &\quad+\Big|\frac{\beta\overline{b}}{b\overline{\beta}_1}\Big|\Big(M+\Big|\frac{v}{a}\Big|\Big)\Big(\frac{2v}{a}T_2\Big)^2\mathrm{e}^{\overline{\alpha}_1\xi_1}\mathrm{e}^{2\overline{\alpha}_1(\xi-\xi_1)}\\ &\quad+\Big|\frac{\beta}{b}\Big||\overline{b}|\Big(M+\Big|\frac{v}{a}\Big|\Big)\frac{2v}{a}T_2\Big(\frac{1}{\overline{\alpha}_1}\Big)\mathrm{e}^{\overline{\alpha}_1\xi_1}\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}+\Big|\frac{\beta}{b}\Big||\overline{\alpha}_1+\overline{\beta}_1|\cdot C_1\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}\\ &=T_6\mathrm{e}^{\overline{\alpha}_1(\xi-\xi_1)}.\end{aligned} \tag{5.24}$$

The above discussion shows: when $-|\frac{v}{b}|\sqrt{2(\beta^2+\delta b)}<r<0$, for the solution $(u(\xi),H(\xi))$ to (1.4) which corresponds to $L(P_2,P_1)$ shown in Figure 3, if we regard $\widetilde{u}(\xi)$ given in (5.19) as the approximate solution of $u(\xi)$ and regard $\widetilde{H}(\xi)$ as the approximate solution of $H(\xi)$, the error between them decreases exponentially. Thus, in the case of $-|\frac{v}{b}|\sqrt{2(\beta^2+\delta b)}<r<0$, it is meaningful for us to regard $(\widetilde{u}(\xi),\widetilde{H}(\xi))$ as the solution $(u(\xi),H(\xi))$ to (1.4) in the interval of $(-\infty,0]$. At the same time, from (5.16) and (5.24), we can obtain that $u(\xi)$ will decrease to $\frac{v}{a}$ exponentially and $H(\xi)$ will decrease to $\frac{v^2}{2ab}$ exponentially as $\xi\to-\infty$.

We use the same method to illustrate the error between the damped oscillatory solution $(u_2(\xi),H_2(\xi))$ to (1.4) corresponding to $L(P_2,P_3)$ in Figure 3 and the approximate solution $(\widetilde{u}_2(\xi),\widetilde{H}_2(\xi))$ given by Theorem 4.3 is $O(\mathrm{e}^{\overline{\alpha}\xi})$ as $\xi\to-\infty$. Similarly, for the case of $0<r<|\frac{v}{b}|\sqrt{2(\beta^2+\delta b)}$, the errors between the damped oscillatory solutions to (1.4) corresponding to $L(P_1,P_2)$ and $L(P_3,P_2)$ in Figure 5 and their approximate solutions constructed in Section 4 are $O(\mathrm{e}^{\overline{\alpha}\xi})$ as $\xi\to-\infty$.

6 Conclusions

In this paper, the bounded traveling wave solutions to the important shallow wave equation, namely the generalized Whitham-Broer-Kaup equation (1.4) with the dissipation terms, and the problem of its solution are studied.

Since under the conditions of hypotheses (2.3)–(2.5), the relation between $H(\xi)$ and $u(\xi)$ in the traveling wave solutions $(u(\xi), H(\xi))$ $(\xi = x - vt)$ to (1.4) satisfies the formula (2.6), while $u(\xi)$ satisfies (2.7), the number, shape, and solution of the bounded traveling wave solutions to (1.4) depend on the study of the solution $u(\xi)$ to (2.7). In this paper, we employ the theory and method of the planar dynamical system to make comprehensive qualitative analyses of (2.7), and give five global phase portraits, the existent conditions and the number of the traveling wave solutions. And then we obtain the existent conditions and the number of the bounded solutions to (1.4) under the conditions of hypotheses (2.3)–(2.5). By studying the dissipation effect on the behavior of the bounded traveling wave solutions to the equation, we obtain a critical value r^*, and prove that a bounded traveling wave solution to (1.4) appears as a damped oscillatory solution if $|r| < r^*$, while it appears as a kink profile solitary wave solution if $|r| \geq r^*$. The kink profile solitary wave solutions to (1.4) for $r = 0$ and $r = \pm\frac{3v}{2b}\sqrt{\beta^2 + b\delta}$ are presented. Based on the evolution relation of orbits in the global phase portraits, by the undetermined coefficients method, we obtain the approximate damped oscillatory solutions under various conditions. More importantly, we also give the error estimates between the approximate solution $(\widetilde{u}(\xi), \widetilde{H}(\xi))$ and the exact solution $(u(\xi), H(\xi))$. We prove that the error is an infinitesimal decreasing exponentially. The key to the proof is as follows. Firstly, we establish the integral equations (5.12) and (5.14) reflecting the relation between $\widetilde{u}(\xi)$ and $u(\xi)$ by using some transformations and the idea of the homogenization principle. And then we give the error estimate (5.19). Furthermore, we obtain the error estimate (5.23) between $\widetilde{H}(\xi)$ and $H(\xi)$ through the relational expression (2.6).

It should be pointed out that this paper gives a method of finding the approximate damped oscillatory solutions to nonlinear evolution equations with the dissipation effect. Firstly, we make shape analysis of an equation. Secondly, we obtain the solutions with no dissipation effect. Finally, according the evolution relation of orbits in global phase portraits the damped oscillatory solutions responding to, we obtain their approximate damped oscillatory solutions. This method can also be applied to find approximate damped oscillatory solutions to other nonlinear evolution equations, including the uncoupled equation, such as KdV-Burgers equation (see [19–29])

$$u_t + auu_x + ru_{xx} + u_{xxx} = 0, \quad r < 0, \tag{6.1}$$

Burgers-BBM equation (see [30–31])

$$u_t + auu_x + ru_{xx} - u_{xxt} = 0, \quad r < 0, \tag{6.2}$$

the nonlinear wave equation (see [32–33])

$$u_{tt} - ku_{xx} + ru_t + a_1u + a_2u^2 + a_3u^3 = 0, \quad r < 0 \tag{6.3}$$

and so on. By the way, it still interests many scholars to study the solution of nonlinear equations, and then many new methods occur, such as F-expansion method (see [34–36]), the similarity transformation method (see [37–38]), generalized Hirota Ansatz method (see [39–40]), phase portraits survey method (see [41]), appropriate reduction method (see [42]), which are all worthy of acquaintance and serve as valuable references for nonlinear researchers.

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