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Symmetry Reduction and Exact Solutions of a Hyperbolic Monge-Ampère Equation*

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Abstract By means of the classical symmetry method, a hyperbolic Monge-Ampère equation is investigated. The symmetry group is studied and its corresponding group invariant solutions are constructed. Based on the associated vector of the obtained symmetry, the authors construct the group-invariant optimal system of the hyperbolic Monge-Ampère equation, from which two interesting classes of solutions to the hyperbolic Monge-Ampère equation are obtained successfully.

Keywords Symmetry reduction, Monge-Ampère equation, Exact solutions
2000 MR Subject Classification 22E50, 22E60, 35L70

1 Introduction

It is known that, for an unknown function

$$z = z(\theta, \tau)$$

defined for $(\theta, \tau) \in \mathbb{R}^2$, the corresponding Monge-Ampère equation (see [1–3]) reads

$$A + Bz_{\tau\tau} + Cz_{\theta\tau} + Dz_{\theta\theta} + E(z_{\tau\tau}z_{\theta\theta} - z_{\theta\tau}^2) = 0, \tag{1.1}$$

where the coefficients A, B, C, D and E depend on θ, τ, S, S_θ and S_τ. We say that (1.1) is τ-hyperbolic for S, if

$$\triangle^2(\theta, \tau, S, S_\theta, S_\tau) \triangleq C^2 - 4BD + 4AE > 0$$

and

$$z_{\theta\theta} + B(\theta, \tau, S, S_\theta, S_\tau) \neq 0.$$

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Recently, Kong, Liu and Wang [2] completely reduced the one-dimensional hyperbolic mean curvature flow for closed plane curves to an initial value problem for a single partial differential equation for its support function. The reduced equation is a hyperbolic Monge-Ampère equation of the following form:

$$S_{\tau\tau} = \frac{S_{\theta\tau}^2 - 1}{S + S_{\theta\theta}}. \tag{1.2}$$

As we know, symmetry group techniques provide one powerful method for obtaining solutions to partial differential equations (see [4–5]). Many mathematicians and physicists have done excellent work for the symmetry and reduction theory and techniques (see [6–12]). The methods of finding group-invariant solutions and generalizing the well-known techniques for finding similarity solutions, provide a systematic computational method for determining large classes of special solutions. These group-invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations. The more symmetrical the solution is, the easier it is to construct. However, there is almost always an infinite number of the subgroups, and we need an optimal system to classify all possible group-invariant solutions to the system (see [4]). Based on the theory of group invariant solutions, we construct the group-invariant optimal system of the hyperbolic Monge-Ampère equation, from which the interesting exact solutions are obtained.

2 Main Results

We consider the one-parameter group of infinitesimal transformations in (θ, τ, S) given by

$$\begin{aligned} \theta^* &= \theta + \epsilon\xi(\theta, \tau, S) + o(\epsilon^2), \\ \tau^* &= \tau + \epsilon\eta(\theta, \tau, S) + o(\epsilon^2), \\ S^* &= S + \epsilon\Psi(\theta, \tau, S) + o(\epsilon^2), \end{aligned} \tag{2.1}$$

where ϵ is a group parameter. It is required that the set of (1.2) should be invariant under the transformation (2.1), and this yields a system of overdetermined, linear equations for the infinitesimals ξ, η and Ψ. Solving these equations, one can get

$$\xi = c_1, \quad \eta = c_2\theta + c_3\tau + c_4, \quad \Psi = c_3 S + c_5 \sin(\theta) + c_6 \cos(\theta),$$

where c_i $(i = 1, 2, \cdots, 6)$ are arbitrary constants. And the associated vector fields for the one-parameter Lie group of infinitesimal transformations are $v_1, v_2, \cdots, v_6$ given by

$$v_1 = \partial_\theta, \quad v_2 = \partial_\tau, \quad v_3 = \theta\partial_\tau, \quad v_4 = \sin(\theta)\partial_S, \quad v_5 = \cos(\theta)\partial_S, \quad v_6 = \tau\partial_\tau + S\partial_S. \tag{2.2}$$

(2.2) shows that the following transformations (given by $\exp(\epsilon v_i)$, $i = 1, 2, \cdots, 6$) of variables

(θ, τ, S) leave the solutions to (1.2) invariant:

$$\begin{aligned}
\exp(\epsilon v_1) &: (\theta, \tau, S) \mapsto (\theta + \epsilon, \tau, S), \\
\exp(\epsilon v_2) &: (\theta, \tau, S) \mapsto (\theta, \tau + \epsilon, S), \\
\exp(\epsilon v_3) &: (\theta, \tau, S) \mapsto (\theta, \tau + \epsilon\theta, S), \\
\exp(\epsilon v_4) &: (\theta, \tau, S) \mapsto (\theta, \tau, S + \epsilon \sin(\theta)), \\
\exp(\epsilon v_5) &: (\theta, \tau, S) \mapsto (\theta, \tau, S + \epsilon \cos(\theta)), \\
\exp(\epsilon v_6) &: (\theta, \tau, S) \mapsto (\theta, \mathrm{e}^{\epsilon}\tau, \mathrm{e}^{\epsilon} S).
\end{aligned} \tag{2.3}$$

And the following theorem holds.

Theorem 2.1 *If $S = p(\theta, \tau)$ is a solution to* (1.2)*, then so are*

$$\begin{aligned}
S^{(1)} &= p(\theta - \epsilon, \tau), \\
S^{(2)} &= p(\theta, \tau - \epsilon), \\
S^{(3)} &= p(\theta, \tau - \epsilon\theta), \\
S^{(4)} &= p(\theta, \tau) + \epsilon \sin(\theta), \\
S^{(5)} &= p(\theta, \tau) + \epsilon \cos(\theta), \\
S^{(6)} &= \mathrm{e}^{\epsilon} p(\theta, \mathrm{e}^{-\epsilon}\tau).
\end{aligned}$$

By exploiting the generators v_i of the Lie-point transformations in (2.2), one can build up exact solutions to (1.2) via the symmetry reduction approach. This allows one to lower the number of independent variables of the system of differential equations under consideration using the invariants associated with a given subgroup of the symmetry group. In the following, we present some reductions leading to exact solutions to the equation of possible physical interest. In general, to each subgroup of the symmetry group, there will be a corresponding family of group-invariant solutions to the equation. It is very complicated to list all possible group-invariant solutions. So it is necessary to find an "optimal system" of group-invariant solutions. By using the method presented in [4], we will find the optimal system of group-invariant solutions.

Table 1 Lie bracket

Lie	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	v_2	v_5	$-v_4$	0
v_2	0	0	0	0	0	v_2
v_3	$-v_2$	0	0	0	0	v_3
v_4	$-v_5$	0	0	0	0	v_4
v_5	v_4	0	0	0	0	v_5
v_6	0	$-v_2$	$-v_3$	$-v_4$	$-v_5$	0

From (2.2), applying the commutator operators $[v_m, v_n] = v_m v_n - v_n v_m$, we get the commutator table listed in Table 1 with the (i, j)-th entry indicating $[v_m, v_n]$. It follows that the proposition below holds.

Proposition 2.1 *The operators v_i $(i = 1, 2, \cdots, 6)$ form a Lie algebra, which is a six-dimensional symmetry algebra.*

Applying the formula

$$\mathrm{Ad}(\exp(\varepsilon v))v_0 = v_0 - \varepsilon[v, v_0] + \frac{1}{2}\varepsilon^2[v, [v, v_0]] - \cdots$$

and Table 1, one can get the adjoint representation listed in Table 2 with the (i, j)-th entry indicating $\mathrm{Ad}(\exp(\varepsilon v_i))v_j$.

Table 2 Adjoint representation

Ad	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_2	$v_3 - \varepsilon v_2$	$\cos(\varepsilon)v_4 - \sin(\varepsilon)v_5$	$\sin(\varepsilon)v_4 + \cos(\varepsilon)v_5$	v_6
v_2	v_1	v_2	v_3	v_4	v_5	$v_6 - \varepsilon v_2$
v_3	$v_1 + \varepsilon v_2$	v_2	v_3	v_4	v_5	$v_6 - \varepsilon v_3$
v_4	$v_1 + \varepsilon v_5$	v_2	v_3	v_4	v_5	$v_6 - \varepsilon v_4$
v_5	$v_1 - \varepsilon v_4$	v_2	v_3	v_4	v_5	$v_6 - \varepsilon v_5$
v_6	v_1	$\mathrm{e}^{\varepsilon}v_2$	$\mathrm{e}^{\varepsilon}v_3$	$\mathrm{e}^{\varepsilon}v_4$	$\mathrm{e}^{\varepsilon}v_5$	v_6

If we set

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6,$$

our task is to simplify as many of the coefficients a_i as possible through judicious application of adjoint maps to v. Suppose first that

$$a_6 \neq 0.$$

Scaling v if necessary, we can assume that

$$a_6 = 1.$$

Referring to Table 2, if we act on v by

$$\mathrm{Ad}\Big(\exp\Big(\frac{a_4 - a_1a_5}{1 + a_1^2}v_4\Big)\Big) \quad \text{and} \quad \mathrm{Ad}\Big(\exp\Big(\frac{a_5 + a_1a_4}{1 + a_1^2}v_5\Big)\Big),$$

respectively, we can make the coefficients of v_4 and v_5 vanish:

$$\begin{aligned} v^{(1)} &= \mathrm{Ad}\Big(\exp\Big(\frac{a_5 + a_1a_4}{1 + a_1^2}v_5\Big)\Big) \circ \mathrm{Ad}\Big(\exp\Big(\frac{a_4 - a_1a_5}{1 + a_1^2}v_4\Big)\Big)v \\ &= a_1v_1 + a_2v_2 + a_3v_3 + v_6. \end{aligned}$$

Next we act on $v^{(1)}$ by $\mathrm{Ad}(\exp(a_3v_3))$ to cancel the coefficient of v_3, leading to $v^{(2)} = a_1v_1 + (a_2 + a_1a_3)v_2 + v_6$, and finally by $\mathrm{Ad}(\exp((a_2 + a_1a_3)v_2))$ to cancel the coefficient of v_2. So v is equivalent to $v_6 + a_1 v_1$ under the adjoint representation. In other words, every one-dimensional subalgebra generated by v with $a_6 \neq 0$ is equivalent to the subalgebra spanned by $v_6 + a_1 v_1$.

The remaining one-dimensional subalgebra is spanned by vector v with $a_6 = 0$. If $a_1 \neq 0$, we scale to make $a_1 = 1$, and then act on v by $\mathrm{Ad}(\exp(-a_2 v_3))$, $\mathrm{Ad}(\exp(-a_5 v_4))$ and $\mathrm{Ad}(\exp(a_4 v_5))$, respectively, so that v is equivalent to

$$v^{(1)} = v_1 + a_3 v_3.$$

We can further act on $v^{(1)}$ by the $\mathrm{Ad}(\exp(\varepsilon v_6))$:

$$v^{(2)} = \mathrm{Ad}(\exp(\varepsilon v_6))v^{(1)} = v_1 + a_3 \mathrm{e}^{\varepsilon} v_3.$$

Depending on the sign of a_3, we can make the coefficient of v_3 either $+1$, -1 or 0. Thus any one-dimensional subalgebra spanned by v with $a_6 = 0$, $a_1 \neq 0$ is equivalent to the subalgebra spanned by either $v_1 + v_3$, $v_1 - v_3$ or v_1. The remaining cases are spanned by vector v with $a_1 = a_6 = 0$. If $a_3 \neq 0$, we scale to make $a_3 = 1$, and then act on v by $\mathrm{Ad}(\exp(a_2 v_1))$. It follows that v is equivalent to $v_3 + m v_4 + n v_5$ for certain scalers m, n depending on a_4 and a_5. The remaining one-dimensional subalgebra is spanned by vector v with $a_1 = a_3 = a_6 = 0$. If $a_5 \neq 0$, we can scale to make $a_5 = 1$. By acting on v by $\mathrm{Ad}(\exp(-\arctan(a_4)v_1))$, v is equivalent to $v_5 + l v_2$ by scaling the coefficient of v_5, where l depends on a_2 and a_4. The remaining cases,

$$a_1 = a_3 = a_5 = a_6 = 0,$$

are similarly seen to be equivalent either to $v_4 + a_2 v_2$ ($a_4 \neq 0$) by scaling a_4 to 1 or to v_2 ($a_1 = a_3 = a_4 = a_5 = a_6 = 0$).

It is also pointed out that by the discrete symmetry $(\theta, \tau, S) \mapsto (\theta, -\tau, S)$, $v_1 - v_3$ is mapped to $v_1 + v_3$. So the following theorem holds.

Theorem 2.2 *The operators generate an optimal system* $\mathcal{S}$

(a) $v_6 + a_1 v_1,\ a_6 \neq 0;$

(b$_1$) $v_1,\ a_6 = 0,\ a_1 \neq 0;$

(b$_2$) $v_1 + v_3,\ a_6 = 0,\ a_1 \neq 0;$

(c) $v_3 + m v_4 + n v_5,\ a_1 = a_6 = 0,\ a_3 \neq 0;$

(d) $v_5 + l v_2,\ a_1 = a_3 = a_6 = 0,\ a_5 \neq 0;$

(e) $v_4 + a_2 v_2,\ a_1 = a_3 = a_5 = a_6 = 0,\ a_4 \neq 0;$

(f) $v_2,\ a_1 = a_3 = a_4 = a_5 = a_6 = 0,\ a_2 \neq 0.$

Making use of Theorem 2.2, we will discuss the reduction and solutions to (1.2).

For case (a), from $a_1 S_\theta + \tau S_\tau - S = 0$, one can get

$$S = F(X)\tau,$$

where $X = \theta - a_1 \ln(\tau)$ and F is an arbitrary function of X. Then (1.2) is reduced to

$$(a_1 F'' - F')(F' + a_1 F) + 1 = 0.$$

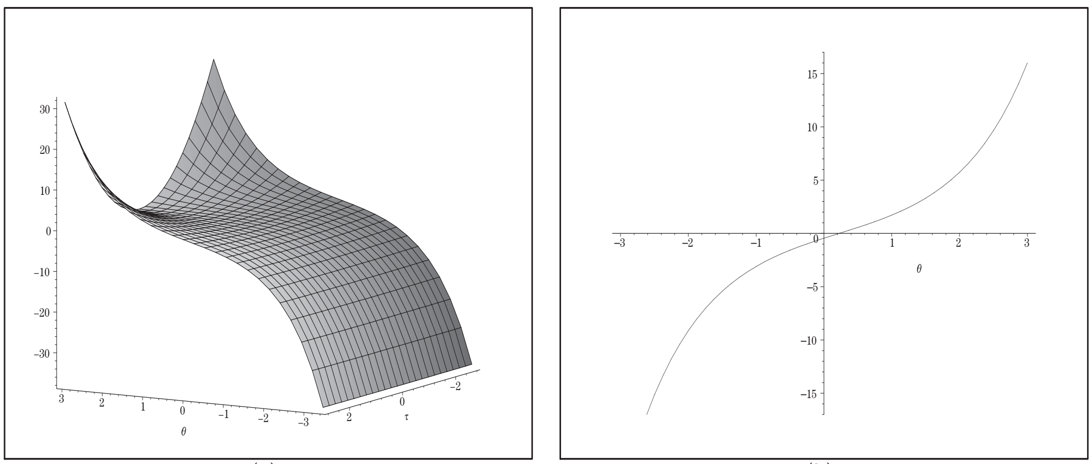
When $a_1^2 = 1$, by solving the above equation, one can obtain

$$F(X) = \sec(x)\sinh(X) + \tan(x)\cosh(X),$$

where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. So we derive a solution to (1.2)

$$S = \frac{\sin(x)+1}{2\cos(x)} \mathrm{e}^{\theta} \tau^{1-a_1} + \frac{\sin(x)-1}{2\cos(x)} \mathrm{e}^{-\theta} \tau^{1+a_1}. \tag{2.4}$$

We depict the shape of S in (2.4) in Figure 1, which shows that it is in a symmetric shape.



(a) (b)

Figure 1 The shape of S with $a_1 = -1$ and $x = -\arcsin(\frac{9}{41})$: (a) 3D-plot, (b) 2D-plot with $\tau = 2$.

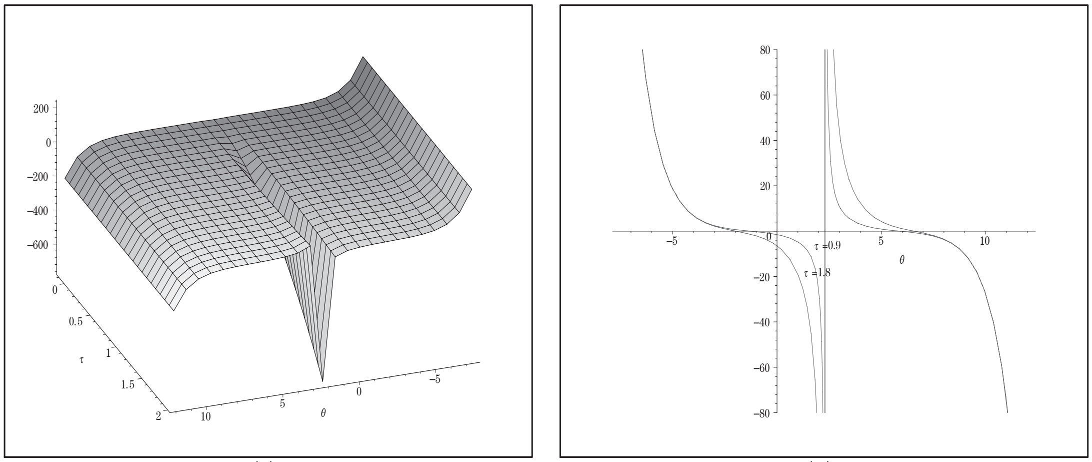
Enlightened by the solution (2.4), we can find the form of the solution to (1.2)

$$S = f(\theta)\tau^2 + g(\theta)\tau + h(\theta). \tag{2.5}$$

Substituting (2.5) into (1.2) and solving the equations composed by the coefficients of τ, we can obtain the solution to (1.2)

$$\begin{aligned} S = &\frac{(2\tau + C_3\theta + C_4)\mathrm{e}^{\theta}\tau}{C_1\mathrm{e}^{2\theta} - C_2} + \frac{(C_3\theta + C_4)^2\mathrm{e}^{\theta}}{8(C_1\mathrm{e}^{2\theta} - C_2)} \\ &- \frac{C_1\mathrm{e}^{2\theta} - C_2}{8\mathrm{e}^{\theta}} + C_5\sin(\theta) + C_6\cos(\theta), \end{aligned} \tag{2.6}$$

where C_i $(i = 1, 2, \cdots, 6)$ are arbitrary constants. We depict the shape of S in (2.6) in Figure 2, which shows that it is not in a symmetric shape.



(c) (d)

Figure 2 The shape of S with $C_1 = 0.01$, $C_2 = 1$, $C_3 = 0.00001$ and $C_4 = C_5 = C_6 = 0$: (a) 3D-plot, (b) 2D-plot with $\tau = 0.9$ and $\tau = 1.8$.

In case (b_2), the solution to $S_\theta + \theta S_\tau = 0$ has the following form:

$$S = F(X),$$

where

$$X = \theta^2 - 2\tau$$

and F is an arbitrary function of X. Substituting it into (1.2) leads to

$$4F''(2F' + F) + 1 = 0,$$

which has the solution

$$\int^{F(X)} \frac{1}{\mathrm{RootOf}(2y + 4Z + \mathrm{e}^{-2Z^2} 2^{\frac{1}{2}} \pi^{\frac{1}{2}} \mathrm{erf}(2^{\frac{1}{2}} Z\mathrm{i})\mathrm{i} - 2C_1 \mathrm{e}^{-2Z^2})} \mathrm{d}y - X - C_2 = 0,$$

where C_i $(i = 1, 2)$ are arbitrary constants. For the other cases in $\mathcal{S}$, you can also use them to reduce (1.2) and get the solutions. Here we do not study any more.

Acting $\exp(\epsilon v_i)$ $(i = 1, 2, \cdots, 6)$ in (2.3) on S in (2.6), respectively, one can obtain the following solution to (1.2) by Theorem 2.1

$$\begin{aligned} S = \mathrm{e}^{\epsilon_6} \Big(& \frac{\sin(x) + 1}{2\cos(x)} \mathrm{e}^{\theta - \epsilon_1} (\mathrm{e}^{-\epsilon_6}\tau - \epsilon_3\theta - \epsilon_2)^{1 - a_1} \\ & + \frac{\sin(x) - 1}{2\cos(x)} \mathrm{e}^{-\theta + \epsilon_1} (\mathrm{e}^{-\epsilon_6}\tau - \epsilon_3\theta - \epsilon_2)^{1 + a_1} + \epsilon_4 \sin(\theta) + \epsilon_5 \cos(\theta) \Big). \end{aligned}$$

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