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Nash and Stackelberg Differential Games*

Alain BENSOUSSAN[1] Jens FREHSE[2] Jens VOGELGESANG[3]

(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract A large class of stochastic differential games for several players is considered in this paper. The class includes Nash differential games as well as Stackelberg differential games. A mix is possible. The existence of feedback strategies under general conditions is proved. The limitations concern the functionals in which the state and the controls appear separately. This is also true for the state equations. The controls appear in a quadratic form for the payoff and linearly in the state equation. The most serious restriction is the dimension of the state equation, which cannot exceed 2. The reason comes from PDE (partial differential equations) techniques used in studying the system of Bellman equations obtained by Dynamic Programming arguments. In the authors' previous work in 2002, there is not such a restriction, but there are serious restrictions on the structure of the Hamiltonians, which are violated in the applications dealt with in this article.

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1 Introduction

The regularity theory of systems of PDE is a remarkable technique to solve stochastic differential games. These systems arise normally in a similar way as Bellman equation is derived for stochastic control. There are some restrictions. The first one is that the control policies of all players are defined by feedbacks on the state. For a single player, one can show that such policies are optimal against any other. In stochastic control, the value function solution to Bellman equation is the infimum of the cost objective, so uniqueness of the solution to the PDE can be obtained under reasonable assumptions. In the case of games, the value function cannot be defined in an intrinsic way. It is in general a saddle point, and depends on the feedbacks of all players. So we cannot hope for uniqueness of solutions to the system of PDE. This is an interpretation of the fact that no uniqueness result can be obtained from PDE techniques.

The regularity theory, which is essential to constructing optimal feedbacks for each player and to giving a meaning to the controlled state equation, is not available as easily as in the case of a scalar equation. In our problem, the controls act on the drift and not on the diffusion

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[1]International Center for Decision and Risk Analysis, School of Management, University of Texas-Dallas, 800 West Camptell Rd, SM30, Richardson, TX 75050-3021, USA; Faculty of Business, The Hong Kong Polytechnic University, Hong Kong, China. E-mail: axb046100@dallas.edu
[2]Department of Mathematics, Institute of Applied Mathematics, University of Bonn, Germany. E-mail: mathfrehse@googlemail.com
[3]PACM, Princeton University, USA; HCM, University of Bonn, Germany. E-mail: jens@vogelgesang.ch
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term. So we have quasi-linear differential operators, with nonlinearity concentrated on the gradient. The nonlinear part is called the Hamiltonian. In previous publications, in particular in the book mentioned above, we have made assumptions on the structure of the various Hamiltonians, which lead to useful regularity results. After having noted that we can combine the equations with linear combination, we fix the set of equations with an order which is important. Indeed the restriction concerns a smallness condition, which states that the dependence of a Hamiltonian with respect to functions of order higher than that of its equation is small. In applications, it is often nondependent. We have many useful examples in which this restriction is satisfied. However, they all concern Nash equilibrium. In this article, following a previous publication (see [2]), we consider structures of the Hamiltonian called non-market interaction for which the smallness condition is not satisfied. We also consider Stackelberg games for which it is not available either. So the techniques of this paper do not require any smallness condition. However, they remain so far limited to dimension two only. Higher dimension require more sophisticated techniques, which are the objective of our current research. To the best of our knowledge, there is no result in the literature of the existence of optimal feedbacks for Stackelberg differential games. Another limitation, which we hope to waive in the near future, is that we assume Neumann boundary conditions. This means that the evolution of the state is modelled by a diffusion reflected at the boundary of a domain. Dirichlet boundary conditions (diffusion stopped at the exit of the domain) yield technical difficulties.

2 Formulation of the Problem

2.1 Differential games

We formulate our differential game without the restriction of dimension. We will explain where this restriction is used. There are N players and the space state is R^n $(n = 1, 2)$. Consider a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ on which a standard Wiener process $w(t)$ in R^n is defined. Let $\mathcal{O}$ be a smooth bounded domain of R^n in which the state will remain. Its evolution is defined by a diffusion reflected at the boundary of the domain. To simplify the notation, we will write the equation of this reflected diffusion as if it were just an ordinary diffusion in R^n. So we write it as follows:

$$\mathrm{d}y = \Big(g(y(t)) \ + \sum_{\nu=1}^{N} A_\nu(y(t))v_\nu(t)\Big)\mathrm{d}t + \sqrt{2}\,\mathrm{d}w(t), \quad y(0) = x. \tag{2.1}$$

The stochastic processes $v_\nu(t)$ are the controls decided by the players. They will be defined following the stucture of the game. For Nash differential games, they are simply defined by a feedback on the state. For Stackelberg differential games, they are more involved and will be discussed later. A feedback is a function $v_\nu(x)$ and one has

$$v_\nu(t) = v_\nu(y(t)).$$

Note that for stochastic differential equations measurability of the feedback functions is sufficient. The solution $y(t)$ to equation (2.1) can then be defined in a weak sense. The diffusion term is very simple. We can put a volatility term $\sigma(y)$ (Lipschitz) in front of the Wiener process, provided that it does not depend on the controls, the matrix $\sigma(x)$ is bounded, invertible and its inverse is also bounded. The coefficient $\sqrt{2}$ is for convenience. In the drift term, the important property is the linearity with respect to the controls. Since in fact the reflected diffusion lies in a bounded domain, we can assume with limited restriction that $g(x)$ and $A_\nu(x)$ are bounded in $\mathcal{O}$. The coefficients $A_\nu(x)$ are matrices. As we have said, we do not write the reflected part. For full details we refer to [3]. The feedbacks $v_\nu(x)$ are not bounded functions, but they are bounded on $\mathcal{O}$. The initial condition $x \in \mathcal{O}$. We also denote by $v = (v_1, \cdots, v_N)$ the vector of all controls, with the corresponding notation $v(t)$ and $v(y)$ for the feedbacks. We also use the notation $v(\cdot)$ to emphasize the function. Each player, $\nu = 1, \cdots, N$, has a functional

$$J_\nu(x, v(\cdot)) = E\Big[\int_0^\infty \mathrm{e}^{-\lambda t}(f_\nu(y(t)) + l_\nu(y(t), v(t)))\mathrm{d}t\Big]. \tag{2.2}$$

The part $l_\nu(y(t), v(t))$ incorporates the direct interaction of the players since all components of $v(t)$ are involved. The interaction through the state $y(t)$ is indirect. This indirect interaction is called market interaction. The direct interaction is called non-market interaction. The structure of the function $l_\nu(x, v)$ is very important and will be made precise later. The function $f_\nu(x)$ is measurable and bounded on $\mathcal{O}$.

2.2 Structure of controls

In a Nash differential game, as said before, each player chooses a feedback $v_\nu(x)$ and we look for a saddle point, namely feedbacks $\widehat{v}_\nu(x)$ such that

$$J_\nu(x, \widehat{v}_1(\cdot), \cdots, \widehat{v}_N(\cdot)) \le J_\nu(x, \widehat{v}_1(\cdot), \cdots, \widehat{v}_{i-1}(\cdot), v_i(\cdot), \widehat{v}_{i+1}(\cdot), \cdots, \widehat{v}_N(\cdot)). \tag{2.3}$$

For a Nash game, the players are on equal footing. For Stackelberg games, there is a hierarchy. Player 1 is the leader, and he chooses a feedback $v_1(x)$. The second player chooses a feedback depending on the state and also on the decision of the leader. He is the first follower. So his feedback is of the form $v_2(x; v_1)$ and the actual feedback will be $v_2(x; v_1(x))$. The third player chooses a feedback $v_3(x; v_1, v_2)$ and the actual feedback is $v_3(x; v_1(x), v_2(x; v_1(x)))$.

The induction is clear although it is heavy. The optimization is done sequentially. Consider simplifying notation of the case $N = 3$. Optimal feedback rules are defined as follows. We begin with the 3rd player. He minimizes

$$J_3(x, v_1(\cdot), v_2(\cdot\,; v_1(\cdot)), v_3(\cdot\,, v_1(\cdot), v_2(\cdot\,, v_1(\cdot))))$$

over $v_3(\cdot\,, v_1(\cdot), v_2(\cdot\,, v_1(\cdot)))$ and defines in this way $\widehat{v}_3(\cdot\,, v_1(\cdot), v_2(\cdot\,, v_1(\cdot)))$. The second player uses this optimal feedback in his own objective function. So his objective function becomes

$$J_2(x, v_1(\cdot), v_2(\cdot\,; v_1(\cdot)), \widehat{v}_3(\cdot\,, v_1(\cdot), v_2(\cdot\,, v_1(\cdot)))),$$

which he minimizes in $v_2(\cdot\,; v_1(\cdot))$. In this way, he defines his optimal feedback $\widehat{v}_2(\cdot\,; v_1(\cdot))$. The objective function of the first player becomes

$$J_1(x, v_1(\cdot), \widehat{v}_2(\cdot\,; v_1(\cdot)), \widehat{v}_3(\cdot\,, v_1(\cdot), \widehat{v}_2(\cdot\,, v_1(\cdot)))),$$

which he minimizes to get the optimal feedback $\widehat{v}_1(\cdot)$.

3 Systems of Partial Differential Equations

3.1 Lagrangians and Hamiltonians

Let $p_\nu \in \mathbb{R}^n$. The Lagrangians are defined by the formulas

$$L_\nu(x, p_\nu, v) = f_\nu(x) + l_\nu(x, v) + p_\nu\Big(g(x) + \sum_{\mu=1}^N A_\mu(x) v_\mu\Big). \tag{3.1}$$

The Hamiltonians are defined by replacing the argument v by a function $\widehat{V}(x,p)$, where

$$p = (p_1, \cdots, p_N).$$

This function will depend on the structure of the game. The Hamiltonians are next defined by the formulas

$$H_\nu(x,p) = L_\nu(x, p_\nu, \widehat{V}(x,p)). \tag{3.2}$$

3.2 Bellman system

With the Hamiltonians constructed by (3.2), we define the Bellman system by

$$-\Delta u_\nu + \lambda u_\nu = H_\nu(x, Du), \quad x \in \mathcal{O} \tag{3.3}$$

with Neumann boundary conditions. Clearly, u stands for the vector $u_1, \cdots, u_N$ and

$$Du = Du_1, \cdots, Du_N.$$

If we consider the feedback vector

$$\widehat{v}(x) = \widehat{V}(x, Du(x)),$$

then the fundamental property which is desired is

$$u_\nu(x) = J(x; \widehat{v}(\cdot)). \tag{3.4}$$

This formula connects the value function of player ν with the solution to the νth equation (3.3). When the solution is sufficiently smooth, like $W^{2,q}(\mathcal{O})$ for q sufficiently large, then the proof is relatively easy and follows the so-called verification property. Regularity is used first to solve the state equation in a convenient way, and then to apply Itô's formula to the process $u_\nu(y(t))$. This is only possible for smooth u_ν. We will not reproduce the argument which is standard. However, this stresses the importance of regularity theory for (3.3). Unlike the situation of scalar equations, in which regularity can be waived by techniques like viscosity solutions, this does not carry over for systems.

4 Main Result

4.1 Statement of the result

In this section, we make assumptions on the Hamiltonians $H_\nu(x,p)$ which will check later on with the Lagrangians. We assume that

$$\begin{gathered} -K|p|^2 - K \le H_\nu(x,p) \le p_\nu \cdot F_\nu(x,p) + K, \\ |F_\nu(x,p)| \le K|B(x,p)| + K, \end{gathered} \tag{4.1}$$

where K is a generic positive constant and $B(x,p)$ is a vector in some Euclidean space satisfying

$$|B(x,p)| \le K(|p| + 1). \tag{4.2}$$

Also there exists a vector in $\mathbb{R}^n, G(x,p)$, such that

$$\sum_\nu H_\nu(x,p) \ge \alpha |B(x,p)|^2 + G(x,p) \cdot \sum_\nu p_\nu - K \tag{4.3}$$

with

$$|G(x,p)| \le K(|B(x,p)|+1). \tag{4.4}$$

Note that since only the norm $|B(x,p)|$ plays a role, the space of the vector $B(x,p)$ is indifferent. From the previous inequalities, we will deduce an important property. We first have, from the 2nd and 3rd inequalities in (4.1),

$$H_\nu(x,p) \le \epsilon |p_\nu|^2 + C_\epsilon(|B(x,p)|^2+1),$$

where ϵ can be taken arbitrarily small and C_ϵ can be large. On the other hand,

$$\begin{aligned} H_\nu(x,p) &= \sum_\mu H_\mu(x,p) - \sum_{\mu\neq\nu} H_\mu(x,p) \\ &\ge \alpha|B(x,p)|^2 - K(|B(x,p)|+1)\Big|\sum_\nu p_\nu\Big| - K - \epsilon|p|^2 - C_\epsilon(|B(x,p)|^2+1) \\ &\ge -\epsilon|p|^2 - C_\epsilon(|B(x,p)|^2+1) - C\Big|\sum_\nu p_\nu\Big|^2. \end{aligned}$$

Collecting results, we can assert that

$$|H_\nu(x,p)| \le \epsilon|p|^2 + C_\epsilon(|B(x,p)|^2+1) + C\Big|\sum_\nu p_\nu\Big|^2. \tag{4.5}$$

In this inequality, the constant C does not depend on ϵ.

As we shall see, these assumptions are satisfied for a large class of Hamiltonians derived from differential games.

We state the following theorem.

Theorem 4.1 *We assume* (4.1), (4.3)–(4.4) *and* $n=2$. *Then there exists a solution to* (3.3) *which belongs to* $W^{2,q}(\mathcal{O})$, $\forall q>0$.

4.2 Verification of assumptions

We consider Hamiltonians defined by Lagrangians, as in (3.1)–(3.2). We assume

$$\sum_\nu l_\nu(x,v) \ge \alpha|v|^2 - K, \quad \alpha>0, \tag{4.6}$$

$$|l_\nu(x,v)| \le K|v|^2 + K. \tag{4.7}$$

We also assume that the feedback strategies $\widehat{V}(x,p)$ satisfy

$$L_\nu(x,p_\nu,\widehat{V}(x,p)) \le p_\nu \cdot F_\nu(x,p) + K, \tag{4.8}$$

$$|\widehat{V}(x,p)| \le K|p| + K, \tag{4.9}$$

$$|F_\nu(x,p)| \le K|p| + K. \tag{4.10}$$

Then the assumptions (4.1) and (4.3)–(4.4) are satisfied with

$$B(x,p) = \widehat{V}(x,p), \quad G(x,p) = \sum_\mu A_\mu(x)\widehat{V}_\mu(x,p) + g(x).$$

5 Proof of the Main Result

5.1 Approximation

We start with an approximation scheme for (3.3). We define

$$H_\nu^\delta(x,p) = \frac{H_\nu(x,p)}{1+\delta|p|^2}.$$

These approximate Hamiltonians are bounded. So we can solve

$$-\Delta u_\nu^\delta + \lambda u_\nu^\delta = H_\nu^\delta(x, Du^\delta), \quad x \in \mathcal{O} \tag{5.1}$$

with Neumann boundary conditions. The solution is smooth. We begin with standard maximum principle arguments. Consider a point of maximum of u_ν^δ, written as x^*. This point is interior because of the Neumann boundary condition. From the second estimate (4.1), by using $Du_\nu^\delta(x^*) = 0$, we deduce immediately $\lambda u_\nu^\delta(x^*) \leq K$. Hence

$$\lambda u_\nu^\delta(x) \leq K, \quad \forall x. \tag{5.2}$$

Using (4.3), we next have

$$-\Delta \sum_\nu u_\nu^\delta + \lambda \sum_\nu u_\nu^\delta \geq \frac{G(x, Du^\delta) \cdot \sum_\nu Du_\nu^\delta}{1+\delta|Du^\delta|^2} - K.$$

If we consider a point of minimum of $\sum_\nu u_\nu^\delta$, it is also an interior point, and hence $\sum_\nu Du_\nu^\delta(x^*) = 0$. Therefore, $\lambda \sum_\nu u_\nu^\delta(x^*) \geq -K$. Hence,

$$\lambda \sum_\nu u_\nu^\delta(x) \geq -K, \quad \forall x. \tag{5.3}$$

Combining (5.2) and (5.3), we obtain easily the estimate

$$\sup_x |u_\nu^\delta(x)| \leq K. \tag{5.4}$$

5.2 Estimates in Sobolev spaces

Let us use the following notation, by dropping the index δ to simplify notation,

$$\phi(x) = \frac{1}{1+\delta|Du^\delta(x)|^2} \quad \text{and} \quad w(x) = \sum_\nu u_\nu^\delta(x).$$

We consider a function $\Phi_0(x) \geq 0$ such that

$$\int_{\mathcal{O}} \frac{|D\Phi_0|^2}{\Phi_0} \mathrm{d}x + \int_{\mathcal{O}} |\Phi_0|^2 \mathrm{d}x = C(\Phi_0) < \infty. \tag{5.5}$$

We have

$$-\Delta w + \lambda w = \varphi \sum_\nu H_\nu(x, Du).$$

We test this equation with $\Phi_0 \exp(-\sigma w)$. Note that $\exp(-\sigma w)$ is bounded. We obtain

$$\begin{aligned}
&-\sigma\int \exp(-\sigma w)|Dw|^2\Phi_0\mathrm{d}x + \int \exp(-\sigma w)\ Dw\cdot D\Phi_0\mathrm{d}x + \lambda\int w\Phi_0\exp(-\sigma w)\ \mathrm{d}x\\
&\geq \int \varphi\Phi_0\exp(-\sigma w)[\alpha|B(Du)|^2 + G(Du)\cdot Dw - K]\mathrm{d}x\\
&\geq \int \varphi\Phi_0\exp(-\sigma w)[\alpha|B(Du)|^2 - K|B(Du)||Dw| - K|Dw| - K]\mathrm{d}x.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\alpha\int \varphi\Phi_0\exp(-\sigma w)|B(Du)|^2\mathrm{d}x + \sigma\int \exp(-\sigma w)\ |Dw|^2\Phi_0\mathrm{d}x\\
&\leq K\int \varphi\Phi_0\exp(-\sigma w)|B(Du)||Dw|\mathrm{d}x + K\int \varphi\Phi_0\exp(-\sigma w)|Dw|\mathrm{d}x\\
&\quad + \int (K\varphi + \lambda w)\Phi_0\exp(-\sigma w)\mathrm{d}x + \int \exp(-\sigma w)\ Dw\cdot D\Phi_0\mathrm{d}x
\end{aligned}$$

By taking σ sufficiently large and independent of Φ_0, we deduce

$$\begin{aligned}
&\int \varphi\Phi_0\exp(-\sigma w)|B(Du)|^2\mathrm{d}x + \int \exp(-\sigma w)\ |Dw|^2\Phi_0\mathrm{d}x\\
&\leq K\int \Phi_0\mathrm{d}x + K\left|\int \exp(-\sigma w)\ Dw\cdot D\Phi_0\mathrm{d}x\right|,
\end{aligned}$$

where the constant K depends on the L^∞ norm of w, but not on Φ_0. It follows, using the fact that $\exp(-\sigma w)$ is bounded below and above

$$\int \varphi\Phi_0|B(Du)|^2\mathrm{d}x + \int\ |Dw|^2\Phi_0\mathrm{d}x \leq c_0C(\Phi_0) + c_1. \tag{5.6}$$

We next proceed by testing the νth equation with $u_\nu\Phi_0$. We obtain

$$\int |Du_\nu|^2\Phi_0\mathrm{d}x + \int u_\nu Du_\nu\cdot D\Phi_0\mathrm{d}x = \int \varphi H_\nu(x, Du)u_\nu\Phi_0\mathrm{d}x. \tag{5.7}$$

Summing up, we get

$$\int |Du|^2\Phi_0\mathrm{d}x + \sum_\nu\int u_\nu Du_\nu\cdot D\Phi_0\mathrm{d}x = \sum_\nu\int \varphi H_\nu(x, Du)u_\nu\Phi_0\mathrm{d}x.$$

Using (4.5), we obtain

$$\begin{aligned}
\frac{1}{2}\int |Du|^2\Phi_0\mathrm{d}x \leq C\int \frac{|D\Phi_0|^2}{\Phi_0}\mathrm{d}x + \epsilon\int |Du|^2\Phi_0\mathrm{d}x + C_\epsilon\int \varphi\Phi_0|B(Du)|^2\mathrm{d}x\\
+ C_\epsilon\int \Phi_0\mathrm{d}x + C\int |Dw|^2\Phi_0\mathrm{d}x.
\end{aligned}$$

Choosing $\epsilon < \frac{1}{2}$, we derive from (5.6) the inequality

$$\int |Du|^2\Phi_0\mathrm{d}x \leq c_0C(\Phi_0) + c_1. \tag{5.8}$$

We can apply this estimate first with $\Phi_0 = 1$ and obtain

$$\int |Du|^2 \mathrm{d}x \leq C. \tag{5.9}$$

A less trivial and important estimate is obtained as follows. Let $\tau(\rho)$ be a C^1 function on $\mathbb{R}^+$, such that

$$0 \leq \tau(\rho) \leq 1, \quad \tau(\rho) = 0 \quad \text{if } \rho \geq \frac{1}{2}, \quad \frac{\tau'^2(\rho)}{\tau(\rho)} \leq C. \tag{5.10}$$

For instance,

$$\tau(\rho) = \Big(\frac{1}{2} - \rho\Big)^{+2}$$

satisfies the conditions. Then we take any point $x_0 \in \mathcal{O}$ and set

$$\Phi_0(x) = \tau(|x - x_0|)|\log|x - x_0||^k, \quad 0 < k < 1.$$

From the condition on τ, we can restrict x so that $|x - x_0| < \frac{1}{2}$. So the only singularity is when $x = x_0$. It is easy to check that condition (5.5) is satisfied. Therefore, we get

$$\int |Du|^2 \tau(|x - x_0|)|\log|x - x_0||^k \mathrm{d}x \leq C. \tag{5.11}$$

Note that the constant does not depend on x_0.

5.3 Small Dirichlet growth and Cacciopoli inequality

We take $R \leq R_0 < \frac{1}{2}$. Define the function

$$\tau_R(\rho) = \begin{cases} 1, & \text{if } 0 \leq \rho \leq R, \\ (a\rho + b)\Big(\frac{1}{2} - \rho\Big)^{+2}, & \text{if } \rho \geq R. \end{cases} \tag{5.12}$$

We adjust the constants a, b, so that

$$\tau_R(R) = 1, \quad \tau_R'(R) = 0.$$

We get

$$a = \frac{2}{\big(\frac{1}{2} - R\big)^3}, \quad aR + b = \frac{1}{\big(\frac{1}{2} - R\big)^2}.$$

We have, for $R \leq \rho$,

$$\tau_R'(\rho) = -3a\Big(\frac{1}{2} - \rho\Big)^+ (\rho - R) \leq 0,$$

and thus $0 \leq \tau_R(\rho) \leq 1$. Moreover, for $R \leq \rho \leq \frac{1}{2}$, one has

$$\frac{(\tau_R'(\rho))^2}{\tau_R(\rho)} = 18 \frac{(\rho - R)^2}{\big(\frac{1}{2} - R\big)^3 \big(\rho - R + \frac{1}{2}\big(\frac{1}{2} - R\big)\big)} \leq \frac{36}{\big(\frac{1}{2} - R\big)^2}.$$

If we restrict $R \leq R_0 < \frac{1}{2}$, we get

$$\frac{(\tau_R'(\rho))^2}{\tau_R(\rho)} \leq \frac{36}{\big(\frac{1}{2} - R_0\big)^2}.$$

We can use the function $\tau_R(|x-x_0|)$ in (5.11), and get a constant independent of R and x_0. In particular, we obtain

$$\int_{\mathcal{O}\cap B_R(x_0)} |Du|^2 |\log|x-x_0||^k \mathrm{d}x \le C, \tag{5.13}$$

where $B_R(x_0)$ represents the ball of radius R and centered in x_0. We recall that $k<1$, and $R \le R_0 < \frac{1}{2}$. The constant C depends on R_0, but not on R or x_0. Since

$$|x-x_0| \le R \le 1,$$

we have

$$|\log|x-x_0|| \ge |\log R|.$$

Therefore,

$$\int_{\mathcal{O}\cap B_R(x_0)} |Du|^2 \mathrm{d}x \le \frac{C}{|\log R|^k}.$$

We deduce that

$$\int_{\mathcal{O}\cap B_R(x_0)} |Du|^2 \mathrm{d}x \le \epsilon, \quad \text{if } R \le R(\epsilon) \tag{5.14}$$

with $R(\epsilon) = \exp\big(-\big(\frac{C}{\epsilon}\big)^{\frac{1}{k}}\big)$. We call this property the small Dirichlet growth. We can, without loss of generality, assume that the condition $R(\epsilon) \le R_0$ is satisfied.

We next state the Cacciopoli inequality. Let τ be in $L^\infty \cap H^1(\mathcal{O})$.

We can test the νth equation with $(u_\nu - c_\nu)\tau^2$, where c_ν are constants to be defined later. Using simply the quadratic growth condition of the Hamiltonian H_ν and standard inequalities, we obtain easily

$$\int |Du|^2\tau^2 \mathrm{d}x \le C_0 \int |u-c|^2 |D\tau|^2 \mathrm{d}x + K\int |u-c||Du|^2\tau^2\mathrm{d}x + K\int \tau^2 \mathrm{d}x, \tag{5.15}$$

where C_0, K are constants. This is Cacciopoli inequality.

5.4 Use of Poincaré inequality

We will derive from Cacciopoli inequality the following inequality:

$$\begin{aligned}\int_{B_{\frac{R}{2}}(x_0)\cap\mathcal{O}} |Du|^2\mathrm{d}x \le \frac{C_0}{1+C_0} &\int_{B_{2(m+1)R}(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x \\ &+ K\mathrm{osc}[u](B_{2(m+1)R}(x_0)\cap\mathcal{O}) \int_{B_{2(m+1)R}(x_0)\cap\mathcal{O}} |Du|^2\mathrm{d}x \\ &+ K_m R^n. \end{aligned}\tag{5.16}$$

In this relation, x_0 is any point of $\mathcal{O}$, R is arbitrary, and m is a fixed integer depending only on the domain and the dimension n (the result does not require $n=2$). Moreover,

$$\mathrm{osc}[u](B_{2(m+1)R}(x_0)\cap\mathcal{O}) = \sqrt{\sum_\nu \Big(\sup_{B_{2(m+1)R}(x_0)\cap\mathcal{O}} u_\nu - \inf_{B_{2(m+1)R}(x_0)\cap\mathcal{O}} u_\nu\Big)^2}.$$

The constant K_m depends on m. The constants K and C_0 do not.

The easy case is when $B_R(x_0) \subset \mathcal{O}$. We apply (5.15) with

$$\tau(x) = \varpi_R(|x - x_0|)$$

and

$$\varpi_R(\rho) = \begin{cases} 1, & \text{if } \rho \leq \dfrac{R}{2}, \\ 0, & \text{if } \rho \geq R \end{cases}$$

with ϖ_R continuously differentiable, $|\varpi_R'(\rho)| \leq \frac{C}{R}$. We obtain

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 \mathrm{d}x \leq C \int_{B_R(x_0) - B_{\frac{R}{2}}(x_0)} \frac{|u-c|^2}{R^2} \mathrm{d}x + K \int_{B_R(x_0)} |u - c||Du|^2 \mathrm{d}x + KR^n.$$

We then take

$$c = \frac{1}{|B_R(x_0) - B_{\frac{R}{2}}(x_0)|} \int_{B_R(x_0) - B_{\frac{R}{2}}(x_0)} u \mathrm{d}x.$$

We can then use Poincaré inequality to obtain

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 \mathrm{d}x \leq C \int_{B_R(x_0) - B_{\frac{R}{2}}(x_0)} |Du|^2 \mathrm{d}x + K \int_{B_R(x_0)} |u - c||Du|^2 \mathrm{d}x + KR^n.$$

Noting that

$$\int_{B_R(x_0)} |u - c||Du|^2 \mathrm{d}x \leq \operatorname{osc}[u](B_R(x_0)) \int_{B_R(x_0)} |Du|^2 \mathrm{d}x$$

and filling the hole, we obtain (5.16) with $m = 0$.

When $B_R(x_0) \cap \mathcal{O}^c \neq \emptyset$, the situation is more delicate and we need bigger balls. Let $\widetilde{x}_0 \in B_R(x_0) \cap \Gamma$, where $\Gamma = \partial\mathcal{O}$. We will use the sphere property (which holds for smooth domains)

$$|B_{2mR}(\widetilde{x}_0) \cap \mathcal{O}| \geq c_0(2mR)^n, \tag{5.17}$$

where c_0 is a fixed constant, not dependent on the point $\widetilde{x}_0$ or the radius of the ball $2mR$. Therefore, if we consider the domain

$$A_R = B_{2mR}(\widetilde{x}_0) - B_{2R}(\widetilde{x}_0),$$

then

$$A_R \cap \mathcal{O} = B_{2mR}(\widetilde{x}_0) \cap \mathcal{O} - B_{2R}(\widetilde{x}_0) \cap \mathcal{O}$$

and

$$|A_R \cap \mathcal{O}| \geq |B_{2mR}(\widetilde{x}_0) \cap \mathcal{O}| - |B_{2R}(\widetilde{x}_0) \cap \mathcal{O}| \geq c_0(2mR)^n - |B_{2R}(\widetilde{x}_0)| \geq c_0(2mR)^n - \omega_n(2R)^n,$$

where ω_n represents the volume of the unit ball. We deduce that for m large enough, $c_0 m^n - \omega_n > 0$, one has

$$|A_R \cap \mathcal{O}| \geq a_0 R^n. \tag{5.18}$$

We next choose numbers c_ν, such that

$$|\{x \in A_R \cap \mathcal{O} \mid u_\nu(x) - c_\nu \geq 0\}| \geq \frac{1}{2}|A_R \cap \mathcal{O}|,$$
$$|\{x \in A_R \cap \mathcal{O} \mid u_\nu(x) - c_\nu \leq 0\}| \geq \frac{1}{2}|A_R \cap \mathcal{O}|.$$

Such numbers exist. Then, from (5.18), we can derive a Poincaré inequality, by using

$$\int_{A_R\cap\mathcal{O}}(u_\nu(x)-c_\nu)^2\mathrm{d}x = \int_{A_R\cap\mathcal{O}}[(u_\nu(x)-c_\nu)^+]^2\mathrm{d}x + \int_{A_R\cap\mathcal{O}}[(u_\nu(x)-c_\nu)^-]^2\mathrm{d}x$$
$$\le c_0R^2\int_{A_R\cap\mathcal{O}}|Du_\nu|^2\mathrm{d}x.$$

Therefore, for this choice of c, we can assert that

$$\int_{A_R\cap\mathcal{O}}\frac{|u(x)-c|^2}{R^2}\mathrm{d}x \le C\int_{A_R\cap\mathcal{O}}|Du|^2\mathrm{d}x. \tag{5.19}$$

We then apply (5.15) with
$$\tau(x) = \varpi_R(|x-\widetilde{x}_0|)$$
where
$$\varpi_R(\rho) = \begin{cases}1, & \text{if } \rho \le 2R,\\ 0, & \text{if } \rho \ge 2mR.\end{cases}$$

Using Poincaré inequality (5.19), we can write

$$\int_{B_{2R}(\widetilde{x}_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x \le C\int_{A_R\cap\mathcal{O}}|Du|^2\mathrm{d}x + K\int_{B_{2mR}(\widetilde{x}_0)\cap\mathcal{O}}|u-c||Du|^2\mathrm{d}x + K_mR^n.$$

Next we use

$$B_{\frac{R}{2}}(x_0)\subset B_{2R}(\widetilde{x}_0),\quad B_{2mR}(\widetilde{x}_0)\subset B_{(2m+2)R}(x_0),\quad A_R\subset B_{(2m+2)R}(x_0)-B_{\frac{R}{2}}(x_0).$$

Therefore, one has

$$\int_{B_{\frac{R}{2}}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x \le C_0\int_{(B_{(2m+2)R}(x_0)-B_{\frac{R}{2}}(x_0))\cap\mathcal{O}}|Du|^2\mathrm{d}x$$
$$+K\int_{B_{(2m+2)R}(x_0)\cap\mathcal{O}}|u-c||Du|^2\mathrm{d}x + K_mR^n.$$

In the integral involving $|u-c|$, we can use

$$|u-c| \le c_0\mathrm{osc}[u](B_{2(m+1)R}(x_0)\cap\mathcal{O}).$$

It remains to fill the hole to obtain (5.16).

5.5 Morrey norm estimate

Remember that (5.16) applies in fact to u^δ which is a smooth function. So we can estimate the oscillation of u with a C^α-norm, with α arbitrary smaller than 1. We get

$$\mathrm{osc}[u](B_{2(m+1)R}(x_0)\cap\mathcal{O}) \le KR^\alpha\|u\|_{C^\alpha(\overline{\mathcal{O}})}.$$

We now divide (5.16) by $\left(\frac{R}{2}\right)^{-2\alpha}$. Set

$$\theta = \frac{C_0}{1+C_0}(4(m+1))^{2\alpha}.$$

We now choose α sufficiently small to have $\theta < 1$. We can then write

$$\begin{aligned}\Big(\frac{R}{2}\Big)^{-2\alpha}\int_{B_{\frac{R}{2}}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x &\leq \theta(2(m+1)R)^{-2\alpha}\int_{B_{2(m+1)R}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x\\ &\quad + KR^{-\alpha}\|u\|_{C^\alpha(\overline{\mathcal{O}})}\int_{B_{2(m+1)R}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x\\ &\quad + K_mR^{n-2\alpha}. \end{aligned}\tag{5.20}$$

We can write

$$\begin{aligned}&R^{-\alpha}\int_{B_{2(m+1)R}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x\\ &= (2(m+1))^\alpha\Big(\int_{B_{2(m+1)R}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big((2(m+1)R)^{-2\alpha}\int_{B_{2(m+1)R}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x\Big)^{\frac{1}{2}}.\end{aligned}$$

Then, incorporating $(2(m+1))^\alpha$ in constant K, we deduce from (5.20), with a slight increase of θ, keeping it strictly less than 1, and changing R into $2R$

$$\begin{aligned}R^{-2\alpha}\int_{B_R(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x &\leq \theta(4(m+1)R)^{-2\alpha}\int_{B_{4(m+1)R}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x\\ &\quad + K\|u\|^2_{C^\alpha(\mathcal{O})}\Psi_m(x_0,R)+K_m\end{aligned}\tag{5.21}$$

with

$$\Psi_m(x_0,R)=\int_{B_{4(m+1)R}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x.$$

We apply this inequality with

$$R=\frac{R_0}{(4(m+1))^{j+1}},$$

and set

$$\varphi_j=\Big(\frac{R_0}{(4(m+1))^j}\Big)^{-2\alpha}\int_{B_{\frac{R_0}{(4(m+1))^j}}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x.$$

We obtain

$$\varphi_{j+1}\leq\theta\varphi_j+K\|u\|^2_{C^\alpha(\overline{\mathcal{O}})}\Psi(x_0,R_0)+K_m,$$

where

$$\Psi(x_0,R_0)=\int_{B_{R_0}(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x.$$

Therefore,

$$\varphi_j\leq\frac{\varphi_0}{1-\theta}+\frac{K\Psi(x_0,R_0)}{1-\theta}\|u\|^2_{C^\alpha(\overline{\mathcal{O}})}+\frac{K_m}{1-\theta}\leq K+K\Psi(x_0,R_0)\|u\|^2_{C^\alpha(\overline{\mathcal{O}})}.$$

Since j is any integer, this inequality implies

$$\sup_{\substack{x_0\\ R\leq R_0}}R^{-2\alpha}\int_{B_R(x_0)\cap\mathcal{O}}|Du|^2\mathrm{d}x\leq K+K\sup_{x_0}\Psi(x_0,R_0)\|u\|^2_{C^\alpha(\overline{\mathcal{O}})}.\tag{5.22}$$

We now use the condition $n = 2$. Note that it has not been used before. Consider the Morrey norm

$$|||u|||^2_{\alpha,\mathcal{O}} = \sup_{x_0 \in \mathcal{O}, R} R^{-2\alpha} \int_{B_R(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x.$$

Then one has

$$\|u\|^2_{C^\alpha(\overline{\mathcal{O}})} \le C|||u|||^2_{\alpha,\mathcal{O}}. \tag{5.23}$$

Also

$$\begin{aligned} |||u|||^2_{\alpha,\mathcal{O}} &\le \sup_{\substack{x_0 \\ R\le R_0}} R^{-2\alpha} \int_{B_R(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x + \sup_{\substack{x_0 \\ R\ge R_0}} R^{-2\alpha} \int_{B_R(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x \\ &\le \sup_{\substack{x_0 \\ R\le R_0}} R^{-2\alpha} \int_{B_R(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x + KR_0^{-2\alpha}. \end{aligned}$$

Using this inequality in (5.22), we obtain

$$\begin{aligned} &\sup_{\substack{x_0 \\ R\le R_0}} R^{-2\alpha} \int_{B_R(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x \\ &\le K + K \sup_{x_0} \Psi(x_0, R_0) \Big[\sup_{\substack{x_0 \\ R\le R_0}} R^{-2\alpha} \int_{B_R(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x + KR_0^{-2\alpha} \Big]. \end{aligned} \tag{5.24}$$

We now use the small Dirichlet growth property, from which we can assert that

$$\sup_{x_0} \Psi(x_0, R_0) \to 0, \quad \text{as } R_0 \to 0.$$

We then fix R_0 sufficiently small so that $K \sup\limits_{x_0} \Psi(x_0, R_0) = \epsilon < 1$. It follows from (5.24) that

$$\sup_{\substack{x_0 \\ R\le R_0}} R^{-2\alpha} \int_{B_R(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x \le \frac{K + \epsilon K R_0^{-2\alpha}}{1-\epsilon}.$$

This provides an a priori bound on $\sup\limits_{\substack{x_0 \\ R\le R_0}} R^{-2\alpha} \int_{B_R(x_0)\cap\mathcal{O}} |Du|^2 \mathrm{d}x$, from which we deduce an a priori bound on the Morrey norm and finally we obtain an a priori bound on $C^\alpha(\overline{\mathcal{O}})$.

5.6 Convergence and end of proof

From the previous sections, we have obtained the estimates

$$\|u^\delta\|_{H^1(\mathcal{O})} \le C, \quad \|u^\delta\|_{C^\alpha(\overline{\mathcal{O}})} \le C.$$

We can then extract a subsequence which converges weakly in $H^1(\mathcal{O})$ and in $C^0(\overline{\mathcal{O}})$ strongly to u. Since $H^\delta_\nu(x, Du^\delta)$ is bounded in $L^1(\mathcal{O})$, we deduce from the equation

$$\sum_\nu \int Du^\delta_\nu . (Du^\delta_\nu - Du) \mathrm{d}x \to 0,$$

from which the strong convergence in $H^1(\mathcal{O})$ follows. We can then deduce, for a new subsequence

$$H^\delta_\nu(x, Du^\delta) \to H_\nu(x, Du) \quad \text{a.e.}$$

For any measurable subset A_ϵ in $\mathcal{O}$, we have

$$\int_{A_\epsilon} |H_\nu^\delta(x, Du^\delta)|\mathrm{d}x \leq K \int_{A_\epsilon} |Du^\delta|^2 \mathrm{d}x + K|A_\epsilon|,$$

and the right-hand side tends to 0 as $|A_\epsilon| \to 0$, unifomly in δ. Therefore, the left-hand side has the same property which implies

$$H_\nu^\delta(x, Du^\delta) \to H_\nu(x, Du) \in L^1(\mathcal{O}).$$

Therefore, we have obtained a solution to (3.3) in the space $H^1(\mathcal{O}) \cap C^\alpha(\overline{\mathcal{O}})$. To go from $H^1(\mathcal{O}) \cap C^\alpha(\overline{\mathcal{O}})$ to $W^{2,q}(\mathcal{O})$ is a standard result (see [4] or [1]).

6 Application to Nash and Stackelberg Games

6.1 Nash games

We assume

$$l_\nu(x, v) = \frac{1}{2} v_\nu \cdot Q_\nu(x) v_\nu + v_\nu \cdot \sum_{\mu \neq \nu} \Gamma_{\nu\mu}(x) v_\mu, \tag{6.1}$$

where the matrices $Q_\nu(x)$ are positive definite and bounded, with bounded inverse. The matrices $\Gamma_{\nu\mu}(x)$ are bounded. In the Nash game, the feedbacks are defined as Nash equilibrium of the Lagrangians

$$L_\nu(x, p_\nu, v) = l_\nu(x, v) + p_\nu \cdot \Big(\sum_\mu A_\mu(x) v_\mu + g(x) \Big).$$

So we make the assumption: the matrix Z with

$$Z_{\nu\nu} = Q_\nu, \quad Z_{\nu\mu} = \Gamma_{\nu\mu}, \quad \text{if } \mu \neq \nu \tag{6.2}$$

is invertible. The feedbacks $\widehat{v}_\nu(x, p)$ are defined by the system

$$Q_\nu(x)\widehat{v}_\nu + \sum_{\mu \neq \nu} \Gamma_{\nu\mu}(x)\widehat{v}_\mu + A_\nu^*(x) p_\nu = 0. \tag{6.3}$$

We can write the Hamiltonians in the form

$$H_\nu(x, p) = -\frac{1}{2}\widehat{v}_\nu \cdot Q_\nu(x)\widehat{v}_\nu + p_\nu \cdot \Big(\sum_\mu A_\mu(x)\widehat{v}_\mu + g(x) \Big) + f_\nu(x). \tag{6.4}$$

The assumptions (4.8)–(4.10) are satisfied with

$$F_\nu(x, p) = \sum_\mu A_\mu(x)\widehat{v}_\mu(x, p) + g(x).$$

Example 6.1 Cyclic non-market interaction
We take

$$Q_\nu = I \quad \text{and} \quad \Gamma_{\nu\mu} = \begin{cases} \theta, & \text{if } \mu = \nu + 1, \\ 0, & \text{if } \mu \neq \nu + 1. \end{cases}$$

It is convenient to consider the index $N+\nu$ with the meaning $N+\nu=\nu$. It is an easy exercise to check that the matrix (6.2) is invertible if $|\theta|\neq 1$ and one has the explicit formulas

$$\widehat{v}_\nu(x,p) = -\frac{1}{1-(-\theta)^N}\sum_{j=0}^{N-1}(-\theta)^j A^*_{\nu+j}(x)p_{\nu+j}. \tag{6.5}$$

We next have

$$\sum_\nu l_\nu(x,v) = \frac{1}{2}|v|^2 + \theta\sum_\nu v_\nu\cdot v_{\nu+1} \geq \Big(\frac{1}{2}-|\theta|\Big)|v|^2.$$

Therefore, the coercivity assumption (4.6) is satified if $|\theta|<\frac{1}{2}$. We can state the following proposition.

Proposition 6.1 *The Nash differential game of Example* 6.1*, i.e., cyclic non-market interaction, has a saddle point if $|\theta|<\frac{1}{2}$ and $n=2$.*

Example 6.2 Symmetric interaction

We consider the situation studied in the book [1], namely we take

$$Q_\nu = I, \quad \Gamma_{\nu\mu} = \begin{cases}\theta, & \text{if } \mu\neq\nu,\\ 0, & \text{if } \mu=\nu.\end{cases} \tag{6.6}$$

Then the matrix (6.2) is invertible provided that $\theta\neq 1,\ \theta\neq -\frac{1}{N-1}$. We have the formulas

$$\widehat{v}_\nu(x,p) = \frac{\theta\sum\limits_\mu A^*_\mu(x)p_\mu}{(1-\theta)(1+(N-1)\theta)} - \frac{A^*_\nu(x)p_\nu}{1-\theta}. \tag{6.7}$$

We next check the coercivity assumption

$$\sum_\nu l_\nu(x,v) = \frac{1}{2}|v|^2 + \theta\sum_\nu v_\nu\cdot\overline{v}_\nu$$

with $\overline{v}_\nu = \sum\limits_{\mu\neq\nu} v_\mu$. Therefore

$$\sum_\nu l_\nu(x,v) = \Big(\frac{1}{2}-\theta\Big)|v|^2 + \theta\Big|\sum_\nu v_\nu\Big|^2 = |v|^2\Big[\frac{1}{2}-\theta+\theta\Big|\sum_\nu \xi_\nu\Big|^2\Big] \tag{6.8}$$

with $\xi_\nu = \frac{v_\nu}{|v|}$. Note that $\sum\limits_\nu |\xi_\nu|^2 = 1$. We then use the properties (easy to check)

$$\min_{|\xi|=1}\Big|\sum_\nu \xi_\nu\Big|^2 = 0, \tag{6.9}$$

$$\max_{|\xi|=1}\Big|\sum_\nu \xi_\nu\Big|^2 = N. \tag{6.10}$$

We deduce that

$$\text{if } \theta\geq 0, \quad \frac{1}{2}-\theta+\theta\Big|\sum_\nu \xi_\nu\Big|^2 \geq \frac{1}{2}-\theta.$$

To get coercivity when $\theta\geq 0$, we need to have $\theta<\frac{1}{2}$. For $\theta<0$, it follows from (6.10) that

$$\text{if } \theta<0, \quad \frac{1}{2}-\theta+\theta\Big|\sum_\nu \xi_\nu\Big|^2 \geq \frac{1}{2}+\theta(N-1),$$

and to get coercivity we need to have $\theta > -\frac{1}{2(N-1)}$. Therefore we can state the next proposition.

Proposition 6.2 *The Nash differential game of Example* 6.2*, i.e., symmetric interaction, has a saddle point if* $-\frac{1}{2(N-1)} < \theta < \frac{1}{2}$ *and* $n = 2$.

Remark 6.1 In the book above, it is shown that we can have a saddle point for more values of θ and n in general. So Proposition 6.2 is not optimal. However, it is the best one within the framework of the techniques of this article, in which we require the coercivity condition, that is a strong condition.

6.2 Stackelberg games

We assume (symmetric interaction)

$$l_\nu(x,v) = \frac{1}{2}|v|^2 + \theta v_\nu \cdot \overline{v}_\nu. \tag{6.11}$$

So the coercivity condition is the same as for Nash games. Therefore $-\frac{1}{2(N-1)} < \theta < \frac{1}{2}$. To define the strategies, we have to solve Stackelberg hierarchical minimization for the Lagrangians. The calculations turn out to be very messy. We have performed them for $N = 3$. We have to define functions $\widehat{v}_3(x; v_1, v_2; p)$, $\widehat{v}_2(x; v_1; p)$, $\widehat{v}_1(x, p)$. We get successively

$$\widehat{v}_3 = -\theta(v_1 + v_2) - A_3^* p_3, \tag{6.12}$$

$$\widehat{v}_2 = -\frac{\theta(1-\theta)}{1-2\theta^2} v_1 + \frac{\theta A_3^* p_3 - (A_2^* - \theta A_3^*) p_2}{1-2\theta^2}, \tag{6.13}$$

$$\begin{aligned}\widehat{v}_1 = {} & \frac{1-2\theta^2}{1+2\theta^4+4\theta^3-6\theta^2} \Big\{ -\Big(A_1^* - \frac{\theta(1-\theta)}{1-2\theta^2} A_2^* - \frac{\theta(1-\theta-\theta^2)}{1-2\theta^2} A_3^* \Big) p_1 \\ & + \frac{\theta(1-\theta)}{1-2\theta^2}(A_2^* - \theta A_3^*) p_2 + \frac{\theta(1-\theta-\theta^2)}{1-2\theta^2} A_3^* p_3 \Big\}. \end{aligned} \tag{6.14}$$

We can check that for θ in the interval $(-\frac{1}{4}, \frac{1}{2})$ the assumptions (4.8)–(4.10) are satisfied. We can state the proposition as follows.

Proposition 6.3 *The Stackelberg differential game with* (6.11)*, i.e., symmetric interaction and* $N = 3, n = 2$*, has a solution if* $\theta \in (-\frac{1}{4}, \frac{1}{2})$.

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