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Global Classical Solutions to Partially Dissipative Quasilinear Hyperbolic Systems with One Weakly Linearly Degenerate Characteristic*

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Abstract For a kind of partially dissipative quasilinear hyperbolic systems without Shizuta-Kawashima condition, in which all the characteristics, except a weakly linearly degenerate one, are involved in the dissipation, the global existence of H^2 classical solution to the Cauchy problem with small initial data is obtained.

Keywords Global classical solution, Quasilinear hyperbolic system, Weak linear degeneracy, Partial dissipation
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1 Introduction

We consider the Cauchy problem for the following one dimensional quasilinear hyperbolic system

$$\begin{cases}\dfrac{\partial u}{\partial t}+A(u)\dfrac{\partial u}{\partial x}=F(u), \quad x\in\mathbb{R},\ t\geq 0, & (1.1)\\ t=0:\ u=u_0(x), \quad x\in\mathbb{R}, & (1.2)\end{cases}$$

where $u=(u_1,\cdots,u_n)^{\mathrm{T}}$ is the unknown vector function of (t,x), $A(u)\in C^3$ is an $n\times n$ matrix and $F(u)=(F_1(u),\cdots,F_n(u))^{\mathrm{T}}\in C^3$. By hyperbolicity, the coefficient matrix $A(u)$ has n real eigenvalues $\lambda_1(u),\cdots,\lambda_n(u)$ and a complete set of left (resp., right) eigenvectors $l_k(u)=(l_{k1}(u),\cdots,l_{kn}(u))$ (resp., $r_k(u)=(r_{1k}(u),\cdots,r_{nk}(u))^{\mathrm{T}}$) $(k=1,\cdots,n)$ with assumption $\lambda_k(u),l_k(u),r_k(u)\in C^3$ $(k=1,\cdots,n)$. Without loss of generality, we assume

$$F(0)=0 \tag{1.3}$$

and

$$l_k(u)r_{\overline{k}}(u)\equiv\delta_{k\overline{k}} \quad \text{for any small } |u|,\ \forall\, 1\leq k,\overline{k}\leq n, \tag{1.4}$$

where $\delta_{k\overline{k}}$ is the Kronecker symbol. The initial data $u_0(x)$ is supposed to be suitably smooth with

$$\theta\overset{\text{def.}}{=}\|u_0\|_{H^2}\leq\theta_0, \tag{1.5}$$

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where $\theta_0 > 0$ is a small constant to be determined.

In this paper, we focus on the H^2 classical solution $u = u(t, x)$ to Cauchy problem (1.1)–(1.2), whose local well-posedness was well discussed in [9, 12]. Generally speaking, this local H^2 classical solution would blow up in a finite time even for small and decaying initial data (see [6, 12]). Weak linear degeneracy and dissipative inhomogeneous terms are two well-known structures that could lead to the global existence of classical solution for small and decaying smooth initial data.

There are mainly two kinds of results on the dissipative system. Hsiao, Li [4], Li [6], and Li, Qin [7] used the method of characteristics to prove the global existence of C^1 classical solution to the Cauchy problem for strictly dissipative quasilinear hyperbolic systems with small C^1 initial data. On the other hand, Hanouzet and Natalini [3] applied the energy method to the hyperbolic conservation laws, and got the global H^2 classical solution for small H^2 initial data under Shizuta-Kawashima condition and the dissipative entropy condition. Then Zhou [19] gave a different proof with slightly different hypotheses. The corresponding case of several space variables and the asymptotic behavior were studied in [15] and [1–2], respectively.

For the homogeneous quasilinear hyperbolic systems with weakly linearly degenerate characteristics, a series of results on global C^1 classical solutions for small and decaying initial data were given in [6, 8, 10, 18, 20].

Besides, Zeng [16–17] discussed the global classical solution to one dimensional gas dynamics in thermal nonequilibrium. This system is partially dissipative in which all the characteristics, except a linearly degenerate one, are involved in the dissipation by Shizuta-Kawashima condition.

By all results mentioned above, it is natural to raise the following conjecture that for a quasilinear hyperbolic system, if a part of the system is involved in the dissipation, while the other part possesses weakly linearly degenerate characteristics, moreover, some suitable conditions are imposed for interactions between these two parts, then the corresponding Cauchy problem should admit a global classical solution for small and decaying smooth initial data. The earliest version of this conjecture occurs in [13, 15]. Mascia and Natalini [13] also gave a series of discussions on the systems of this kind. Then Liu and Qu [11] proved that for systems one part of whose characteristics is involved in the dissipation in the sense of strict row diagonal dominance, the other part of whose characteristics is weakly linearly degenerate, and that moreover, the interactions of these two parts are restricted by some suitable conditions, the corresponding Cauchy problem admits a unique global classical solution for small and decaying initial data. In [11], it also provided the pointwise decay estimate of the solution and gave a finite-time singularity for some examples with worse interactions.

We continue the study on this kind of partially dissipative quasilinear hyperbolic systems, but to be different from [11], we focus on the case that the partial dissipation is in the sense of positive definiteness other than diagonal dominance, and the number of weakly linearly degenerate characteristics is only one.

The main result of this paper is presented in the rest part of this section, and then proved in Sections 2–3, in which a new set of wave decomposition formulas are deduced and analyzed in Subsection 2.3, that plays an important role in the proof. Finally, we give some remarks in Section 4.

Now, let us give the detailed description of our main result. As mentioned before, for Cauchy problem (1.1)–(1.2), we study the case that one characteristic is weakly linearly degenerate. Without loss of generality, let it be the nth characteristic $\lambda_n(u)$, namely, we have

$$\lambda_n(u^{(n)}(s)) \equiv \lambda_n(0) \quad \text{for any small } |s|, \tag{1.6}$$

where $u = u^{(k)}(s)$ is the kth characteristic trajectory passing through the origin,

$$\frac{\mathrm{d}u^{(k)}(s)}{\mathrm{d}s} = r_k(u^{(k)}(s)), \quad u^{(k)}(0) = 0, \quad k = 1, \cdots, n.$$

Remark 1.1 The weak linear degeneracy (1.6) is weaker than the linear degeneracy proposed by Lax,

$$\nabla\lambda_n(u) r_n(u) \equiv 0.$$

For more details of the weak linear degeneracy, one may refer to [6, 8].

For other $n-1$ characteristics, we assume the following dissipation condition: there exists a constant $\delta_0 > 0$, such that

$$-\sum_{i,p=1}^{n-1} \xi_i G_{ip}(0) \xi_p \geq \delta_0 \sum_{i=1}^{n-1} |\xi_i|^2, \quad \forall\, \xi = (\xi_1, \cdots, \xi_{n-1})^{\mathrm{T}} \in \mathbb{R}^{n-1} \tag{1.7}$$

with

$$G(u) = L(u) \nabla F(u) R(u), \tag{1.8}$$

where

$$L(u) = \begin{pmatrix} l_1(u) \\ \vdots \\ l_n(u) \end{pmatrix} \quad \text{and} \quad R(u) = (r_1(u), \cdots, r_n(u)) \tag{1.9}$$

are the matrices composed of left and right eigenvectors, respectively. By (1.4), we obviously have

$$L(u)R(u) \equiv I \qquad \text{for any small } |u|.$$

Moreover, we need the following condition to restrict interactions between these two parts of characteristics:

$$F(u^{(n)}(s)) \equiv 0 \quad \text{for any small } |s|. \tag{1.10}$$

Remark 1.2 Obviously, (1.10) implies (1.3).

Under these assumptions, we have our main result.

Theorem 1.1 *Under hypotheses* (1.4), (1.6)–(1.7) *and* (1.10), *there exists a constant* $\theta_0 > 0$ *so small that for any given* θ *with* $0 \leq \theta \leq \theta_0$, *Cauchy problem* (1.1)–(1.2) *admits a unique global* H^2 *classical solution* $u = u(t, x)$ *on* $t \geq 0$ *for any initial data* $u_0(x)$ *satisfying* (1.5).

2 Preliminaries

In this section, we give some preliminaries for the proof of Theorem 1.1. First, the normalized coordinates and the equivalent theorem of Theorem 1.1 in normalized coordinates are introduced in Subsection 2.1. Then, the formulas of wave decomposition are reduced to a suitable form in Subsection 2.2. Finally, a new set of wave decomposition formulas is presented and analyzed in Subsection 2.3 for high-order derivatives of the solution.

2.1 The equivalent form of Theorem 1.1 in normalized coordinates

As in [6, 8, 10], we introduce (generalized) normalized coordinates in u-space. Denote $\widetilde{u}=\widetilde{u}(u)$ with $\widetilde{u}(0)=0$ as the corresponding C^4-diffeomorphism, and

$$J(u)=\frac{\partial \widetilde{u}}{\partial u} \tag{2.1}$$

as its Jacobi matrix. By the properties of normalized coordinates given in [6, 8], we have

$$J^{-1}(u(\widetilde{u}))|_{\widetilde{u}=0}=R(0) \tag{2.2}$$

and

$$(Jr_k)(u(s\widetilde{e}_k)) \parallel \widetilde{e}_k \quad \text{for any small } |s|,\ \forall 1\le k\le n, \tag{2.3}$$

where $\widetilde{e}_k$ stands for the kth unit vector in the normalized coordinates $\widetilde{u}$.

As in [11], we list the equivalent forms of the formulas given in Section 1 in normalized coordinates $\widetilde{u}$. First, it is easy to see that the original system (1.1) can be rewritten as

$$\frac{\partial \widetilde{u}}{\partial t}+\widetilde{A}(\widetilde{u})\frac{\partial \widetilde{u}}{\partial x}=\widetilde{F}(\widetilde{u}), \tag{2.4}$$

where

$$\widetilde{A}(\widetilde{u})=J(u(\widetilde{u}))A(u(\widetilde{u}))J^{-1}(u(\widetilde{u})), \tag{2.5}$$

$$\widetilde{F}(\widetilde{u})=J(u(\widetilde{u}))F(u(\widetilde{u})), \tag{2.6}$$

while, the eigenvalues and matrices of eigenvectors of $\widetilde{A}(\widetilde{u})$ are, respectively,

$$\widetilde{\lambda}_k(\widetilde{u})=\lambda_k(u(\widetilde{u})), \quad \forall\, 1\le k\le n, \tag{2.7}$$

$$\widetilde{R}(\widetilde{u})=J(u(\widetilde{u}))R(u(\widetilde{u})), \tag{2.8}$$

$$\widetilde{L}(\widetilde{u})=L(u(\widetilde{u}))J^{-1}(u(\widetilde{u})). \tag{2.9}$$

Thus, (1.4) and (1.3) can be equivalently rewritten as

$$\widetilde{l}_k(\widetilde{u})\widetilde{r}_{\overline{k}}(\widetilde{u})\equiv\delta_{k\overline{k}} \quad \text{for any small } |\widetilde{u}|,\ \forall 1\le k,\overline{k}\le n, \tag{2.10}$$

$$\widetilde{F}(0)=0. \tag{2.11}$$

Moreover, noting (2.2), we have

$$\widetilde{L}(0)=\widetilde{R}(0)=I. \tag{2.12}$$

For

$$\widetilde{G}(\widetilde{u}) = \widetilde{L}(\widetilde{u})\nabla_{\widetilde{u}}\widetilde{F}(\widetilde{u})\widetilde{R}(\widetilde{u}), \tag{2.13}$$

by (2.2), (2.6) and (2.11)–(2.12), we have

$$\widetilde{G}(0) = \nabla_{\widetilde{u}}\widetilde{F}(0) = L(0)\nabla_u F(0)R(0) = G(0). \tag{2.14}$$

Then (1.7) can be equivalently rewritten as

$$-\sum_{i,p=1}^{n-1} \xi_i \widetilde{G}_{ip}(0)\xi_p \geq \delta_0 \sum_{i=1}^{n-1} |\xi_i|^2, \quad \forall\, \xi = (\xi_1, \cdots, \xi_{n-1})^{\mathrm{T}} \in \mathbb{R}^{n-1}. \tag{2.15}$$

By (2.3) and (2.8), without loss of generality, we can assume

$$\widetilde{r}_k(s\widetilde{e}_k) \equiv \widetilde{e}_k \quad \text{ for any small } |s|,\ \forall\, 1 \leq k \leq n. \tag{2.16}$$

Thus, we have

$$\widetilde{u}(u^{(k)}(s)) = s\widetilde{e}_k \quad \text{for any small } |s|,\ \forall\, 1 \leq k \leq n.$$

So, noting (2.6)–(2.7), (1.6) and (1.10) are equivalent to

$$\widetilde{\lambda}_n(\widetilde{u}_n\widetilde{e}_n) \equiv \widetilde{\lambda}_n(0) \quad \text{for any small } |s|, \tag{2.17}$$

$$\widetilde{F}(\widetilde{u}_n\widetilde{e}_n) \equiv 0 \quad \text{for any small } |s|. \tag{2.18}$$

Moreover, the initial condition (1.2) can be rewritten as

$$t = 0:\ \widetilde{u} = \widetilde{u}_0(x), \quad x \in \mathbb{R}, \tag{2.19}$$

where $\widetilde{u}_0(x) = \widetilde{u}(u_0(x))$ and

$$\widetilde{\theta} \overset{\text{def.}}{=} \|\widetilde{u}_0\|_{H^2} \leq \widetilde{\theta}_0, \tag{2.20}$$

and furthermore, for suitably small $\widetilde{\theta}_0$, we have $K_0^{-1}\theta_0 \leq \widetilde{\theta}_0 \leq K_0\theta_0$ for a constant $K_0 \geq 1$.

Through the above analysis, we get the following equivalent form of Theorem 1.1.

Theorem 2.1 *In normalized coordinates, under hypotheses* (2.10), (2.15) *and* (2.17)–(2.18), *there exists a constant* $\widetilde{\theta}_0 > 0$ *so small that for any given* $\widetilde{\theta}$ *with* $0 \leq \widetilde{\theta} \leq \widetilde{\theta}_0$ *and any initial data* $\widetilde{u}_0(x)$ *satisfying* (2.20), *Cauchy problem* (2.4) *and* (2.19) *admits a unique global* H^2 *classical solution* $\widetilde{u} = \widetilde{u}(t, x)$ *on* $t \geq 0$.

2.2 Formulas of wave decomposition and their reduction

In this subsection, we introduce the formulas of wave decomposition and reduce them to a desired form. For convenience, we omit the sign "$\sim$" in Subsections 2.2–2.3 and Section 3 for all the functions in normalized coordinates.

As in [5–6, 8, 10], we introduce the formulas of wave decomposition. Setting

$$w_k = l_k(u)\frac{\partial u}{\partial x}, \quad k = 1, \cdots, n, \tag{2.21}$$

by (2.10), we have

$$\frac{\partial u}{\partial x}=\sum_{r=1}^{n} w_r r_r(u). \tag{2.22}$$

The corresponding formulas of wave decomposition are

$$\frac{\partial u_k}{\partial t}+\frac{\partial(\lambda_k(u)u_k)}{\partial x}=\sum_{r=1}^{n} B_{kr}(u)w_r+\sum_{r=1}^{n}\Xi_{kr}(u)u_k w_r+F_k(u), \quad k=1,\cdots,n, \tag{2.23}$$

$$\frac{\partial u_k}{\partial t}+\lambda_k(u)\frac{\partial u_k}{\partial x}=\sum_{r=1}^{n} B_{kr}(u)w_r+F_k(u), \quad k=1,\cdots,n, \tag{2.24}$$

$$\frac{\partial w_k}{\partial t}+\frac{\partial(\lambda_k(u)w_k)}{\partial x}=\sum_{\substack{1\le r,l\le n\\ r\neq l}}\Gamma_{krl}(u)w_r w_l+\sum_{r=1}^{n} K_{kr}(u)w_r, \quad k=1,\cdots,n, \tag{2.25}$$

$$\begin{aligned}\frac{\partial w_k}{\partial t}+\lambda_k(u)\frac{\partial w_k}{\partial x}=&\sum_{\substack{1\le r,l\le n\\ r\neq l}}\Gamma_{krl}(u)w_r w_l-\sum_{r=1}^{n}\Xi_{kr}(u)w_k w_r\\ &+\sum_{r=1}^{n} K_{kr}(u)w_r, \quad k=1,\cdots,n,\end{aligned} \tag{2.26}$$

where

$$B_{kr}(u)=(\lambda_k(u)-\lambda_r(u))r_{kr}(u)\in C^3, \quad k,r=1,\cdots,n, \tag{2.27}$$

$$\Xi_{kr}(u)=\nabla\lambda_k(u)r_r(u)\in C^2, \quad k,r=1,\cdots,n, \tag{2.28}$$

$$\Gamma_{krl}(u)=(\lambda_r(u)-\lambda_l(u))l_k(u)\nabla r_l(u)r_r(u)\in C^2, \quad k,r,l=1,\cdots,n, \tag{2.29}$$

$$K_{kr}(u)=G_{kr}(u)-l_k(u)\nabla r_r(u)F(u)\in C^2, \quad k,r=1,\cdots,n. \tag{2.30}$$

To reduce these formulas, we need the following lemmas.

Lemma 2.1 *Suppose that a function $a(u)\in C^{m+1}\,(m\in\mathbb{N})$ satisfies*

$$a(u_{\overline{k}}e_{\overline{k}})\equiv 0 \quad \text{for any small } |u_{\overline{k}}|$$

for an index $\overline{k}$ $(1\le\overline{k}\le n)$. Then there exist functions $b_k(u)\in C^m\,(k=1,\cdots,n)$, such that

$$a(u)=\sum_{\substack{1\le k\le n\\ k\neq\overline{k}}} b_k(u)u_k \quad \text{for any small } |u|.$$

Proof It is a direct consequence of Hadamard's formula.

Lemma 2.2 *Suppose that a function $a(u)\in C^{m+2}\,(m\in\mathbb{N})$ satisfies*

$$a(u_n e_n)\equiv 0 \quad \text{for any small } |u_n|. \tag{2.31}$$

Then there exist functions $b_{rl}(u)\in C^m\,(r,l=1,\cdots,n)$, such that

$$a(u)=\sum_{p=1}^{n-1}\frac{\partial a}{\partial u_p}(0)u_p+\sum_{\substack{1\le r,l\le n\\ (r,l)\neq(n,n)}} b_{rl}(u)u_r u_l \quad \text{for any small } |u|.$$

Proof Taking Taylor expansion to $a(u)$ at the point $(u_n e_n)$, we get that there exist functions $b_{p\overline{p}}(u) \in C^m$ $(p, \overline{p} = 1, \cdots, n-1)$, such that

$$a(u) = a(u_n e_n) + \sum_{p=1}^{n-1} \frac{\partial a}{\partial u_p}(u_n e_n) u_p + \sum_{p,\overline{p}=1}^{n-1} b_{p\overline{p}}(u) u_p u_{\overline{p}}.$$

Then the Taylor expansion to $\partial_{u_p} a(u_n e_n)$ at the point 0 gives

$$\frac{\partial a}{\partial u_p}(u_n e_n) = \frac{\partial a}{\partial u_p}(0) + b_{pn}(u) u_n, \quad \forall\, 1 \le p \le n-1$$

with some functions $b_{pn}(u) \in C^m$ $(p = 1, \cdots, n-1)$. Noting (2.31), these two formulas lead directly to the conclusion of the lemma.

Remark 2.1 Lemma 2.2 is similar to [11, Lemma 3.2]. However, by a different proof, we only need a weaker hypothesis on the regularity of function $a(u)$.

As in [11], we reduce the coefficients in the formulas (2.23)–(2.26) by means of the above lemmas. Our aims of the reduction are as follows:

(1) The coefficients of the first order terms are all constants.

(2) No second order term has the index pair (n, n), i.e., the index pair of every second order term is of the form $(p, \overline{p})$, (p, n) or (n, p) for $p, \overline{p} = 1, \cdots, n-1$.

In what follows in this subsection and in Subsection 2.3, we always take $|u|$ to be suitably small.

First, by (2.16) and (2.27), we have

$$B_{kr}(u_r e_r) = (\lambda_k(u_r e_r) - \lambda_r(u_r e_r)) \cdot \delta_{kr} \equiv 0, \quad \forall\, 1 \le k, r \le n.$$

Then by Lemma 2.1, there exist functions $\psi_{krl}(u) \in C^2$ $(k, r, l = 1, \cdots, n)$, such that

$$B_{kr}(u) w_r = \sum_{\substack{1 \le l \le n \\ l \ne r}} \psi_{krl}(u) u_l w_r, \quad \forall\, 1 \le k, r \le n. \tag{2.32}$$

By Lemma 2.2 and noting (2.18), there exist functions $\Phi_{krl}(u) \in C^1$ $(k, r, l = 1, \cdots, n)$, such that

$$F_k(u) = \sum_{p=1}^{n-1} \frac{\partial F_k}{\partial u_p}(0) u_p + \sum_{\substack{1 \le r, l \le n \\ (r,l) \ne (n,n)}} \Phi_{krl}(u) u_r u_l, \quad \forall\, 1 \le k \le n.$$

Then, noting the first equality of (2.14), we have

$$F_k(u) = \sum_{p=1}^{n-1} G_{kp}(0) u_p + \sum_{\substack{1 \le r, l \le n \\ (r,l) \ne (n,n)}} \Phi_{krl}(u) u_r u_l, \quad \forall\, 1 \le k \le n. \tag{2.33}$$

Using (2.32)–(2.33), we can reduce (2.24) into

$$\begin{aligned} \frac{\partial u_k}{\partial t} + \lambda_k(u) \frac{\partial u_k}{\partial x} = &\sum_{\substack{1 \le r, l \le n \\ r \ne l}} \psi_{krl}(u) u_l w_r + \sum_{\substack{1 \le r, l \le n \\ (r,l) \ne (n,n)}} \Phi_{krl}(u) u_r u_l \\ &+ \sum_{p=1}^{n-1} G_{kp}(0) u_p, \quad k = 1, \cdots, n. \end{aligned}$$

For the convenience of discussion, we split this formula into

$$\begin{aligned}\frac{\partial u_i}{\partial t}+\lambda_i(u)\frac{\partial u_i}{\partial x}=&\sum_{\substack{1\le r,l\le n\\ r\neq l}}\psi_{irl}(u)u_lw_r+\sum_{\substack{1\le r,l\le n\\ (r,l)\neq(n,n)}}\Phi_{irl}(u)u_ru_l\\ &+\sum_{p=1}^{n-1}G_{ip}(0)u_p,\quad i=1,\cdots,n-1\end{aligned}\tag{2.34}$$

and

$$\frac{\partial u_n}{\partial t}+\lambda_n(u)\frac{\partial u_n}{\partial x}=\sum_{\substack{1\le r,l\le n\\ r\neq l}}\psi_{nrl}(u)u_lw_r+\sum_{\substack{1\le r,l\le n\\ (r,l)\neq(n,n)}}\Phi_{nrl}(u)u_ru_l+\sum_{p=1}^{n-1}G_{np}(0)u_p.\tag{2.35}$$

Moreover, by (2.16)–(2.17) and (2.28), we have

$$\Xi_{nn}(u_ne_n)=\nabla\lambda_n(u_ne_n)r_n(u_ne_n)=\frac{\partial}{\partial u_n}\lambda_n(u_ne_n)=0.$$

So by Lemma 2.1, there exist functions $\eta_{np}(u)\in C^1$ $(p=1,\cdots,n-1)$, such that

$$\Xi_{nn}(u)=\sum_{p=1}^{n-1}\eta_{np}(u)u_p.\tag{2.36}$$

Thus, by using (2.32)–(2.33) and (2.36), (2.23) can be reduced into

$$\begin{aligned}\frac{\partial u_i}{\partial t}+\frac{\partial(\lambda_i(u)u_i)}{\partial x}=&\sum_{\substack{1\le r,l\le n\\ r\neq l}}\psi_{irl}(u)u_lw_r+\sum_{r=1}^{n}\Xi_{ir}(u)u_iw_r+\sum_{\substack{1\le r,l\le n\\ (r,l)\neq(n,n)}}\Phi_{irl}(u)u_ru_l\\ &+\sum_{p=1}^{n-1}G_{ip}(0)u_p,\quad i=1,\cdots,n-1\end{aligned}\tag{2.37}$$

and

$$\begin{aligned}\frac{\partial u_n}{\partial t}+\frac{\partial(\lambda_n(u)u_n)}{\partial x}=&\sum_{\substack{1\le r,l\le n\\ r\neq l}}\psi_{nrl}(u)u_lw_r+\sum_{p=1}^{n-1}\Xi_{np}(u)u_nw_p+\sum_{p=1}^{n-1}(u_n\eta_{np}(u))u_pw_n\\ &+\sum_{\substack{1\le r,l\le n\\ (r,l)\neq(n,n)}}\Phi_{nrl}(u)u_ru_l+\sum_{p=1}^{n-1}G_{np}(0)u_p.\end{aligned}\tag{2.38}$$

Remark 2.2 In (2.38), we treat $u_n\eta_{np}(u)$ in the third order term $\eta_{np}(u)u_nu_pw_n$ as a coefficient. Then this third order term can be regarded as a second order one without the index pair (n,n). We will use the similar tricks many times in what follows.

By (2.13), (2.16), (2.18) and (2.30), we have

$$\begin{aligned}K_{kn}(u_ne_n)&=G_{kn}(u_ne_n)\\ &=l_k(u_ne_n)\nabla F(u_ne_n)r_n(u_ne_n)\\ &=l_k(u_ne_n)\frac{\partial}{\partial u_n}F(u_ne_n)\\ &=0,\quad\forall\, 1\le k\le n\end{aligned}$$

and

$$K_{kp}(0) = G_{kp}(0), \quad \forall\, 1 \le p \le n-1, \forall\, 1 \le k \le n.$$

So by Lemma 2.1 and Hadamard's formula, there exist functions $\xi_{krl}(u) \in C^1\,(k, r, l = 1, \cdots, n)$, such that

$$K_{kn}(u) = \sum_{p=1}^{n-1} \xi_{knp}(u) u_p, \quad \forall\, 1 \le k \le n,$$
$$K_{kp}(u) = G_{kp}(0) + \sum_{l=1}^{n} \xi_{kpl}(u) u_l, \quad \forall\, 1 \le p \le n-1, \forall\, 1 \le k \le n.$$

Thus, we have

$$\sum_{r=1}^{n} K_{kr}(u) w_r = \sum_{\substack{1 \le r,l \le n \\ (r,l) \ne (n,n)}} \xi_{krl}(u) u_l w_r + \sum_{p=1}^{n-1} G_{kp}(0) w_p, \quad \forall\, 1 \le k \le n. \tag{2.39}$$

Using this equality, we can reduce (2.25) into

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u) w_i)}{\partial x} = \sum_{\substack{1 \le r,l \le n \\ r \ne l}} \Gamma_{irl}(u) w_r w_l + \sum_{\substack{1 \le r,l \le n \\ (r,l) \ne (n,n)}} \xi_{irl}(u) u_l w_r + \sum_{p=1}^{n-1} G_{ip}(0) w_p, \quad i = 1, \cdots, n-1, \tag{2.40}$$

$$\frac{\partial w_n}{\partial t} + \frac{\partial(\lambda_n(u) w_n)}{\partial x} = \sum_{\substack{1 \le r,l \le n \\ r \ne l}} \Gamma_{nrl}(u) w_r w_l + \sum_{\substack{1 \le r,l \le n \\ (r,l) \ne (n,n)}} \xi_{nrl}(u) u_l w_r + \sum_{p=1}^{n-1} G_{np}(0) w_p. \tag{2.41}$$

Moreover, using (2.36) and (2.39), we can reduce (2.26) into

$$\frac{\partial w_i}{\partial t} + \lambda_i(u) \frac{\partial w_i}{\partial x} = \sum_{\substack{1 \le r,l \le n \\ r \ne l}} \Gamma_{irl}(u) w_r w_l - \sum_{r=1}^{n} \Xi_{ir}(u) w_i w_r + \sum_{\substack{1 \le r,l \le n \\ (r,l) \ne (n,n)}} \xi_{irl}(u) u_l w_r + \sum_{p=1}^{n-1} G_{ip}(0) w_p \quad i = 1, \cdots, n-1, \tag{2.42}$$

$$\frac{\partial w_n}{\partial t} + \lambda_n(u) \frac{\partial w_n}{\partial x} = \sum_{\substack{1 \le r,l \le n \\ r \ne l}} \Gamma_{nrl}(u) w_r w_l - \sum_{p=1}^{n-1} \Xi_{np}(u) w_n w_p - \sum_{p=1}^{n-1} (w_n \eta_{np}(u)) u_p w_n + \sum_{\substack{1 \le r,l \le n \\ (r,l) \ne (n,n)}} \xi_{nrl}(u) u_l w_r + \sum_{p=1}^{n-1} G_{np}(0) w_p. \tag{2.43}$$

2.3 Formulas of wave decomposition for higher order derivatives

In this subsection, we derive formulas of wave decomposition for higher-order derivatives of the solution.

Let

$$y_k = \frac{\partial u_k}{\partial x}, \quad k = 1, \cdots, n, \tag{2.44}$$

$$z_k = \frac{\partial w_k}{\partial x}, \quad k = 1, \cdots, n. \tag{2.45}$$

By (2.22), we have

$$y_k = \sum_{r=1}^{n} w_r r_{kr}(u), \quad \forall\, 1 \le k \le n. \tag{2.46}$$

Noting (2.16), we have

$$r_{in}(u_n e_n) \equiv 0, \quad \forall\, 1 \le i \le n-1.$$

So by Lemma 2.1, there exist functions $\phi_{inp}(u) \in C^2$ $(i, p = 1, \cdots, n-1)$, such that

$$r_{in}(u) = \sum_{p=1}^{n-1} \phi_{inp}(u) u_p, \quad \forall\, 1 \le i \le n-1.$$

Substituting this into (2.46), we obtain

$$y_i = \sum_{p=1}^{n-1} r_{ip}(u) w_p + \sum_{p=1}^{n-1} \phi_{inp}(u) u_p w_n, \quad \forall\, 1 \le i \le n-1. \tag{2.47}$$

In order to get the formulas of wave decomposition for z_i, taking the derivative with respect to x on both sides of (2.42), we have

$$\begin{aligned}
\frac{\partial z_i}{\partial t} + \frac{\partial(\lambda_i(u) z_i)}{\partial x} = {} & \sum_{\substack{1 \le r,l \le n \\ r \ne l}} \Gamma_{irl}(u)(z_r w_l + w_r z_l) + \sum_{\substack{1 \le r,l \le n \\ r \ne l}} \Big(\sum_{h=1}^{n} y_h \frac{\partial}{\partial u_h} \Gamma_{irl}(u) \Big) w_r w_l \\
& - \sum_{r=1}^{n} \Xi_{ir}(u)(z_i w_r + w_i z_r) - \sum_{r=1}^{n} \Big(\sum_{h=1}^{n} y_h \frac{\partial}{\partial u_h} \Xi_{ir}(u) \Big) w_i w_r \\
& + \sum_{\substack{1 \le r,l \le n \\ (r,l) \ne (n,n)}} \xi_{irl}(u)(y_l w_r + u_l z_r) + \sum_{\substack{1 \le r,l \le n \\ (r,l) \ne (n,n)}} \Big(\sum_{h=1}^{n} y_h \frac{\partial}{\partial u_h} \xi_{irl}(u) \Big) u_l w_r \\
& + \sum_{p=1}^{n-1} G_{ip}(0) z_p, \quad i = 1, \cdots, n-1.
\end{aligned} \tag{2.48}$$

Similarly, taking the derivative with respect to x on both sides of (2.43), we get

$$\begin{aligned}\frac{\partial z_n}{\partial t}+\frac{\partial(\lambda_n(u)z_n)}{\partial x}=&\sum_{\substack{1\le r,l\le n\\ r\ne l}}\Gamma_{nrl}(u)(z_rw_l+w_rz_l)+\sum_{\substack{1\le r,l\le n\\ r\ne l}}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\Gamma_{nrl}(u)\Big)w_rw_l\\&-\sum_{p=1}^{n-1}\Xi_{np}(u)(z_nw_p+w_nz_p)-\sum_{p=1}^{n-1}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\Xi_{np}(u)\Big)w_nw_p\\&-\sum_{p=1}^{n-1}(w_n\eta_{np}(u))(y_pw_n+2u_pz_n)-\sum_{p=1}^{n-1}\Big(w_n\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\eta_{np}(u)\Big)u_pw_n\\&+\sum_{\substack{1\le r,l\le n\\ (r,l)\ne(n,n)}}\xi_{nrl}(u)(y_lw_r+u_lz_r)+\sum_{\substack{1\le r,l\le n\\ (r,l)\ne(n,n)}}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\xi_{nrl}(u)\Big)u_lw_r\\&+\sum_{p=1}^{n-1}G_{np}(0)z_p.\end{aligned}\tag{2.49}$$

Noting (2.28) and (2.36), we can easily use (2.48)–(2.49) to obtain another two formulas of wave decomposition for z_k as

$$\begin{aligned}\frac{\partial z_i}{\partial t}+\lambda_i(u)\frac{\partial z_i}{\partial x}=&\sum_{\substack{1\le r,l\le n\\ r\ne l}}\Gamma_{irl}(u)(z_rw_l+w_rz_l)+\sum_{\substack{1\le r,l\le n\\ r\ne l}}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\Gamma_{irl}(u)\Big)w_rw_l\\&-\sum_{r=1}^{n}\Xi_{ir}(u)(2z_iw_r+w_iz_r)-\sum_{r=1}^{n}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\Xi_{ir}(u)\Big)w_iw_r\\&+\sum_{\substack{1\le r,l\le n\\ (r,l)\ne(n,n)}}\xi_{irl}(u)(y_lw_r+u_lz_r)+\sum_{\substack{1\le r,l\le n\\ (r,l)\ne(n,n)}}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\xi_{irl}(u)\Big)u_lw_r\\&+\sum_{p=1}^{n-1}G_{ip}(0)z_p,\quad i=1,\cdots,n-1,\end{aligned}\tag{2.50}$$

$$\begin{aligned}\frac{\partial z_n}{\partial t}+\lambda_n(u)\frac{\partial z_n}{\partial x}=&\sum_{\substack{1\le r,l\le n\\ r\ne l}}\Gamma_{nrl}(u)(z_rw_l+w_rz_l)+\sum_{\substack{1\le r,l\le n\\ r\ne l}}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\Gamma_{nrl}(u)\Big)w_rw_l\\&-\sum_{p=1}^{n-1}\Xi_{np}(u)(2z_nw_p+w_nz_p)-\sum_{p=1}^{n-1}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\Xi_{np}(u)\Big)w_nw_p\\&-\sum_{p=1}^{n-1}(w_n\eta_{np}(u))(y_pw_n+3u_pz_n)-\sum_{p=1}^{n-1}\Big(w_n\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\eta_{np}(u)\Big)u_pw_n\\&+\sum_{\substack{1\le r,l\le n\\ (r,l)\ne(n,n)}}\xi_{nrl}(u)(y_lw_r+u_lz_r)+\sum_{\substack{1\le r,l\le n\\ (r,l)\ne(n,n)}}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\xi_{nrl}(u)\Big)u_lw_r\\&+\sum_{p=1}^{n-1}G_{np}(0)z_p.\end{aligned}\tag{2.51}$$

Remark 2.3 All the coefficients in formulas (2.48)–(2.51) have already matched the requirements of reduction given in Subsection 2.2.

Remark 2.4 By taking the derivative with respect to x on both sides of (2.34)–(2.35) and noting (2.36), we can similarly obtain the formulas of wave decomposition for y_k. However, since relations (2.46)–(2.47) can be simply used to establish corresponding a priori estimates instead of these formulas, we omit them here.

3 Proof of the Main Theorem

In this section, we use the reduced formulas of wave decomposition, given in Subsections 2.2–2.3, to get a series of a priori estimates of the solution and complete the proof of Theorem 2.1, which leads to the validity of our main result, Theorem 1.1.

For any given $T \geq 0$, set

$$
\begin{aligned}
U_{D,a,b}(T) &= \max_{1\leq i\leq n-1} \|u_i\|_{L^a(0,T;L^b(\mathbb{R}))}, \quad a=1,\infty,\ b=2,\infty, &(3.1)\\
W_{D,a,b}(T) &= \max_{1\leq i\leq n-1} \|w_i\|_{L^a(0,T;L^b(\mathbb{R}))}, \quad a=1,\infty,\ b=2,\infty, &(3.2)\\
Y_{D,a,b}(T) &= \max_{1\leq i\leq n-1} \|y_i\|_{L^a(0,T;L^b(\mathbb{R}))}, \quad a=1,\infty,\ b=2,\infty, &(3.3)\\
Z_{D,a,2}(T) &= \max_{1\leq i\leq n-1} \|z_i\|_{L^a(0,T;L^2(\mathbb{R}))}, \quad a=1,\infty, &(3.4)\\
U_{L,\infty,b}(T) &= \|u_n\|_{L^\infty(0,T;L^b(\mathbb{R}))}, \quad b=2,\infty, &(3.5)\\
W_{L,\infty,b}(T) &= \|w_n\|_{L^\infty(0,T;L^b(\mathbb{R}))}, \quad b=2,\infty, &(3.6)\\
Y_{L,\infty,b}(T) &= \|y_n\|_{L^\infty(0,T;L^b(\mathbb{R}))}, \quad b=2,\infty, &(3.7)\\
Z_{L,\infty,2}(T) &= \|z_n\|_{L^\infty(0,T;L^2(\mathbb{R}))}, &(3.8)\\
U_{\infty,b}(T) &= \max\{U_{D,\infty,b}(T), U_{L,\infty,b}(T)\}, \quad b=2,\infty, &(3.9)\\
W_{\infty,b}(T) &= \max\{W_{D,\infty,b}(T), W_{L,\infty,b}(T)\}, \quad b=2,\infty, &(3.10)\\
Y_{\infty,b}(T) &= \max\{Y_{D,\infty,b}(T), Y_{L,\infty,b}(T)\}, \quad b=2,\infty, &(3.11)\\
Z_{\infty,2}(T) &= \max\{Z_{D,\infty,2}(T), Z_{L,\infty,2}(T)\}, &(3.12)\\
I(T) &= U_{\infty,2}(T) + U_{\infty,\infty}(T) + U_{D,1,2}(T)\\
&\quad + W_{\infty,2}(T) + W_{\infty,\infty}(T) + W_{D,1,2}(T)\\
&\quad + Y_{\infty,2}(T) + Y_{\infty,\infty}(T) + Y_{D,1,2}(T)\\
&\quad + Z_{\infty,2}(T) + Z_{D,1,2}(T). &(3.13)
\end{aligned}
$$

By (2.20)–(2.21) and (2.44)–(2.45), we have

$$I(0) \leq C\theta. \tag{3.14}$$

Here and hereafter, C stands for a positive constant independent of θ and T, but possibly depending on θ_0.

Now, we use a bootstrap argument to prove that there exists a constant $\theta_0\,(0<\theta_0<1)$ so small that for any given $\theta\,(0\leq\theta\leq\theta_0)$ and any given $T\geq 0$, we have

$$I(T) \leq C\theta^{\frac{3}{4}}, \tag{3.15}$$

i.e., for any given $T \geq 0$, we try to get (3.15) under the assumption

$$I(T) \leq \theta^{\frac{1}{2}}. \tag{3.16}$$

First, by (3.16) and noting (2.44)–(2.45), the Sobolev embedding theorem can be easily used to get

$$U_{D,1,\infty}(T) \leq C(U_{D,1,2}(T) + Y_{D,1,2}(T)) \leq C\theta^{\frac{1}{2}}, \tag{3.17}$$

$$W_{D,1,\infty}(T) \leq C(W_{D,1,2}(T) + Z_{D,1,2}(T)) \leq C\theta^{\frac{1}{2}}. \tag{3.18}$$

Then by (2.47), we have

$$Y_{D,1,\infty}(T) \leq CW_{D,1,\infty}(T) + CW_{\infty,\infty}(T)U_{D,1,\infty}(T) \leq C\theta^{\frac{1}{2}}. \tag{3.19}$$

Next, we estimate the dissipative part of the solution. Multiplying u_i on both sides of (2.34) and (2.37), summing up and integrating over $x \in \mathbb{R}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} u_i \cdot \Big(2 \sum_{\substack{1\leq r,l\leq n\\ r\neq l}} \psi_{irl}(u)u_l w_r + \sum_{r=1}^{n} \Xi_{ir}(u)u_i w_r + 2 \sum_{\substack{1\leq r,l\leq n\\ (r,l)\neq(n,n)}} \Phi_{irl}(u)u_r u_l \\ + 2\sum_{p=1}^{n-1} G_{ip}(0)u_p\Big)\mathrm{d}x, \quad i = 1,\cdots,n-1.\end{aligned}$$

Summing them up for $1 \leq i \leq n-1$ and noting (2.15), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{i=1}^{n-1}\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\Big) + 2\delta_0\Big(\sum_{i=1}^{n-1}\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\Big) \\ \leq{}& 2\sum_{i=1}^{n-1}\Big(\|u_i(t,\cdot)\|_{L^2(\mathbb{R})} \cdot \Big\|\Big(\sum_{\substack{1\leq r,l\leq n\\ r\neq l}} \psi_{irl}(u)u_l w_r + \frac{1}{2}\sum_{r=1}^{n}\Xi_{ir}(u)u_i w_r \\ &+ \sum_{\substack{1\leq r,l\leq n\\ (r,l)\neq(n,n)}} \Phi_{irl}(u)u_r u_l\Big)(t,\cdot)\Big\|_{L^2(\mathbb{R})}\Big).\end{aligned}$$

Similarly, by (2.40), (2.42) and (2.48), (2.50), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{i=1}^{n-1}\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\Big) + 2\delta_0\Big(\sum_{i=1}^{n-1}\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\Big) \\ \leq{}& 2\sum_{i=1}^{n-1}\Big(\|w_i(t,\cdot)\|_{L^2(\mathbb{R})} \cdot \Big\|\Big(\sum_{\substack{1\leq r,l\leq n\\ r\neq l}} \Gamma_{irl}(u)w_r w_l - \frac{1}{2}\sum_{r=1}^{n}\Xi_{ir}(u)w_i w_r \\ &+ \sum_{\substack{1\leq r,l\leq n\\ (r,l)\neq(n,n)}} \xi_{irl}(u)u_l w_r\Big)(t,\cdot)\Big\|_{L^2(\mathbb{R})}\Big)\end{aligned}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{i=1}^{n-1}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\Big)+2\delta_0\Big(\sum_{i=1}^{n-1}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\Big)\\
&\le 2\sum_{i=1}^{n-1}\Big(\|z_i(t,\cdot)\|_{L^2(\mathbb{R})}\cdot\Big\|\Big(\sum_{\substack{1\le r,l\le n\\ r\ne l}}\Gamma_{irl}(u)(z_rw_l+w_rz_l)\\
&\quad+\sum_{\substack{1\le r,l\le n\\ r\ne l}}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\Gamma_{irl}(u)\Big)w_rw_l-\sum_{r=1}^n\Xi_{ir}(u)\Big(\frac{3}{2}z_iw_r+w_iz_r\Big)\\
&\quad-\sum_{r=1}^n\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\Xi_{ir}(u)\Big)w_iw_r+\sum_{\substack{1\le r,l\le n\\ (r,l)\ne(n,n)}}\xi_{irl}(u)(y_lw_r+u_lz_r)\\
&\quad+\sum_{\substack{1\le r,l\le n\\ (r,l)\ne(n,n)}}\Big(\sum_{h=1}^n y_h\frac{\partial}{\partial u_h}\xi_{irl}(u)\Big)u_lw_r\Big)(t,\cdot)\Big\|_{L^2(\mathbb{R})}\Big),
\end{aligned}$$

respectively. Summing them up gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)\\
&\quad+2\delta_0\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)\\
&\le C(1+Y_{\infty,\infty}(T))\Big((U_{\infty,\infty}(T)+W_{\infty,\infty}(T)+Y_{\infty,\infty}(T))\\
&\quad\cdot\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|y_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)\\
&\quad+Z_{L,\infty,2}(T)\Big(\sum_{i=1}^{n-1}(\|z_i(t,\cdot)\|_{L^2(\mathbb{R})})\Big)\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|_{L^\infty(\mathbb{R})}+\|w_i(t,\cdot)\|_{L^\infty(\mathbb{R})})\Big)\Big).
\end{aligned}$$

By the Sobolev inequality and (2.47), we have

$$\begin{aligned}
&\|u_i(t,\cdot)\|_{L^\infty(\mathbb{R})}+\|w_i(t,\cdot)\|_{L^\infty(\mathbb{R})}\\
&\le C(\|u_i(t,\cdot)\|_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|_{L^2(\mathbb{R})}+\|y_i(t,\cdot)\|_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|_{L^2(\mathbb{R})})
\end{aligned}$$

and

$$\|y_i(t,\cdot)\|_{L^2(\mathbb{R})}\le C(\|w_i(t,\cdot)\|_{L^2(\mathbb{R})}+\|u_i(t,\cdot)\|_{L^2(\mathbb{R})}).$$

Thus,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)\\
&\quad+2\delta_0\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)\\
&\le C(1+Y_{\infty,\infty}(T))\Big((U_{\infty,\infty}(T)+W_{\infty,\infty}(T)+Y_{\infty,\infty}(T)+Z_{L,\infty,2}(T))
\end{aligned}$$

$$\cdot\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)\Big).$$

Noting (3.16), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)\\&+\delta_0\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)\le 0.\end{aligned}$$

Now we get the exponential decay of the dissipative part of the solution,

$$\begin{aligned}&\Big(\sum_{i=1}^{n-1}(\|u_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})})\Big)^{\frac12}\\&\le \mathrm{e}^{-\frac{\delta_0 t}{2}}\Big(\sum_{i=1}^{n-1}(\|u_i(0,\cdot)\|^2_{L^2(\mathbb{R})}+\|w_i(0,\cdot)\|^2_{L^2(\mathbb{R})}+\|z_i(0,\cdot)\|^2_{L^2(\mathbb{R})})\Big)^{\frac12},\quad \forall t\in[0,T],\end{aligned}$$

which obviously leads to

$$\begin{aligned}&U_{D,\infty,2}(T)+W_{D,\infty,2}(T)+Z_{D,\infty,2}(T)+U_{D,1,2}(T)+W_{D,1,2}(T)+Z_{D,1,2}(T)\\&\le C(U_{D,\infty,2}(0)+W_{D,\infty,2}(0)+Z_{D,\infty,2}(0))\le C\theta.\end{aligned}\tag{3.20}$$

By (2.47), we have

$$Y_{D,\infty,2}(T)+Y_{D,1,2}(T)\le C\theta.\tag{3.21}$$

Then by the Sobolev embedding theorem, it follows from (3.20)–(3.21) that

$$U_{D,\infty,\infty}(T)+U_{D,1,\infty}(T)\le C\theta,\tag{3.22}$$

$$W_{D,\infty,\infty}(T)+W_{D,1,\infty}(T)\le C\theta.\tag{3.23}$$

Using (2.47) again, we get

$$Y_{D,\infty,\infty}(T)+Y_{D,1,\infty}(T)\le C\theta.\tag{3.24}$$

At last, we estimate the nondissipative and weakly linearly degenerate part of the solution. Multiplying u_n on both sides of (2.35) and (2.38), summing up and integrating over $x\in\mathbb{R}$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&=\int_{\mathbb{R}}u_n\cdot\Big(2\sum_{1\le r,l\le n}\psi_{nrl}(u)u_lw_r+\sum_{p=1}^{n-1}\Xi_{np}(u)u_nw_p+\sum_{p=1}^{n-1}(u_n\eta_{np}(u))u_pw_n\\&\quad+2\sum_{\substack{1\le r,l\le n\\(r,l)\ne(n,n)}}\Phi_{nrl}(u)u_ru_l+2\sum_{p=1}^{n-1}G_{np}(0)u_p\Big)\mathrm{d}x\end{aligned}$$

$$\leq 2\|u_n(t,\cdot)\|_{L^2(\mathbb{R})}\cdot\Big(\Big\|\Big(\sum_{1\leq r,l\leq n}\psi_{nrl}(u)u_lw_r+\frac{1}{2}\sum_{p=1}^{n-1}\Xi_{np}(u)u_nw_p+\frac{1}{2}\sum_{p=1}^{n-1}(u_n\eta_{np}(u))u_pw_n$$
$$+\sum_{\substack{1\leq r,l\leq n\\(r,l)\neq(n,n)}}\Phi_{nrl}(u)u_ru_l\Big)(t,\cdot)\Big\|_{L^2(\mathbb{R})}+\Big\|\sum_{p=1}^{n-1}G_{np}(0)u_p(t,\cdot)\Big\|_{L^2(\mathbb{R})}\Big).$$

Integrating with respect to t from 0 to T, we obtain

$$\begin{aligned}U^2_{L,\infty,2}(T)&\leq U^2_{L,\infty,2}(0)+CU_{L,\infty,2}(T)((U_{\infty,\infty}(T)+W_{\infty,\infty}(T))\\&\quad\cdot(U_{D,1,2}(T)+W_{D,1,2}(T))+U_{D,1,2}(T))\\&\leq C\theta^{\frac{3}{2}}.\end{aligned}$$

Similar calculations for (2.41), (2.43), (2.49) and (2.51) lead to

$$\begin{aligned}W^2_{L,\infty,2}(T)&\leq W^2_{L,\infty,2}(0)+CW_{L,\infty,2}(T)((U_{\infty,\infty}(T)+W_{\infty,\infty}(T))\\&\quad\cdot(U_{D,1,2}(T)+W_{D,1,2}(T))+W_{D,1,2}(T))\\&\leq C\theta^{\frac{3}{2}},\\Z^2_{L,\infty,2}(T)&\leq Z^2_{L,\infty,2}(0)+CZ_{L,\infty,2}(T)((U_{\infty,\infty}(T)+W_{\infty,\infty}(T)+Y_{\infty,\infty}(T))\\&\quad\cdot(U_{D,1,2}(T)+W_{D,1,2}(T)+Y_{D,1,2}(T)+Z_{D,1,2}(T))\\&\quad+Z_{L,\infty,2}(T)(U_{D,1,\infty}(T)+W_{D,1,\infty}(T))+Z_{D,1,2}(T))\\&\leq C\theta^{\frac{3}{2}}.\end{aligned}$$

Consequently, we have

$$U_{L,\infty,2}(T)\leq C\theta^{\frac{3}{4}}, \tag{3.25}$$
$$W_{L,\infty,2}(T)\leq C\theta^{\frac{3}{4}}, \tag{3.26}$$
$$Z_{L,\infty,2}(T)\leq C\theta^{\frac{3}{4}}. \tag{3.27}$$

Then by (2.46), we have

$$Y_{L,\infty,2}(T)\leq C(W_{D,\infty,2}(T)+W_{L,\infty,2}(T))\leq C\theta^{\frac{3}{4}}. \tag{3.28}$$

Thus, we can use the Sobolev embedding theorem again to get

$$U_{L,\infty,\infty}(T)\leq C\theta^{\frac{3}{4}}, \tag{3.29}$$
$$W_{L,\infty,\infty}(T)\leq C\theta^{\frac{3}{4}}. \tag{3.30}$$

Finally, by (2.46) and (3.23), we have

$$Y_{L,\infty,\infty}(T)\leq C\theta^{\frac{3}{4}}. \tag{3.31}$$

Inequalities (3.20)–(3.31) give our desired estimate (3.15) which finishes the bootstrap argument and completes the proof of Theorem 2.1 and so Theorem 1.1.

4 Remarks

In this section, we give some remarks on several aspects of our main result, Theorem 1.1.

First, because of condition (1.10), as mentioned in [11], Shizuta-Kawashima condition proposed in [14], which plays an important role in [1–3, 13–18] et al., is violated. Moreover, from the finite-time singularity of the classical solution to the following problem given in [11],

$$\begin{cases} u_{1t} = u_1^2 u_2^2, \\ u_{2t} + u_{2x} = -u_2 + \dfrac{1}{2} u_1^m, \quad m \in \mathbb{N}, \\ t = 0: \ u_1 = \begin{cases} \varepsilon \mathrm{e}^{\frac{1}{|x|^2 - 1}}, & |x| \le 1, \\ 0, & |x| \ge 1, \end{cases} \quad u_2 = 0, \end{cases}$$

we know that (1.10) is indispensable for our result.

We can also treat (1.10) as a part of null condition that (1.6) and (1.10) guarantee that the nth characteristic satisfies the generalized null condition given in [6], i.e., in normalized coordinates, each nth traveling wave solution

$$\widetilde{u} = \widetilde{\varphi}(x - \widetilde{\lambda}_n(0)t)\widetilde{r}_n(0), \quad \widetilde{\varphi} \in C^1$$

to the linearized system

$$\frac{\partial \widetilde{u}}{\partial t} + \widetilde{A}(0) \frac{\partial \widetilde{u}}{\partial x} = 0$$

is also a solution to the corresponding quasilinear system (2.4).

Secondly, besides a condition similar to (1.10), there is another hypothesis in [11] to restrict the interactions between two parts of characteristics. In this paper, that hypothesis reads as

$$l_n(u^{(i)}(s))F(u^{(i)}(s)) \equiv 0 \quad \text{for any small } |s|, \forall\, 1 \le i \le n-1, \tag{4.1}$$

which can lead to one more reduction in Subsections 2.2–2.3 that no first order action from dissipative waves to nondissipative one exists, i.e.,

$$\widetilde{G}_{np}(0) = 0, \quad \forall\, 1 \le p \le n-1.$$

In this paper, we do not need such a hypothesis to get our result on the global well-posedness of the classical solution.

At last, we compare Theorem 1.1 with other results on dissipative systems. Comparing with [1–3, 13, 15, 19], we do not require the hypotheses on the structure of conservation laws or strictly convex entropy, as well as Shizuta-Kawashima condition. Comparing with [4, 6–7], we have only one characteristic that is not involved in the strict dissipation, and the dissipation is in the sense of positive definiteness other than diagonal dominance. Comparing with [11], we use the dissipation in the sense of positiveness other than diagonal dominance, and we need fewer assumptions on the interactions between two parts of characteristics, but the number of nondissipative wave is only one.

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