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Applications of the Kinetic Formulation for Scalar Conservation Laws with a Zero-Flux Type Boundary Condition*

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Abstract The authors are concerned with a zero-flux type initial boundary value problem for scalar conservation laws. Firstly, a kinetic formulation of entropy solutions is established. Secondly, by using the kinetic formulation and kinetic techniques, the uniqueness of entropy solutions is obtained. Finally, the parabolic approximation is studied and an error estimate of order $\eta^{\frac{1}{3}}$ between the entropy solution and the viscous approximate solutions is established by using kinetic techniques, where η is the size of artificial viscosity.

Keywords Scalar conservation laws, Entropy solutions, Kinetic formulation, Uniqueness, Error estimate
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1 Introduction

We are concerned with a zero-flux type initial boundary value problem for scalar conservation laws

$$\partial_t u + \mathrm{div} f(u) = 0, \quad (t,x) \in Q := (0,+\infty) \times \Omega, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$f(u) \cdot \mathbf{n} = 0, \quad (t,x) \in \Sigma := (0,+\infty) \times \partial\Omega, \tag{1.3}$$

where Ω is a bounded spatial domain in $\mathbb{R}^d$ with deformable Lipschitz boundary $\partial\Omega$ (see [8]), and $\mathbf{n}$ is the unit outer normal vector to the boundary $\partial\Omega$. $u = u(t,x)$ is an unknown function that is sought. The flux $f(u)$ is a smooth vector function which is genuinely nonlinear in the following sense:

$$\forall (\tau,\zeta) \in \mathbb{R} \times \mathbb{R}^d, \quad (\tau,\zeta) \neq (0,0) : \mathcal{L}(\{\xi \mid \tau + \zeta \cdot f'(\xi) = 0\}) = 0, \tag{1.4}$$

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where $\mathcal{L}$ denotes the one-dimensional Lebesgue measure.

Equation (1.1) arises in a variety of physical theories, primarily in dynamics, continuum mechanics and optics. As the flux is nonlinear, solutions to (1.1) may blow up in finite time, and it is also well-known that weak solutions to (1.1) are in general not uniquely determined by their initial data. In order to find the physically relevant discontinuous solution to (1.1), we should consider it in the sense of entropy solutions (see [14]).

A well-studied boundary condition is the Dirichlet boundary condition

$$u(t,x) = u^b(t,x), \quad (t,x) \in \Sigma, \tag{1.5}$$

where $u^b \in L^\infty(\Sigma)$. Since the value of the solution to (1.1) is a constant along characteristics and the characteristics from the interior of Ω may intersect $\partial\Omega$, (1.5) may not be assumed pointwise, and we should read (1.5) as an entropy condition on the boundary. The first analysis on existence and uniqueness of BV solutions to problem (1.1)–(1.2) and (1.5) is due to Bardos et al [2]. The BV property ensures the existence of boundary traces, which is crucial for the uniqueness result. In order to define a setting for more general data (namely, L^∞ data), a new definition has been given by Otto [20]. In this definition, the boundary condition is required to be held in an integral form by introducing appropriate boundary entropy-entropy flux pairs. These results were also extended to strongly degenerate parabolic-hyperbolic equations (see [7, 17–18] and the references cited therein). In [23], it is shown that L^∞ entropy solutions to (1.1) always have traces at the boundary of Q, no matter what initial and boundary conditions are assigned. Thus, L^∞ entropy solutions to problem (1.1)–(1.2) and (1.5) can also be defined as in [2], and the notion of entropy solutions used by Otto [20] can be avoided (see [15]).

Another kind of boundary condition that is prescribed to some physical problems is the zero-flux boundary condition (1.3), such as the sedimentation of suspensions in closed vessels (see [3, 5–6]) and the dispersal of a single species of animals in a finite territory (see [19]). R. Bürger et al [4] utilized the existence of strong traces of L^∞ entropy solutions to (1.1) to give the definition of L^∞ entropy solutions to (1.1)–(1.3). They proved the uniqueness of entropy solutions by using the Kružkov's device of doubling variables, and obtained the existence of entropy solutions by using the vanishing viscosity method.

A kinetic formulation of entropy solutions for scalar conservation law was first obtained by Lions et al [16] to the Cauchy problem, and Perthame [21] showed that the kinetic formulation supplies a good technical framework to easily prove the L^1-contraction property of entropy solutions and the error estimate with regard to the parabolic approximation, without using the Kružkov's device of doubling variables. For Dirichlet problem (1.1)–(1.2) and (1.5), Imbert and Vovelle [12] got the kinetic formulation of entropy solutions and proved the uniqueness of entropy solutions. An error estimate for the parabolic approximation of Dirichlet problem (1.1)–(1.2) and (1.5) was obtained by Droniou et al [11] under kinetic framework. The analogous results can be developed to the Cauchy problem of anisotropic degenerate parabolic-hyperbolic equation (see [9–10]), as well as the Dirichlet boundary problem of isotropic degenerate parabolic-hyperbolic equation (see [13]).

In this paper, we develop a kinetic formulation of entropy solutions to problem (1.1)–(1.3), and prove the uniqueness of entropy solutions. Meanwhile, we are also interested in the following parabolic approximation of problem (1.1)–(1.3):

$$\begin{cases} \partial_t v^\eta + \mathrm{div} f(v^\eta) = \eta \triangle v^\eta, & (t,x) \in Q, \\ v^\eta(0,x) = u_0(x), & x \in \Omega, \\ (f(v^\eta) - \eta \nabla v^\eta) \cdot \mathbf{n} = 0, & (t,x) \in \Sigma, \end{cases} \tag{1.6}$$

where η (> 0) is the size of artificial viscosity, and v^η is called a viscous approximate solution to (1.1)–(1.3). We use the kinetic formulation and kinetic techniques to obtain an error estimate of order $\eta^{\frac{1}{3}}$, which is the first result about the error estimate between the viscous approximate solution v^η and the entropy solution u. Hereafter, for narrative simplicity, we drop the superscript η in v^η.

The remaining part of this paper is organized as follows. Section 2 is devoted to some notations and assumptions. In Section 3, we introduce the definition of entropy solutions to problem (1.1)–(1.3) and establish the kinetic formulation of entropy solutions. In Section 4, the uniqueness of entropy solutions is proved. Section 5 is devoted to the study of the error estimate between the viscous approximate solution v and the entropy solution u.

2 Preliminaries

In this section, we give some notations and some assumptions that are used throughout the paper.

For Kružkov entropy $|u-k|$, the entropy fluxes are defined by

$$F(u,k) = \mathrm{sgn}(u-k)(f(u) - f(k)).$$

Set

$$\mathrm{sgn}^+(r) = \begin{cases} 1, & r > 0, \\ 0, & r \le 0, \end{cases}$$

$$\mathrm{sgn}^-(r) = \begin{cases} -1, & r < 0, \\ 0, & r \ge 0, \end{cases}$$

and $r^\pm = \mathrm{sgn}^\pm(r) r$. The semi-Kružkov entropies are the convex functions defined by

$$\eta_k^\pm(r) = (u-k)^\pm, \quad k \in \mathbb{R},$$

and the corresponding entropy fluxes are defined by

$$F^\pm(u,k) = \mathrm{sgn}^\pm(u-k)(f(u) - f(k)).$$

We define a kinetic function χ_u associated with the function u as in [12]:

$$\chi_u(t,x,\xi) = \begin{cases} 1, & 0 < \xi < u(t,x), \\ -1, & u(t,x) < \xi < 0, \\ 0, & \text{otherwise}. \end{cases}$$

Such a kinetic function is a so-called equilibrium function. Let

$$\chi_u^\pm(t,x,\xi) = \mathrm{sgn}^\pm(u(t,x) - \xi).$$

Then

$$\chi_u(t,x,\xi) = \chi_u^{\pm}(t,x,\xi) + \operatorname{sgn}^{\mp}(\xi). \tag{2.1}$$

We assume that Ω is a C^2 bounded open subset in $\mathbb{R}^d$. Thus, we can find a finite open cover $\{\mathcal{B}_i\}_{i=0}^N$ of $\overline{\Omega}$ and a partition of unity $\{\lambda_i\}_{i=0}^N$ on $\overline{\Omega}$ subordinate to $\{\mathcal{B}_i\}_{i=0}^N$ with $\mathcal{B}_0 \subseteq \Omega$ and $\{\mathcal{B}_i\}_{i=1}^N$ an open cover of the boundary of Ω, such that, for $i \geq 1$, up to a change of coordinates represented by an orthogonal matrix T_i, the set $\Omega \cap \mathcal{B}_i$ is the epigraph of a C^2 function $h_i : \mathbb{R}^{d-1} \to \mathbb{R}$, that is,

$$\begin{aligned}\Omega_{\lambda_i} &= \Omega \cap \mathcal{B}_i = \{x \in \mathcal{B}_i : (T_i x)_d > h_i(\overline{T_i x})\},\\ \partial\Omega_{\lambda_i} &= \partial\Omega \cap \mathcal{B}_i = \{x \in \mathcal{B}_i : (T_i x)_d = h_i(\overline{T_i x})\},\end{aligned}$$

where $x = (\overline{x}, x_d) \in \mathbb{R}^d$ and $\overline{x} = (x_1, \cdots, x_{d-1})$. For simplicity, we suppose that the change of coordinates is trivial: $T_i = \mathrm{Id}$. Below we will drop the index i for convenience. We also write $Q_\lambda = (0,+\infty)\times\Omega_\lambda$, $\Sigma_\lambda = (0,+\infty)\times\partial\Omega_\lambda$, $\Pi_\lambda = \{\overline{x} : x \in \operatorname{supp}(\lambda)\cap\Omega\}$, and $\Theta_\lambda = (0,+\infty)\times\Pi_\lambda$. We denote by $\mathbf{n}(\overline{x})$ the outward unit normal to $\partial\Omega_\lambda$ at a point $(\overline{x}, h(\overline{x}))$ of $\partial\Omega_\lambda$ and by $\mathrm{d}\sigma(\overline{x})$ the $(d-1)$-dimensional area element in $\partial\Omega_\lambda$ as follows:

$$\begin{aligned}\mathbf{n}(\overline{x}) &= (1 + |\nabla_{\overline{x}} h(\overline{x})|^2)^{-\frac{1}{2}}(\nabla_{\overline{x}} h(\overline{x}), -1),\\ \mathrm{d}\sigma(\overline{x}) &= (1 + |\nabla_{\overline{x}} h(\overline{x})|^2)^{\frac{1}{2}}\mathrm{d}\overline{x}.\end{aligned}$$

We introduce as in [12] a right-decentered regularizing kernel $\theta_\alpha(s) = \frac{1}{\alpha}\theta(\frac{s}{\alpha})$, where $\theta \in C_c^\infty((\varrho,1);\mathbb{R}^+)$ satisfies

$$\int_{\mathbb{R}} \theta(s)\mathrm{d}s = 1,$$

where $0 < \varrho < 1$. Set

$$\gamma_{\alpha,\epsilon}(t,x) = \theta_\alpha(t)\gamma_\epsilon(x) = \theta_\alpha(t)\widetilde{\gamma}_{\epsilon_1}(\overline{x})\theta_{\epsilon_2}(x_d),$$

where $\widetilde{\gamma}_{\epsilon_1}(\overline{x}) = \prod_{i=1}^{d-1}\theta_{\epsilon_1}(x_i)$. Consider now a function H defined on Q_λ and a function $\overline{H}$ defined on Σ_λ. Their regularized functions are, respectively, defined by

$$\begin{aligned}H^{\alpha,\epsilon}(t,x) &:= (H \times 1_{Q_\lambda}) * \gamma_{\alpha,\epsilon}(t,x) = \int_{Q_\lambda} H(s,y)\gamma_{\alpha,\epsilon}(t-s,x-y)\mathrm{d}s\mathrm{d}y,\\ \overline{H}^{\alpha,\epsilon}(t,x) &:= (\overline{H} \times 1_{\Sigma_\lambda}) * \gamma_{\alpha,\epsilon}(t,x) = \int_{\Sigma_\lambda} \overline{H}(s,y)\gamma_{\alpha,\epsilon}(t-s,x-y)\mathrm{d}s\mathrm{d}\sigma(\overline{y}),\end{aligned}$$

where 1_X stands for the characteristic function of the set X. We expect that these two functions vanish outside of Q^λ. For this purpose, in the proof of the uniqueness of entropy solutions, we choose ϵ_1 and ϵ_2 to satisfy the condition $\varrho\epsilon_2 \geq \sqrt{d}\epsilon_1\mathrm{Lip}(h)$ (see [12]).

3 Definitions of Entropy Solutions, Kinetic Formulation

In this section, we introduce three kinds of equivalent definitions of L^∞ entropy solutions to problem (1.1)–(1.3) and establish a kinetic formulation of entropy solutions.

According to [4], entropy solutions to problem (1.1)–(1.3) can be defined as follows.

Definition 3.1 *A function $u \in L^\infty(Q)$ is called an entropy solution to* (1.1)–(1.3) *if the following entropy inequality holds:*

$$\begin{aligned}&\int_Q \{|u-k|\partial_t\varphi + F(u,k)\cdot\nabla\varphi\}\mathrm{d}t\mathrm{d}x + \int_\Omega |u_0-k|\varphi(0,x)\mathrm{d}x \\ &+\int_\Sigma \mathrm{sgn}(u^\tau-k)f(k)\cdot\mathbf{n}\varphi\mathrm{d}t\mathrm{d}\sigma(\overline{x}) \geq 0 \end{aligned} \tag{3.1}$$

for any $k\in\mathbb{R}$ and $\varphi \in C_0^\infty(\overline{Q})$ with $\varphi \geq 0$, where $\overline{Q} := [0,+\infty)\times\overline{\Omega}$, and u^τ is the strong trace of u on Σ.

Definition 3.2 *A function $u \in L^\infty(Q)$ is called an entropy solution to* (1.1)–(1.3) *if the following conditions are staisfied:*

(D.1) (Interior Entropy Condition) $\forall k\in\mathbb{R}$ *and* $\forall\varphi\in C_0^\infty(Q)$ *with* $\varphi\geq 0$,

$$\int_Q \{|u-k|\partial_t\varphi + F(u,k)\cdot\nabla\varphi\}\mathrm{d}t\mathrm{d}x \geq 0. \tag{3.2}$$

(D.2) (Initial Condition) *The initial condition is assumed in the following strong L^1 sense:*

$$\operatorname*{esslim}_{t\to 0}\|u(t,\cdot)-u_0(\cdot)\|_{L^1(\mathbb{R}^d)} = 0.$$

(D.3) (Boundary Condition) *The boundary condition is satisfied in the following pointwise sense:*

$$f(u^\tau)\cdot\mathbf{n} = 0, \quad \textit{a.e. on } \Sigma. \tag{3.3}$$

Remark 3.1 The existence of strong traces for entropy solutions to (1.1) has been proved in [23].

Similarly as in [12] for the case of Dirichlet boundary condition, we can define entropy solutions to (1.1)–(1.3) with the aid of subsolution and supersolution.

Definition 3.3 *Consider a function $u\in L^\infty(Q)$.*

(D.1) *The function u is an entropy subsolution* (*resp. entropy supersolution*) *to* (1.1)–(1.3) *if*

$$\begin{aligned}&\int_Q \{(u-k)^\pm\partial_t\varphi + F^\pm(u,k)\cdot\nabla\varphi\}\mathrm{d}t\mathrm{d}x + \int_\Omega (u_0-k)^\pm\varphi(0,x)\mathrm{d}x \\ &+\int_\Sigma \mathrm{sgn}^\pm(u^\tau-k)f(k)\cdot\mathbf{n}\varphi\mathrm{d}t\mathrm{d}\sigma(\overline{x}) \geq 0, \end{aligned} \tag{3.4}$$

$\forall k\in\mathbb{R}$ *and* $\forall\varphi\in C_0^\infty(\overline{Q})$ *with* $\varphi\geq 0$.

(D.2) *The function u is an entropy solution to* $(1.1)-(1.3)$ *if it is both an entropy subsolution and an entropy supersolution.*

Remark 3.2 It is obvious that these three kinds of definitions of entropy solutions are equivalent to each other.

Now we establish the kinetic formulation of entropy solutions.

Proposition 3.1 *Let u be an entropy solution to* (1.1)–(1.3)*. Then there exists a bounded nonnegative entropy defect measure $m \in \mathcal{M}^+(Q \times \mathbb{R})$ such that m vanishes for $|\xi| \gg 1$, and for any $\phi \in C_0^\infty(\overline{Q} \times \mathbb{R})$,*

$$\begin{aligned}&\int_{Q\times\mathbb{R}} \chi_u(t,x,\xi)(\partial_t\phi + \mathbf{a}\cdot\nabla_x\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\Omega\times\mathbb{R}} \chi_{u_0}(x,\xi)\phi(0,x,\xi)\mathrm{d}x\mathrm{d}\xi \\ &+ \int_{\Sigma\times\mathbb{R}} \chi_{u^\tau}(t,x,\xi)(-\mathbf{a}\cdot\mathbf{n})\phi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi = \int_{Q\times\mathbb{R}} \partial_\xi\phi\mathrm{d}m,\end{aligned} \tag{3.5}$$

where $\mathbf{a} = (\overline{\mathbf{a}}, a_d) = f'$.

Proof Let us fix $k \in \mathbb{R}$ and define a linear form $m_\pm^k$ on $C_0^\infty(\overline{Q})$:

$$\begin{aligned}m_\pm^k(\varphi) :=& \int_Q \{(u-k)^\pm\partial_t\varphi + F^\pm(u,k)\cdot\nabla\varphi\}\mathrm{d}t\mathrm{d}x + \int_\Omega (u_0-k)^\pm\varphi(0,x)\mathrm{d}x \\ &+ \int_\Sigma \mathrm{sgn}^\pm(u^\tau - k)f(k)\cdot\mathbf{n}\varphi\mathrm{d}t\mathrm{d}\sigma(\overline{x}).\end{aligned} \tag{3.6}$$

Since u is an entropy solution, it is obvious that $m_\pm^k$ is nonnegative for any k and φ (≥ 0). Define measures $m^\pm(t,x,\xi)$ by

$$\int_{\overline{Q}\times\mathbb{R}} \phi\mathrm{d}m^\pm(t,x,\xi) = \int_{\mathbb{R}} m_\pm^\xi(\phi)\mathrm{d}\xi$$

for any $\phi \in C_0^\infty(\overline{Q}\times\mathbb{R})$. It is easy to obtain that $m^\pm$ is nonnegative measures on $\overline{Q}\times\mathbb{R}$ and $m^\pm$ vanish for $|\xi| \gg 1$.

For any $\phi \in C_0^\infty(\overline{Q}\times\mathbb{R})$, we have

$$\begin{aligned}&\int_{\overline{Q}\times\mathbb{R}} \partial_\xi\phi(t,x,\xi)\mathrm{d}m^\pm(t,x,\xi) \\ =& \int_{Q\times\mathbb{R}} \{(u-\xi)^\pm\partial_t\partial_\xi\phi + F^\pm(u,\xi)\cdot\nabla\partial_\xi\phi\}\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\Omega\times\mathbb{R}} (u_0-\xi)^\pm\partial_\xi\phi|_{t=0}\mathrm{d}x\mathrm{d}\xi \\ &+ \int_{\Sigma\times\mathbb{R}} \mathrm{sgn}^\pm(u^\tau-\xi)f(\xi)\cdot\mathbf{n}\partial_\xi\phi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi \\ =& \int_{Q\times\mathbb{R}} \mathrm{sgn}^\pm(u-\xi)(\partial_t\phi + \mathbf{a}\cdot\nabla\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\Omega\times\mathbb{R}} \mathrm{sgn}^\pm(u_0-\xi)\phi|_{t=0}\mathrm{d}x\mathrm{d}\xi \\ &+ \int_{\Sigma\times\mathbb{R}} \mathrm{sgn}^\pm(u^\tau-\xi)(-\mathbf{a}\cdot\mathbf{n})\phi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi \\ =& \int_{Q\times\mathbb{R}} \chi_u^\pm(t,x,\xi)(\partial_t\phi + \mathbf{a}\cdot\nabla\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\Omega\times\mathbb{R}} \chi_{u_0}^\pm(x,\xi)\phi|_{t=0}\mathrm{d}x\mathrm{d}\xi \\ &+ \int_{\Sigma\times\mathbb{R}} \chi_{u^\tau}^\pm(t,x,\xi)(-\mathbf{a}\cdot\mathbf{n})\phi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi.\end{aligned}$$

From property (2.1) of the equilibrium function, it is easy to deduce that $m^+ = m^-$. Thus we take the entropy defect measure $m = m^+ = m^-$, and we have

$$\begin{aligned}&\int_{Q\times\mathbb{R}} \chi_u(t,x,\xi)(\partial_t\phi + \mathbf{a}\cdot\nabla_x\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\Omega\times\mathbb{R}} \chi_{u_0}(x,\xi)\phi|_{t=0}\mathrm{d}x\mathrm{d}\xi \\ &+ \int_{\Sigma\times\mathbb{R}} \chi_{u^\tau}(t,x,\xi)(-\mathbf{a}\cdot\mathbf{n})\phi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi = \int_{\overline{Q}\times\mathbb{R}} \partial_\xi\phi\mathrm{d}m\mathrm{d}\xi.\end{aligned} \tag{3.7}$$

Now we only need to prove that $m|_{\{0\}\times\Omega\times\mathbb{R}}=0$ and $m|_{\Sigma\times\mathbb{R}}=0$. For any $\psi\in C_0^\infty(Q\times\mathbb{R})$, from (3.7) we have

$$\int_{Q\times\mathbb{R}}\chi_u(t,x,\xi)(\partial_t\psi+\mathbf{a}\cdot\nabla\psi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi=\int_{Q\times\mathbb{R}}\partial_\xi\psi\mathrm{d}m. \tag{3.8}$$

Let $\omega_\alpha(t)=\int_0^t\theta_\alpha(s)\mathrm{d}s$ and $\phi\in C_0^\infty([0,T)\times\Omega\times\mathbb{R})$. Applying the test function $\omega_\alpha(t)\phi(t,x,\xi)$ to (3.8), we have

$$\begin{aligned}&\int_{Q\times\mathbb{R}}\omega_\alpha(t)\chi_u(t,x,\xi)(\partial_t\phi+\mathbf{a}\cdot\nabla\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi+\int_{Q\times\mathbb{R}}\theta_\alpha(t)\chi_u(t,x,\xi)\phi\mathrm{d}t\mathrm{d}x\mathrm{d}\xi\\&=\int_{Q\times\mathbb{R}}\omega_\alpha(t)\partial_\xi\phi\mathrm{d}m.\end{aligned}$$

Letting $\alpha\to0+$ and using the Lebesgue dominated convergence theorem, we can obtain

$$\begin{aligned}&\int_{Q\times\mathbb{R}}\chi_u(t,x,\xi)(\partial_t\phi+\mathbf{a}\cdot\nabla\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi+\int_{\Omega\times\mathbb{R}}\Big(\lim_{t\to0}\chi_u(t,x,\xi)\Big)\phi|_{t=0}\mathrm{d}x\mathrm{d}\xi\\&=\int_{Q\times\mathbb{R}}\partial_\xi\phi\mathrm{d}m.\end{aligned} \tag{3.9}$$

Next we set $\omega_\alpha(x)=\int_0^{x_d-h(\overline{x})}\theta_\alpha(s)\mathrm{d}s$, $\phi\in C_0^\infty(\overline{Q}\times\mathbb{R})$ and $\phi^\lambda=\phi\lambda$. Applying the test function $\phi^\lambda\omega_\alpha$ to (3.9), we have

$$\begin{aligned}&\int_{Q_\lambda\times\mathbb{R}}\omega_\alpha(x)\chi_u(t,x,\xi)(\partial_t\phi^\lambda+\mathbf{a}\cdot\nabla\phi^\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi+\int_{Q_\lambda\times\mathbb{R}}\phi^\lambda\chi_u(t,x,\xi)\mathbf{a}\cdot\nabla\omega_\alpha(x)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi\\&+\int_{\Omega_\lambda\times\mathbb{R}}\omega_\alpha(x)\Big(\lim_{t\to0}\chi_u(t,x,\xi)\Big)\phi^\lambda\Big|_{t=0}\mathrm{d}x\mathrm{d}\xi=\int_{Q_\lambda\times\mathbb{R}}\omega_\alpha(x)\partial_\xi\phi^\lambda\mathrm{d}m.\end{aligned} \tag{3.10}$$

As $\alpha\to0$, (3.10) implies that

$$\begin{aligned}&\int_{Q_\lambda\times\mathbb{R}}\chi_u(t,x,\xi)(\partial_t\phi^\lambda+\mathbf{a}\cdot\nabla_x\phi^\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi+\int_{\Omega_\lambda\times\mathbb{R}}\Big(\lim_{t\to0}\chi_u(t,x,\xi)\Big)\phi^\lambda\Big|_{t=0}\mathrm{d}x\mathrm{d}\xi\\&+\int_{\Sigma_\lambda\times\mathbb{R}}\Big(\lim_{x_d\to h(\overline{x})}\chi_u(t,x,\xi)\Big)\phi^\lambda(-\mathbf{a}\cdot\mathbf{n})\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi=\int_{Q_\lambda\times\mathbb{R}}\partial_\xi\phi^\lambda\mathrm{d}m.\end{aligned} \tag{3.11}$$

By the existence of strong trace in [23] and the same arguments in [12], from (3.7) and (3.11), we deduce that $\partial_\xi m|_{\{0\}\times\Omega\times\mathbb{R}}=0$ and $\partial_\xi m|_{\Sigma\times\mathbb{R}}=0$, which implies that both $m|_{\{0\}\times\Omega\times\mathbb{R}}$ and $m|_{\Sigma\times\mathbb{R}}$ are constants with respect to ξ. Along with the facts that $m|_{\{0\}\times\Omega\times\mathbb{R}}=0$ and $m|_{\Sigma\times\mathbb{R}}=0$ as $|\xi|\gg1$, we can deduce that $m|_{\{0\}\times\Omega\times\mathbb{R}}\equiv0$ and $m|_{\Sigma\times\mathbb{R}}\equiv0$. Thus we can obtain (3.5) from (3.7).

Remark 3.3 It is obvious that

$$\begin{aligned}&\int_{Q\times\mathbb{R}}\chi_u^\pm(t,x,\xi)(\partial_t\phi+\mathbf{a}\cdot\nabla_x\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi+\int_{\Omega\times\mathbb{R}}\chi_{u_0}^\pm(x,\xi)\phi|_{t=0}\mathrm{d}x\mathrm{d}\xi\\&+\int_{\Sigma\times\mathbb{R}}\chi_{u^\tau}^\pm(t,x,\xi)(-\mathbf{a}\cdot\mathbf{n})\phi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi=\int_{Q\times\mathbb{R}}\partial_\xi\phi\mathrm{d}m.\end{aligned} \tag{3.12}$$

In [11], a kinetic formulation was established for the viscous approximate solution to problem (1.1)–(1.2) and (1.5), while we can obtain a kinetic formulation for the viscous approximate solution to problem (1.1)–(1.3), namely, the solution to problem (1.6), in which the Dirichlet boundary condition in [11] is replaced by the natural zero-flux boundary condition in (1.6).

Proposition 3.2 *Suppose that v is a solution to* (1.6)*. Then for any $\varphi \in C_0^\infty(\overline{Q})$ and $\psi \in C_0^\infty(\mathbb{R})$,*

$$\begin{aligned}&\int_{Q\times\mathbb{R}} \chi_v^\pm(t,x,\xi)(\partial_t\phi + \mathbf{a}\cdot\nabla_x\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi - \int_{Q\times\mathbb{R}} \eta\delta(v-\xi)\nabla v\cdot\nabla\phi\mathrm{d}t\mathrm{d}x\mathrm{d}\xi\\&+\int_{\Omega\times\mathbb{R}} \chi_{u_0}^\pm \phi|_{t=0}\mathrm{d}x\mathrm{d}\xi + \int_{\Sigma\times\mathbb{R}} \mathcal{G}_\pm\phi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi = \int_{Q\times\mathbb{R}} \partial_\xi\phi\mathrm{d}p,\end{aligned} \tag{3.13}$$

where $\phi(t,x,\xi) = \varphi(t,x)\psi(\xi)$, $\mathcal{G}_\pm(t,x,\xi) = (-\mathbf{a}\cdot\mathbf{n})\chi_v^\pm + \eta\delta(v-\xi)\nabla v\cdot\mathbf{n}$ and $\mathrm{d}p = \eta\delta(v-\xi)|\nabla v|^2\mathrm{d}t\mathrm{d}x\mathrm{d}\xi$. Here δ is the Dirac function.

Proof Let

$$\begin{aligned}E(\zeta) &= \int_{\mathbb{R}} \psi(\xi)\mathrm{sgn}^\pm(\zeta-\xi)\mathrm{d}\xi,\\ H(\zeta) &= \int_{\mathbb{R}} \mathbf{a}(\xi)\psi(\xi)\mathrm{sgn}^\pm(\zeta-\xi)\mathrm{d}\xi.\end{aligned}$$

It is easy to know that $E' = \psi$ and $H' = E'\mathbf{a}$. Multipling equation (1.6) by $\varphi(t,x)\psi(v(t,x))$, and integrating over Q, we have

$$\begin{aligned}&\int_Q (E(v)\partial_t\varphi + H(v)\cdot\nabla\varphi)\mathrm{d}t\mathrm{d}x + \int_\Omega E(u_0)\varphi(0,x)\mathrm{d}x\\&-\int_\Sigma H(v)\cdot\mathbf{n}\varphi\mathrm{d}t\mathrm{d}\sigma(\overline{x}) + \int_\Sigma \eta E'(v)\nabla v\cdot\mathbf{n}\varphi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\\&= \int_Q \eta E'(v)\nabla v\cdot\nabla\varphi\mathrm{d}t\mathrm{d}x + \int_Q E''(v)|\nabla v|^2\varphi\mathrm{d}t\mathrm{d}x.\end{aligned} \tag{3.14}$$

Using the definition of E and H, (3.13) follows from (3.14).

4 Uniqueness of Entropy Solutions

In this section, we prove the uniqueness of entropy solutions to problem (1.1)–(1.3) under the kinetic framework.

Theorem 4.1 *Let u and v be entropy solutions to* (1.1)–(1.3) *with the initial data u_0 and v_0, respectively. Then*

$$\int_\Omega |u(t,x)-v(t,x)|\mathrm{d}x \le \int_\Omega |u_0(x)-v_0(x)|\mathrm{d}x \tag{4.1}$$

for a.e. $t\in(0,+\infty)$. In particular, the entropy solution to (1.1)–(1.3) *is unique.*

Proof Since u is an entropy solution to (1.1)–(1.3), for any $\phi \in C_0^\infty(\overline{Q}\times\mathbb{R})$, we have

$$\begin{aligned}&\int_{Q\times\mathbb{R}} \chi_u^+(t,x,\xi)(\partial_t\phi + \mathbf{a}\cdot\nabla_x\phi)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\Omega\times\mathbb{R}} \chi_{u_0}^+(x,\xi)\phi|_{t=0}\mathrm{d}x\mathrm{d}\xi\\&+\int_{\Sigma\times\mathbb{R}} \chi_{u^\tau}^+(t,x,\xi)(-\mathbf{a}\cdot\mathbf{n})\phi\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi = \int_{Q\times\mathbb{R}} \partial_\xi\phi\mathrm{d}m_u,\end{aligned} \tag{4.2}$$

where m_u denotes the entropy defect measure with respect to u.

We choose $\phi^\lambda * \check{\gamma}_{\alpha,\epsilon}$ as the test function in (4.2), where $\check{\gamma}_{\alpha,\epsilon}(t,x,\xi) = \gamma_{\alpha,\epsilon}(-t,-x,-\xi)$ and $\phi^\lambda = \lambda\phi$ with $\phi \in C_0^\infty(\overline{Q} \times \mathbb{R})$ and $\phi \geq 0$. Thus, (4.2) implies

$$\int_{\mathbb{R}^{d+2}} (\chi_u^+(t,x,\xi))^{\alpha,\epsilon}(\partial_t\phi^\lambda + \mathbf{a}\cdot\nabla_x\phi^\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\mathbb{R}^{d+2}} (\chi_{u_0}^+(x,\xi))^\epsilon\theta_\alpha\phi^\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^{d+2}} (\chi_{u^\tau}^+(t,x,\xi)(-\mathbf{a}\cdot\mathbf{n}))^{\alpha,\epsilon}\phi^\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi = \int_{\mathbb{R}^{d+2}} \partial_\xi\phi^\lambda\mathrm{d}m_u^{\alpha,\epsilon}. \tag{4.3}$$

Similarly, for the entropy solution v, we have

$$\int_{\mathbb{R}^{d+2}} (\chi_v^-(t,x,\xi))^{\beta,\mu}(\partial_t\phi^\lambda + \mathbf{a}\cdot\nabla_x\phi^\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\mathbb{R}^{d+2}} (\chi_{v_0}^-(x,\xi))^\mu\theta_\beta\phi^\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^{d+2}} (\chi_{v^\tau}^-(t,x,\xi)(-\mathbf{a}\cdot\mathbf{n}))^{\beta,\mu}\phi^\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi = \int_{\mathbb{R}^{d+2}} \partial_\xi\phi^\lambda\mathrm{d}m_v^{\beta,\mu}. \tag{4.4}$$

Next we take $\phi = -(\chi_v^-)^{\beta,\mu}\varphi$ and $\phi = -(\chi_u^+)^{\alpha,\epsilon}\varphi$ $(0 \leq \varphi \in C_0^\infty(\overline{Q}))$ in (4.3) and (4.4), respectively, and add them together. From the fact that $-(\chi_u^+)^{\alpha,\epsilon}$ and $-(\chi_v^-)^{\beta,\mu}$ are non-decreasing with respect to ξ, it is easy to have

$$\int_{\mathbb{R}^{d+2}} (-(\chi_u^+)^{\alpha,\epsilon}(\chi_v^-)^{\beta,\mu})(\partial_t\varphi^\lambda + \mathbf{a}\cdot\nabla_x\varphi^\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^{d+2}} (-(\chi_{u_0}^+)^\epsilon(\chi_v^-)^{\beta,\mu}\theta(\alpha) - (\chi_{v_0}^+)^\mu(\chi_u^+)^{\alpha,\epsilon}\theta_\beta)\varphi^\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^{d+2}} (-(\chi_{u^\tau}^+(-\mathbf{a}\cdot\mathbf{n}))^{\alpha,\epsilon}(\chi_v^-)^{\beta,\mu} - (\chi_{v^\tau}^-(-\mathbf{a}\cdot\mathbf{n}))^{\beta,\mu}(\chi_u^+)^{\alpha,\epsilon})\varphi^\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi \geq 0. \tag{4.5}$$

Letting successively β, μ_1 and μ_2 go to $0+$, and using the facts that the right-decentered regularized functions vanish at $t = 0$ and on the boundary, we obtain

$$\int_{\mathbb{R}^{d+2}} (-(\chi_u^+)^{\alpha,\epsilon}\chi_v^-)(\partial_t\varphi^\lambda + \mathbf{a}\cdot\nabla_x\varphi^\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\mathbb{R}^{d+2}} -(\chi_{u_0}^+)^\epsilon\chi_v^-\theta(\alpha)\varphi^\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^{d+2}} -(\chi_{u^\tau}^+(-\mathbf{a}\cdot\mathbf{n}))^{\alpha,\epsilon}\chi_v^-\varphi^\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi \geq 0. \tag{4.6}$$

Letting successively α, ϵ_1 and ϵ_2 go to $0+$, we have

$$\int_{Q_\lambda\times\mathbb{R}} (-\chi_u^+\chi_v^-)(\partial_t\varphi^\lambda + \mathbf{a}\cdot\nabla_x\varphi^\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\Omega_\lambda\times\mathbb{R}} (-\chi_{u_0}^+\chi_{v_0}^-)\varphi^\lambda(0,x)\mathrm{d}x\mathrm{d}\xi$$
$$+ \int_{\Sigma_\lambda\times\mathbb{R}} (-\mathbf{a}\cdot\mathbf{n})(-\chi_{u^\tau}^+\chi_{v^\tau}^-)\varphi^\lambda\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi \geq 0.$$

Since

$$\int_{Q_\lambda\times\mathbb{R}} (-\chi_u^+\chi_v^-)(\partial_t\varphi^\lambda + \mathbf{a}\cdot\nabla_x\varphi^\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi = \int_{Q_\lambda} ((u-v)^+\partial_t\varphi^\lambda + F^+(u,v)\cdot\nabla_x\varphi^\lambda)\mathrm{d}t\mathrm{d}x,$$
$$\int_{\Sigma_\lambda\times\mathbb{R}} (-\mathbf{a}\cdot\mathbf{n})(-\chi_{u^\tau}^+\chi_{v^\tau}^-)\varphi^\lambda\mathrm{d}t\mathrm{d}\sigma(\overline{x})\mathrm{d}\xi = \int_{\Sigma_\lambda} \mathrm{sgn}^+(u^\tau - v^\tau)(f(u^\tau) - f(v^\tau))\cdot\mathbf{n}\varphi^\lambda\mathrm{d}t\mathrm{d}\sigma(\overline{x}),$$
$$\int_{\Omega_\lambda\times\mathbb{R}} (-\chi_{u_0}^+\chi_{v_0}^-)\varphi^\lambda(0,x)\mathrm{d}x\mathrm{d}\xi = \int_{\Omega_\lambda} (u_0 - v_0)^+\varphi^\lambda(0,x)\mathrm{d}x,$$

and $f(u^\tau)\cdot\mathbf{n} = 0$, $f(v^\tau)\cdot\mathbf{n} = 0$ a.e. on Σ, we have

$$\int_{Q_\lambda} ((u-v)^+\partial_t\varphi^\lambda + F^+(u,v)\cdot\nabla_x\varphi^\lambda)\mathrm{d}t\mathrm{d}x + \int_{\Omega_\lambda} (u_0 - v_0)^+\varphi^\lambda(0,x)\mathrm{d}x \geq 0. \tag{4.7}$$

Summing over λ in (4.7), we get

$$\int_Q ((u-v)^+\partial_t\varphi + F^+(u,v)\cdot\nabla_x\varphi)\mathrm{d}t\mathrm{d}x + \int_\Omega (u_0-v_0)^+\varphi(0,x)\mathrm{d}x \geq 0 \tag{4.8}$$

for any $\varphi \in C_0^\infty(\overline{Q})$ with $\varphi \geq 0$.

Now we choose a sequence of functions $0 \leq \varphi_n \in C_0^\infty((0,+\infty)\times\overline{\Omega})$ which converges to $1_{(0,\tau)\times\overline{\Omega}}$. It is easy to get

$$\lim_{n\to+\infty}\partial_t\varphi_n = \delta(t)-\delta(t-\tau), \quad \lim_{n\to+\infty}\partial_{x_i}\varphi_n = 0$$

for $1 \leq i \leq d$. Letting φ_n be the test function in (4.8) and sending $n \to +\infty$ lead to

$$-\int_\Omega (u(\tau,x)-v(\tau,x))^+\mathrm{d}x + \int_\Omega (u_0-v_0)^+\mathrm{d}x \geq 0,$$

i.e., for a.e. $t \in (0,+\infty)$,

$$\int_\Omega (u(t,x)-v(t,x))^+\mathrm{d}x \leq \int_\Omega (u_0(x)-v_0(x))^+\mathrm{d}x. \tag{4.9}$$

Similarly, we also have, for a.e. $t \in (0,+\infty)$,

$$\int_\Omega (u(t,x)-v(t,x))^-\mathrm{d}x \leq \int_\Omega (u_0(x)-v_0(x))^-\mathrm{d}x. \tag{4.10}$$

Therefore, (4.1) is proved.

5 An Error Estimate

In this section, we establish an error estimate between the viscous approximate solution v and the entropy solution u under the kinetic framework.

In [4], the existence of entropy solution to (1.1)–(1.3) is obtained as the limit of solutions of the corresponding regularized problems (1.6). Now, we want to consider the convergence rate of the viscous approximate solution to the entropy solution. To this end, we make some assumptions as follows.

Assumption 5.1 *Let $f \in C^2(\mathbb{R})$ and $u_0 \in C^2(\overline{\Omega})$. There exists a constant C depending only on (Ω, u_0, f, T), such that*

(A.1) $\|u\|_{L^\infty} \leq C$, $\|v\|_{L^\infty} \leq C$,

(A.2) $\forall t \in (0,T)$, $\int_\Omega |\partial_t u(t,x)|\mathrm{d}x \leq C$ *and* $\int_\Omega |\partial_t v(t,x)|\mathrm{d}x \leq C$,

(A.3) $\forall t \in (0,T)$, $|u(t,\cdot)|_{BV(\Omega)} \leq C$ *and* $|v(t,\cdot)|_{BV(\Omega)} \leq C$.

Hereinafter, C is a generic positive constant.

Theorem 5.1 *Let u and v be the entropy solutions to problem* (1.1)–(1.3) *and problem* (1.6)*, respectively, satisfying Assumption* 5.1. *Let $T > 0$. Then there exists a positive constant C depending only on (Ω, u_0, f, T), such that for a.e. $t \in (0,T)$,*

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1(\Omega)} \leq C\eta^{\frac{1}{3}}. \tag{5.1}$$

Proof Estimate in the interior of the domain can be obtained in a way similar to that of problem (1.1)–(1.2) and (1.5) (see [11]), and then for a.e. $t' \in (0, T)$, we have

$$\int_\Omega (u(t', x) - v(t', x))^+ \lambda_0(x) \mathrm{d}x \le C\eta^{\frac{1}{2}} + C \int_0^{t'} \int_\Omega (u(t, x) - v(t, x))^+ \mathrm{d}t\mathrm{d}x. \tag{5.2}$$

Now we only need to derive the estimate near the boundary of the domain. We choose a chart $(\Omega_\lambda, h, \lambda)$, and for simplicity, we only consider the special case $h \equiv 0$ and $\mathbf{n} = (0, \cdots, 0, -1)$. General cases can be treated by using the same arguments as in [11].

Since u is an entropy solution to (1.1)–(1.3), from (4.3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} (\chi_u^+(t, x, \xi))^{\alpha,\epsilon} (\partial_t \phi^\lambda + \mathbf{a} \cdot \nabla_x \phi^\lambda) \mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\mathbb{R}^{d+2}} (\chi_{u_0}^+(x, \xi))^\epsilon \theta_\alpha(t) \phi^\lambda \mathrm{d}t\mathrm{d}x\mathrm{d}\xi \\
&+ \int_{\mathbb{R}^{d+2}} (a_d \chi_{u^\tau}^+(t, x, \xi))^{\alpha,\epsilon_1} \theta_{\epsilon_2}(x_d) \phi^\lambda \mathrm{d}t\mathrm{d}x\mathrm{d}\xi = \int_{\mathbb{R}^{d+2}} \partial_\xi \phi^\lambda \mathrm{d}m^{\alpha,\epsilon}
\end{aligned} \tag{5.3}$$

for any $\phi \in C_0^\infty(\overline{Q} \times \mathbb{R})$.

Similarly, from (3.13), we can also get

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} (\chi_v^-(t, x, \xi))^{\beta,\mu} (\partial_t \phi^\lambda + \mathbf{a} \cdot \nabla_x \phi^\lambda) \mathrm{d}t\mathrm{d}x\mathrm{d}\xi + \int_{\mathbb{R}^{d+2}} (\chi_{u_0}^-(x, \xi))^\mu \theta_\beta(t) \phi^\lambda \mathrm{d}t\mathrm{d}x\mathrm{d}\xi \\
&+ \int_{\mathbb{R}^{d+2}} (a_d \chi_{v^\tau}^-(t, x, \xi) - \eta\delta(v^\tau - \xi)\partial_{x_d} v)^{\beta,\mu_1} \theta_{\mu_2}(x_d) \phi^\lambda \mathrm{d}t\mathrm{d}x\mathrm{d}\xi \\
&- \int_{Q_\lambda \times \mathbb{R}} \eta\delta(v - \xi) \nabla v \cdot \nabla(\phi^\lambda * \check{\gamma}_{\beta,\mu}) \mathrm{d}t\mathrm{d}x\mathrm{d}\xi = \int_{\mathbb{R}^{d+2}} \partial_\xi \phi^\lambda \mathrm{d}p^{\beta,\mu}.
\end{aligned} \tag{5.4}$$

Suppose that $\varphi \in C_0^\infty(\overline{Q})$ with $\varphi \ge 0$, take $\phi = -(\chi_v^-)^{\beta,\mu}\varphi$ and $\phi = -(\chi_u^+)^{\alpha,\epsilon}\varphi$ in (5.3) and (5.4), respectively, and add them together. By the fact that $-(\chi_u^+)^{\alpha,\epsilon}$ and $-(\chi_v^-)^{\beta,\mu}$ are non-decreasing with respect to ξ, it is easy to deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} (-(\chi_u^+)^{\alpha,\epsilon} (\chi_v^-)^{\beta,\mu}) (\partial_t \varphi^\lambda + \mathbf{a} \cdot \nabla \varphi^\lambda) \mathrm{d}t\mathrm{d}x\mathrm{d}\xi \\
&+ \int_{\mathbb{R}^{d+2}} (-(a_d \chi_{u^\tau}^+)^{\alpha,\epsilon_1} \theta_{\epsilon_2} (\chi_v^-)^{\beta,\mu} \varphi^\lambda - (a_d \chi_{v^\tau}^- - \eta\delta(v^\tau - \xi)\partial_{x_d} v)^{\beta,\mu_1} \theta_{\mu_2} (\chi_u^+)^{\alpha,\epsilon} \varphi^\lambda) \mathrm{d}t\mathrm{d}x\mathrm{d}\xi \\
&+ \int_{\mathbb{R}^{d+2}} (-(\chi_{u_0}^+)^\epsilon (\chi_v^-)^{\beta,\mu} \theta_\alpha - (\chi_{u_0}^-)^\mu (\chi_u^+)^{\alpha,\epsilon} \theta_\beta) \varphi^\lambda \mathrm{d}t\mathrm{d}x\mathrm{d}\xi \\
&- \eta \int_{Q_\lambda \times \mathbb{R}} \delta(v - \xi) \nabla v \cdot \nabla((-(\chi_u^+)^{\alpha,\epsilon} \varphi^\lambda) * (\check{\gamma}_\mu \check{\theta}_\beta)) \ge 0.
\end{aligned} \tag{5.5}$$

Since $(\chi_u^+)^{\alpha,\epsilon}$ and $(\chi_v^-)^{\beta,\mu}$ vanish outside of Q_λ, letting $\beta, \mu_1, \mu_2 \to 0$, we have

$$\begin{aligned}
&\int_{Q_\lambda \times \mathbb{R}} (-(\chi_u^+)^{\alpha,\epsilon} \chi_v^- (\partial_t \varphi^\lambda + \mathbf{a} \cdot \nabla \varphi^\lambda) - (\chi_{u_0}^+)^\epsilon \chi_v^- \theta_\alpha \varphi^\lambda - (a_d \chi_{u^\tau}^+)^{\alpha,\epsilon_1} \theta_{\epsilon_2} \chi_v^- \varphi^\lambda) \mathrm{d}t\mathrm{d}x\mathrm{d}\xi \\
&- \eta \int_{Q_\lambda \times \mathbb{R}} \delta(v - \xi) \nabla v \cdot \nabla(-(\chi_u^+)^{\alpha,\epsilon} \varphi^\lambda) \mathrm{d}t\mathrm{d}x\mathrm{d}\xi \ge 0.
\end{aligned} \tag{5.6}$$

Now we choose a sequence of functions φ^λ (that converges to $\lambda \times 1_{(0,t')}$) in (5.6) to have

$$T_1 \le T_2 + T_3 + T_4 + T_5,$$

where

$$\begin{aligned}
T_1 &= \int_{\Omega_\lambda\times\mathbb{R}} (-(\chi_u^+)^{\alpha,\epsilon}\chi_v^-)|_{t=t'}\lambda(x)\mathrm{d}x\mathrm{d}\xi,\\
T_2 &= \int_0^{t'}\int_{\Omega_\lambda\times\mathbb{R}} (-(\chi_u^+)^{\alpha,\epsilon}\chi_v^-)\mathbf{a}\cdot\nabla\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi,\\
T_3 &= \int_0^{t'}\int_{\Omega_\lambda\times\mathbb{R}} (-(\chi_{u_0}^+)^{\epsilon}\chi_v^-)\theta_\alpha\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi,\\
T_4 &= -\eta\int_0^{t'}\int_{\Omega_\lambda\times\mathbb{R}} \delta(v-\xi)\nabla v\cdot\nabla(-(\chi_u^+)^{\alpha,\epsilon}\lambda)\mathrm{d}t\mathrm{d}x\mathrm{d}\xi,\\
T_5 &= -\int_0^{t'}\int_{\Omega_\lambda\times\mathbb{R}} (a_d\chi_{u^\tau}^+)^{\alpha,\epsilon_1}\theta_{\epsilon_2}\chi_v^-\lambda\mathrm{d}t\mathrm{d}x\mathrm{d}\xi.
\end{aligned}$$

Now we estimate T_i $(i=1,\cdots,5)$.

The estimates of $\boldsymbol{T_1}$, $\boldsymbol{T_2}$, $\boldsymbol{T_3}$ It is easy to see that

$$\begin{aligned}
T_1 &= \int_{\Omega_\lambda\times\mathbb{R}}\int_{Q_\lambda} -\mathrm{sgn}^+(u(s,y)-\xi)\mathrm{sgn}^-(v(t',x)-\xi)\theta_\alpha(t'-s)\gamma_\epsilon(x-y)\lambda(x)\mathrm{d}s\mathrm{d}y\mathrm{d}x\mathrm{d}\xi\\
&= \int_{\Omega_\lambda}\int_{Q_\lambda} (u(s,y)-v(t',x))^+\theta_\alpha(t'-s)\gamma_\epsilon(x-y)\lambda(x)\mathrm{d}s\mathrm{d}y\mathrm{d}x\\
&\geq \int_{\Omega_\lambda}\int_{Q_\lambda} ((u(t',x)-v(t',x))^+ - (u(s,y)-u(t',x))^+)\\
&\quad\cdot\theta_\alpha(t'-s)\gamma_\epsilon(x-y)\lambda(x)\mathrm{d}s\mathrm{d}y\mathrm{d}x\\
&\geq \int_{\Omega_\lambda} (u(t',x)-v(t',x))^+\lambda(x)\mathrm{d}x\\
&\quad-\int_{Q_\lambda} (u(s,y)-u(t',x))^+\theta_\alpha(t'-s)\gamma_\epsilon(x-y)\lambda(x)\mathrm{d}s\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Since $|u(t,\cdot)|_{BV(\Omega)}\leq C$ and $\int_\Omega|\partial_t u(t,\cdot)|\mathrm{d}x\leq C$, we can derive that

$$\int_{Q_\lambda} (u(s,y)-u(t',x))^+\theta_\alpha(t'-s)\gamma_\epsilon(x-y)\lambda(x)\mathrm{d}s\mathrm{d}y\mathrm{d}x \leq C(\alpha+\epsilon_1+\epsilon_2).$$

Then

$$T_1 \geq \int_{\Omega_\lambda} (u(t',x)-v(t',x))^+\lambda(x)\mathrm{d}x - C(\alpha+\epsilon_1+\epsilon_2). \tag{5.7}$$

Similarly,

$$\begin{aligned}
T_2 &= \int_0^{t'}\int_{\Omega_\lambda\times Q_\lambda} \mathrm{sgn}^+(u(s,y)-v(t,x))(f(u)-f(v))\nabla\lambda\theta_\alpha(t-s)\gamma_\epsilon(x-y)\mathrm{d}s\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\leq C\int_0^{t'}\int_{\Omega_\lambda\times Q_\lambda} |u(s,y)-v(t,x)|\theta_\alpha(t-s)\gamma_\epsilon(x-y)\mathrm{d}s\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\leq C\int_0^{t'}\int_{\Omega_\lambda} |u(t,x)-v(t,x)|\mathrm{d}x\mathrm{d}t + C(\alpha+\epsilon_1+\epsilon_2)
\end{aligned}$$

and

$$\begin{aligned}T_3 &= \int_0^{t'}\int_{\Omega_\lambda\times Q_\lambda}(u_0(y)-v(t,x))^+\lambda\theta_\alpha(t)\gamma_\epsilon(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}t\\ &\le C\int_0^{t'}\int_{\Omega_\lambda\times Q_\lambda}(u_0(y)-v(t,x))^+\theta_\alpha(t)\gamma_\epsilon(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}t\\ &\le C(\alpha+\epsilon_1+\epsilon_2).\end{aligned}$$

The estimate of T_4 For all $(t,x,\xi)\in Q_\lambda\times\mathbb{R}$, we have

$$\begin{aligned}&|\nabla((\chi_u^+)^{\alpha,\epsilon}\lambda)(t,x,\xi)|\\ =&\Big|\int_{Q_\lambda}\mathrm{sgn}^+(u(s,y)-\xi)\theta_\alpha(t-s)(\nabla\gamma_\epsilon(x-y)\lambda(x)+\nabla\lambda(x)\gamma_\epsilon(x-y))\mathrm{d}y\mathrm{d}s\Big|\\ \le& C|\nabla\gamma_\epsilon|_{L^1(\mathbb{R}^d)}+C|\nabla\lambda|_{L^\infty(\mathbb{R}^d)}\le C\Big(\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\Big).\end{aligned}$$

Since $|v(t,\cdot)|_{BV(\Omega)}\le C$, it is easy to get

$$T_4\le C\Big(\frac{\eta}{\epsilon_1}+\frac{\eta}{\epsilon_2}\Big).$$

The estimate of T_5 We have

$$T_5=\int_0^{t'}\int_{\Omega_\lambda\times\mathbb{R}}-\chi_v^-(t,x,\xi)\Phi_0(t,x,\xi)\mathrm{d}x\mathrm{d}t\mathrm{d}\xi,$$

where

$$\Phi_0(t,x,\xi)=a_d(\xi)\theta_{\epsilon_2}(x_d)\lambda(x)\int_{\Theta_\lambda}\mathrm{sgn}^+(u^\tau-\xi)\theta_\alpha(t-s)\widetilde{\gamma}_{\epsilon_1}(\overline{x}-\overline{y})\mathrm{d}\overline{y}\mathrm{d}s.$$

Let

$$\Phi(t,x,\xi)=a_d(\xi)\theta_{\epsilon_2}(x_d)\int_{\Theta_\lambda}\mathrm{sgn}^+(u^\tau-\xi)\theta_\alpha(t-s)\widetilde{\gamma}_{\epsilon_1}(\overline{x}-\overline{y})\lambda(\overline{y})\mathrm{d}\overline{y}\mathrm{d}s.$$

Because λ is Lipschitz continuous, we can deduce that

$$|\Phi(t,x,\xi)-\Phi_0(t,x,\xi)|\le C(\epsilon_1+\epsilon_2)\theta_{\epsilon_2}(x_d).$$

Letting $\Psi_{\epsilon_2}(x_d)=\int_0^{x_d}\theta_{\epsilon_2}(r)\mathrm{d}r$, we have $0\le\Psi_{\epsilon_2}\le 1$, $\Psi_{\epsilon_2}=1$ on $[\epsilon_2,+\infty]$, and

$$\int_0^{+\infty}(1-\Psi_{\epsilon_2}(r))\mathrm{d}r\le\epsilon_2. \tag{5.8}$$

Set

$$\Gamma(t,x,\xi)=(1-\Psi_{\epsilon_2}(x_d))\int_{\Theta_\lambda}\mathrm{sgn}^+(u^\tau-\xi)\theta_\alpha(t-s)\widetilde{\gamma}_{\epsilon_1}(\overline{x}-\overline{y})\lambda(\overline{y})\mathrm{d}\overline{y}\mathrm{d}s.$$

It is obvious that $a_d(\xi)\partial_{x_d}\Gamma(t,x,\xi)=-\Phi(t,x,\xi)$. Thus we have

$$T_5\le S_5+C(\epsilon_1+\epsilon_2),$$

where

$$S_5 = \int_0^{t'} \int_{\Omega_\lambda \times \mathbb{R}} \chi_v^-(t,x,\xi) a_d(\xi) \partial_{x_d} \Gamma(t,x,\xi) \mathrm{d}x\mathrm{d}t\mathrm{d}\xi.$$

Now we estimate S_5. By choosing the test function $\Gamma(t,x,\xi)\omega_\beta(t)$ in (3.13), where $\omega_\beta(t) = \int_{t-t'}^{\infty} \theta_\beta(r)\mathrm{d}r$, and letting $\beta \to 0$, we can see that

$$\begin{aligned} S_5 \leq & -\int_0^{t'} \int_{\Omega_\lambda \times \mathbb{R}} (\chi_v^- \partial_t \Gamma + \chi_v^- \overline{\mathbf{a}}(\xi) \nabla_{\overline{x}} \Gamma - \eta\delta(v-\xi)\nabla v \cdot \nabla \Gamma) \mathrm{d}x\mathrm{d}\xi\mathrm{d}t + \int_{\Omega_\lambda \times \mathbb{R}} (\chi_v^- \Gamma)|_{t=t'} \mathrm{d}x\mathrm{d}\xi \\ & + \int_0^{t'} \int_{\Theta_\lambda \times \mathbb{R}} (a_d \chi_{v^\tau}^- + \eta\delta(v^\tau - \xi)(\partial_{x_d} v)|_{x_d=0}) \Gamma|_{x_d=0} \mathrm{d}\overline{x}\mathrm{d}t\mathrm{d}\xi. \end{aligned} \tag{5.9}$$

Since $\Gamma \geq 0$ and $\chi_v^- \leq 0$, we have

$$\int_{\Omega_\lambda \times \mathbb{R}} (\chi_v^- \Gamma)|_{t=t'} \mathrm{d}x\mathrm{d}\xi \leq 0.$$

From $|\partial_t \Gamma(t,x,\xi)| \leq \frac{C}{\alpha}(1 - \Psi_{\epsilon_2}(x_d))$, it is easy to have

$$-\int_0^{t'} \int_{\Omega_\lambda \times \mathbb{R}} \chi_v^- \partial_t \Gamma \mathrm{d}x\mathrm{d}\xi\mathrm{d}t \leq C\frac{\epsilon_2}{\alpha}.$$

Similarly, we also have

$$-\int_0^{t'} \int_{\Omega_\lambda \times \mathbb{R}} \chi_v^- \overline{\mathbf{a}}(\xi) \nabla_{\overline{x}} \Gamma \mathrm{d}x\mathrm{d}\xi\mathrm{d}t \leq C\Big(\epsilon_2 + \frac{\epsilon_2}{\epsilon_1}\Big)$$

and

$$\int_0^{t'} \int_{\Omega_\lambda \times \mathbb{R}} \eta\delta(v-\xi)\nabla v \cdot \nabla\Gamma \mathrm{d}x\mathrm{d}\xi\mathrm{d}t \leq C\Big(\frac{\eta}{\epsilon_1} + \frac{\eta}{\epsilon_2}\Big).$$

For the last term on the right-hand side of (5.9), we have

$$\begin{aligned} & \int_0^{t'} \int_{\Theta_\lambda \times \mathbb{R}} (a_d \chi_{v^\tau}^- + \eta\delta(v^\tau - \xi)(\partial_{x_d} v)|_{x_d=0}) \Gamma|_{x_d=0} \mathrm{d}\overline{x}\mathrm{d}t\mathrm{d}\xi \\ = & \int_0^{t'} \int_{\Theta_\lambda \times \Theta_\lambda} \mathrm{sgn}^+(u^\tau(s,\overline{y}) - v^\tau(t,\overline{x}))(A_d(u^\tau) - A_d(v^\tau) + \eta(\partial_{x_d} v)|_{x_d=0}) \\ & \times \theta_\alpha(t-s)\widetilde{\gamma}_{\epsilon_1}(\overline{x} - \overline{y})\lambda(\overline{y}) \mathrm{d}s\mathrm{d}\overline{y}\mathrm{d}t\mathrm{d}\overline{x}, \end{aligned}$$

where $A_d'(\cdot) = a_d(\cdot)$. Since $f(u^\tau) \cdot \mathbf{n} = 0$, and $(f(v) - \eta\nabla v) \cdot \mathbf{n} = 0$ on $\partial\Omega$, we obtain

$$\int_0^{t'} \int_{\Theta_\lambda \times \mathbb{R}} (a_d \chi_{v^\tau}^- + \eta\delta(v^\tau - \xi)(\partial_{x_d} v)|_{x_d=0}) \Gamma|_{x_d=0} \mathrm{d}\overline{x}\mathrm{d}t\mathrm{d}\xi = 0.$$

Thus we arrive at

$$T_5 \leq C\Big(\epsilon_1 + \epsilon_2 + \frac{\epsilon_2}{\epsilon_1} + \frac{\eta}{\epsilon_1} + \frac{\eta}{\epsilon_2}\Big).$$

Combining these estimates of T_i $(i = 1, \cdots, 5)$, we have

$$\begin{aligned} & \int_{\Omega_\lambda} (u(t',x) - v(t',x))^+ \lambda(x) \mathrm{d}x \\ \leq & C\Big(\alpha + \epsilon_1 + \epsilon_2 + \frac{\epsilon_2}{\epsilon_1} + \frac{\eta}{\epsilon_1} + \frac{\eta}{\epsilon_2}\Big) + C\int_0^{t'} \int_{\Omega_\lambda} |u(t,x) - v(t,x)| \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.10}$$

Minimizing on α, ϵ_1 and ϵ_2, (5.10) implies that

$$\int_{\Omega_\lambda} (u(t',x) - v(t',x))^+ \lambda(x) \mathrm{d}x \leq C\eta^{\frac{1}{3}} + C \int_0^{t'} \int_{\Omega_\lambda} |u(t,x) - v(t,x)| \mathrm{d}x \mathrm{d}t. \tag{5.11}$$

By (5.2) and (5.11), we get

$$\int_{\Omega} (u(t',x) - v(t',x))^+ \mathrm{d}x \leq C\eta^{\frac{1}{3}} + C \int_0^{t'} \int_{\Omega} |u(t,x) - v(t,x)| \mathrm{d}x \mathrm{d}t. \tag{5.12}$$

By the Gronwall's inequality, we have that (5.12) implies

$$\int_{\Omega} (u(t',x) - v(t',x))^+ \mathrm{d}x \leq C\eta^{\frac{1}{3}}.$$

Similarly, we also have

$$\int_{\Omega} (u(t',x) - v(t',x))^- \mathrm{d}x \leq C\eta^{\frac{1}{3}}.$$

Thus we complete the proof of Theorem 5.1.

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