Chin. Ann. Math.
33B(3), 2012, 367–384
DOI: 10.1007/s11401-012-0713-4

Chinese Annals of Mathematics, Series B

Products of Distributions, Conservation Laws and the Propagation of δ'-Shock Waves*

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Abstract This paper contains a study of propagation of singular travelling waves $u(x,t)$ for conservation laws $u_t + [\phi(u)]_x = \psi(u)$, where ϕ, ψ are entire functions taking real values on the real axis. Conditions for the propagation of wave profiles $\beta + m\delta$ and $\beta + m\delta'$ are presented (β is a real continuous function, $m \neq 0$ is a real number and δ' is the derivative of the Dirac measure δ). These results are obtained with a consistent concept of solution based on our theory of distributional products. Burgers equation $u_t + (\frac{u^2}{2})_x = 0$, the diffusionless Burgers-Fischer equation $u_t + a(\frac{u^2}{2})_x = ru(1-\frac{u}{k})$ with a, r, k being positive numbers, Leveque and Yee equation $u_t + u_x = \mu u(1-u)(u-\frac{1}{2})$ with $\mu \neq 0$, and some other examples are studied within such a setting. A "tool box" survey of the distributional products is also included for the sake of completeness.

Keywords Conservations laws, Travelling waves, δ'-shock waves, δ-shock waves, δ-solitons, Propagation of distributional wave profiles
2000 MR Subject Classification 46F10, 35D

1 Introduction

Let us consider the conservation law

$$u_t + [\phi(u)]_x = \psi(u), \tag{1.1}$$

where $\phi, \psi : \mathbb{C} \to \mathbb{C}$ are entire functions taking real values on the real axis, $x \in \mathbb{R}$ is the space variable, $t \in \mathbb{R}$ is the time variable, and $u(x,t)$ represents the physical state. As we will see, our theory of distributional products allows us to consider this equation in the realm of distributions. Then, it can be proved (see Section 6) that, if $\psi = 0$ and $\phi'' \neq 0$, the unique continuous travelling waves $u(x,t)$ are the constant states. Thus, if we ask for travelling waves for the conservation law

$$u_t + [\phi(u)]_x = 0 \tag{1.2}$$

with $\phi'' \neq 0$, we have to seek them among distributions which are not continuous functions.

In [21] we established that wave profiles $m\delta$ ($m \in \mathbb{R}\backslash\{0\}$ and δ is the Dirac measure) can emerge in models ruled by the conservative inviscid Burgers equation

$$u_t + \Big(\frac{u^2}{2}\Big)_x = 0, \tag{1.3}$$

Manuscript received September 14, 2011. Revised January 10, 2012.
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*Project supported by Fundação para a Ciência e a Tecnologia, PEst OE/MAT/UI0209/2011.

with any speed. In [22] we studied, for (1.1), the propagation of wave profiles which are C^1-functions with one jump discontinuity and we get the well-known Rankine-Hugoniot conditions.

In the present paper we will analyze, for (1.2), the propagation of wave profiles $\beta+m\delta$ and $\beta+m(D\delta)$ (where $\beta:\mathbb{R}\to\mathbb{R}$ is continuous, $m\in\mathbb{R}\setminus\{0\}$, and D denotes the derivative operator in distributional sense), and we compute their wave speeds. For example, we are able to prove that, for Burgers equation (1.3), wave profiles $\beta+m(D\delta)$ can emerge only if $\beta=b$ is a constant function and the wave speed is b. Also for each $b\in\mathbb{R}$, wave profiles $b+m(D\delta)$ can emerge in models ruled by the conservation law $u_t+(u^4)_x=0$, with a wave speed $4b^3$, or $-20b^3$.

For (1.1), we also analyze the propagation of wave profiles $b+m\delta$ and $b+m(D\delta)$, where $b\in\mathbb{R}$ and $m\in\mathbb{R}\setminus\{0\}$. For instance, when we apply these results to the equation $u_t+u_x=\mu u(1-u)(u-\frac{1}{2})$, introduced by LeVeque and Yee [11], we conclude that, if $\mu\neq 0$, wave profiles $b+m(D\delta)$ may appear, but only for $b=\frac{1}{2}$ and with a wave speed 1. For the diffusionless Burger-Fischer equation

$$u_t+a\Big(\frac{u^2}{2}\Big)_x=ru\Big(1-\frac{u}{k}\Big),$$

where a,r,k are positive numbers, wave profiles $b+m(D\delta)$ cannot emerge, but wave profiles $m\delta$ and $k+m\delta$ can both exist with a wave speed $\frac{ak}{2}$.

Different solution concepts are employed in the literature: the measure theoretic method (see [1, 2, 4, 9]), the use of smooth function nets and weighted measure spaces (see [10]), split delta functions (see [14, 16]), Colombeau generalized functions (see [15]), the week asymptotic method (see [7–8]) and others. Meanwhile, in the present paper, we use the concept of α-solution to (1.1), a consistent extension of the concept of a classical solution to this equation. This concept is defined within the framework of our distributional products, where the outcome of the product of distributions is always a distribution. Such a product depends upon the choice of a certain function α that encodes the indeterminacy inherent to products. We stress that this indeterminacy is not, in general, avoidable and in many questions it has a physical meaning. Concerning this point let us mention [3, 5–6] and also [18, Section 4].

Within our framework, phenomena such as "narrow soliton solutions" in the sense of Maslov, Omel'yanov and Tsupin, arise directly in distributional sense (see [19]). We also proved that δ-waves under collision behave as classical soliton collision (as in the Kortweg-de Vries equation) in models ruled by a singular perturbation of Burgers conservative equation in [20].

Let us summarize the contents of the present paper. In Sections 2, 3 and 4, we present a survey of our distributional product and we display some formulas that will be applied in the sequel. The main ideas of this theory can be seen in [18] and the details are given in [17]. Powers of certain distributions and the composition of entire functions with those distributions were introduced in [21] but we proceed in order to keep computations self-contained. In Sections 5 and 6, we apply the concept of α-solution to (1.1) to establish necessary and sufficient conditions for the propagation of a distributional wave profile. Finally, in Sections 7 and 8, we study the propagation of wave profiles $\beta+m\delta$ and $\beta+m(D\delta)$ for equation (1.2), and $b+m\delta$ and $b+m(D\delta)$ for equation (1.1) with examples given.

2 Products of Distributions

Let $\mathcal{D}$ be the space of compactly supported indefinitely differentiable complex functions defined on $\mathbb{R}$, and let $\mathcal{D}'$ be the space of Schwartz distributions. In our theory of products, given $T, S \in \mathcal{D}'$, and once fixed an even real function $\alpha \in \mathcal{D}$ with $\int_{-\infty}^{+\infty} \alpha = 1$, we can always define the product of T with S. But in general, consistence with the usual Schwartz products of distributions with functions cannot be granted. When this consistence is important, as in the sequel, the α-products must be restricted to certain subspaces of distributions to be singled out. We give formulas for two kinds of α-products which will be noted by the unique symbol $T\dot{\alpha}S$, because they are mutually consistent.

The first one can be evaluated by the formula

$$T\dot{\alpha}S = T\beta + (T * \alpha)f \tag{2.1}$$

for $T \in \mathcal{D}'^p$ and $S = \beta + f \in C^p \oplus \mathcal{D}'_\mu$, where $p \in \{0, 1, 2, \cdots, \infty\}$, $\mathcal{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz ($\mathcal{D}'^\infty$ means $\mathcal{D}'$), $\mathcal{D}'_\mu$ is the space of distributions whose support has Lebesgue measure zero, and $T\beta$ is the usual Schwartz product of $\mathcal{D}'^p$ distribution with a C^p-function. For instance, we have, for any α,

$$\delta\dot{\alpha}\delta = \delta\dot{\alpha}(0 + \delta) = (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta,$$
$$H\dot{\alpha}\delta = (H * \alpha)\delta = \Big[\int_{-\infty}^{+\infty} \alpha(-\tau)H(\tau)\mathrm{d}\tau\Big]\delta = \frac{1}{2}\delta,$$

where H stands for the Heaviside function.

The second one is to be computed by the formula

$$T\dot{\alpha}S = D(TF) - (DT)F + (T * \alpha)f, \tag{2.2}$$

where $T \in \mathcal{D}'^{-1}$ and $S = w + f \in L^1_{\mathrm{loc}} \oplus \mathcal{D}'_\mu$, where $\mathcal{D}'^{-1}$ stands for the space of distributions $T \in \mathcal{D}'$ the distributional derivative of which is in $\mathcal{D}'^0$ (so that, locally, T is a function of bounded variation), and $F \in C^0$ is a primitive for w, i.e., $DF = w$. In [18], we proved that $T\dot{\alpha}S$ given by (2.2) is independent of the choice of the function $F \in C^0$. For instance, taking $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 0$ for $x \leq 0$ and $F(x) = x$ for $x > 0$, we have $F \in C^0$, $DF = H$, and so

$$H\dot{\alpha}H = D(HF) - (DH)F + (H * \alpha)0 = DF - \delta F = H - 0 = H,$$

because $H \in \mathcal{D}'^{-1}$ and $H = H + 0 \in L^1_{\mathrm{loc}} \oplus \mathcal{D}'_\mu$.

We stress that in (2.1) and (2.2), the convolution $T * \alpha$ is not to be understood as an approximation of T. Formulas (2.1) and (2.2) are to be considered as exact ones. These α-products are bilinear and have unit element (the constant function taking the value 1 seen as a distribution); also they are transformed as usual by translations and by the symmetry $t \to -t$ from $\mathbb{R}$ onto $\mathbb{R}$. In general, associativity (recall that the usual Schwartz product of distributions with functions is not associative), or commutativity, does not hold, but we have

$$\int_{\mathbb{R}} T\dot{\alpha}S = \int_{\mathbb{R}} S\dot{\alpha}T$$

for any α, if $T, S \in \mathcal{D}'_\mu$ and one of them has compact support (in this trace type formula, $\int_{\mathbb{R}} U$ means, as usual, $\langle U, 1\rangle$, for distributions U with compact support). In general, the α-products cannot be completely localized: note that $\text{supp}(T_{\dot\alpha} S) \subset \text{supp} S$ as for usual functions, but it may happen that $\text{supp}(T_{\dot\alpha} S) \not\subset \text{supp} T$. Thus, in the following, the α-products are to be considered as global entities and when applied to differential equations, the solutions are naturally viewed as global solutions. Products (2.1) and (2.2) are consistent with Schwartz products of $\mathcal{D}'^p$ distributions with C^p-functions (if these ones are placed at the right-hand side), and satisfy the usual differential rules. Leibniz formula must be written in the form

$$D(T_{\dot\alpha} S) = (DT)_{\dot\alpha} S + T_{\dot\alpha}(DS).$$

For instance, applying the derivative operator to both sides of the equality $H_{\dot\alpha}\delta = \frac{1}{2}\delta$, we obtain $\delta_{\dot\alpha}\delta + H_{\dot\alpha}(D\delta) = \frac{1}{2}(D\delta)$ and so $H_{\dot\alpha}(D\delta) = \frac{1}{2}(D\delta) - \alpha(0)\delta$, as can be directly checked by (2.1).

3 Powers of Distributions

Taking advantage of the α-products (2.1), it is possible to define powers of certain distributions. Thus, if $T_1 = \beta_1 + f_1 \in C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$, $T_2 = \beta_2 + f_2 \in C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$, we have $T_1, T_2 \in \mathcal{D}'^p$ and $T_1, T_2 \in C^p \oplus \mathcal{D}'_\mu$, so that

$$\begin{aligned} T_1 {}_{\dot\alpha} T_2 &= T_1\beta_2 + (T_1 * \alpha) f_2 = (\beta_1 + f_1)\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \\ &= \beta_1\beta_2 + f_1\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \in C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu). \end{aligned}$$

Thus, we can define α-powers T^n_α ($n \geq 0$ is an integer) by the recurrence relation

$$T^0_\alpha = 1, \quad T^n_\alpha = (T^{n-1}_\alpha)_{\dot\alpha} T. \tag{3.1}$$

Since our distributional products are consistent with the Schwartz products of distributions with functions when these ones are placed at the right-hand side, we have $\beta^n_\alpha = \beta^n$ for all $\beta \in C^0$, which means that this definition is consistent with the definition of the usual powers of C^0-functions. For instance, if $m \in \mathbb{C}\backslash\{0\}$, we have $(m\delta)^0_\alpha = 1$, $(m\delta)^1_\alpha = m\delta$, and for $n \geq 2$, $(m\delta)^n_\alpha = m^n[\alpha(0)]^{n-1}\delta$, as may be easily seen by induction.

The α-products (2.2) also afford powers of $\mathcal{D}'^{-1}$ functions in the distributional sense. If $T_1, T_2 \in \mathcal{D}'^{-1}$, we can write $T_2 = T_2 + 0 \in L^1_{\text{loc}} \oplus \mathcal{D}'_\mu$ because $\mathcal{D}'^{-1} \subset L^1_{\text{loc}}$. By applying Leibniz formula, we have

$$\begin{aligned} T_1 {}_{\dot\alpha} T_2 &= D(T_1 F) - (DT_1)F + (T_1 * \alpha)0 \\ &= (DT_1)F + T_1(DF) - (DT_1)F = T_1 T_2 \in \mathcal{D}'^{-1}, \end{aligned}$$

where $F \in C^0$ is such that $DF = T_2$. Thus, we can define the α-powers T^n_α ($n \geq 0$ is an integer) again, by the recurrence relation (3.1) and clearly we obtain, for $T \in \mathcal{D}'^{-1}$,

$$T^n_\alpha = T^n.$$

In any situation, it is clear that $(\tau_a T)^n_\alpha = \tau_a(T^n_\alpha)$, where τ_a is the translation operator determined by $a \in \mathbb{R}$, in the sense of distributions. So, in the sequel we simplify the notation by writing T^n instead of T^n_α, supposing α fixed.

The following results will be used.

Theorem 3.1 *Let $\beta : \mathbb{R} \to \mathbb{C}$ be a continuous function and let $m \in \mathbb{C}$. Then, for all α and all integers $n \geq 1$, we have*

$$(\beta + m\delta)^n = \beta^n + mx_n\delta, \tag{3.2}$$

where the sequence $x_n \in \mathbb{C}$ is defined by the recurrence relation

$$x_1 = 1, \quad x_{n+1} = (\beta^n * \alpha)(0) + [\beta(0) + m\alpha(0)]x_n. \tag{3.3}$$

Proof By induction. Clearly (3.2) is true for $n = 1$. Let us suppose that (3.2) is true for n. Then

$$\begin{aligned}
(\beta + m\delta)^{n+1} &= (\beta + m\delta)^n_{\dot\alpha}(\beta + m\delta) \\
&= (\beta^n + mx_n\delta)_{\dot\alpha}(\beta + m\delta) \\
&= \beta^{n+1} + mx_n\beta(0)\delta + [(\beta^n + mx_n\delta) * \alpha]m\delta \\
&= \beta^{n+1} + m[(\beta^n * \alpha)(0) + (\beta(0) + m\alpha(0))x_n]\delta \\
&= \beta^{n+1} + mx_{n+1}\delta.
\end{aligned}$$

Theorem 3.2 *Let $\beta : \mathbb{R} \to \mathbb{C}$ be a C^1-function and let $m \in \mathbb{C}$. Then, for any α and any integer $n \geq 1$, we have*

$$[\beta + m(D\delta)]^n = \beta^n + x_n\delta + y_n(D\delta), \tag{3.4}$$

where the sequences $x_n, y_n \in \mathbb{C}$ are defined by the recurrence relations

$$\begin{cases} x_1 = 0, & x_{n+1} = -m(\beta^n * \alpha)'(0) + \beta(0)x_n - [\beta'(0) + m\alpha''(0)]y_n, \\ y_1 = m, & y_{n+1} = m(\beta^n * \alpha)(0) + m\alpha(0)x_n + \beta(0)y_n. \end{cases} \tag{3.5}$$

Proof By induction. It is easy to see that (3.4) is true for $n = 1$. Let us suppose that (3.4) is true for n. Then, since $\alpha'(0) = 0$, we have

$$\begin{aligned}
[\beta + m(D\delta)]^{n+1} &= [\beta + m(D\delta)]^n_{\dot\alpha}[\beta + m(D\delta)] \\
&= [\beta^n + x_n\delta + y_n(D\delta)]_{\dot\alpha}[\beta + m(D\delta)] \\
&= \beta^{n+1} + x_n\beta(0)\delta + y_n(D\delta)\beta + [(\beta^n * \alpha) + x_n\alpha + y_n\alpha']mD\delta \\
&= \beta^{n+1} + [x_n\beta(0) - y_n\beta'(0) - m(\beta^n * \alpha)'(0) - my_n\alpha''(0)]\delta \\
&\quad + [y_n\beta(0) + m(\beta^n * \alpha)(0) + mx_n\alpha(0)](D\delta) \\
&= \beta^{n+1} + x_{n+1}\delta + y_{n+1}(D\delta).
\end{aligned}$$

4 Composition of an Entire Function with a Distribution

Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function. Then we have

$$\phi(u) = a_0 + a_1 u + a_2 u^2 + \cdots$$

for the sequence $a_n = \frac{\phi^{(n)}(0)}{n!}$ of complex numbers and all $u \in \mathbb{C}$. If $T \in [C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)] \cup \mathcal{D}'^{-1}$, we define the composition $\phi \circ T$ by the formula

$$\phi \circ T = a_0 + a_1 T + a_2 T^2 + \cdots$$

whenever this series converges in $\mathcal{D}'$. Clearly, this definition is consistent with the usual meaning of $\phi \circ T$ when T is a function, and we have $\tau_a(\phi \circ T) = \phi \circ (\tau_a T)$ if $\phi \circ T$, or $\phi \circ (\tau_a T)$ is well-defined. Remember that $\phi \circ T$ depends on α. In the sequel, we need to compute $\phi \circ T$ for $T = b + m\delta$ and $T = b + m(D\delta)$, with $b, m \in \mathbb{C}$. This requires the following two lemmas.

Lemma 4.1 *Let the sequence $x_n \in \mathbb{C}$ be given by the recurrence relation* (*this recurrence relation corresponds to* (3.3) *when $\beta = b$ is a constant function*)

$$x_1 = 1, \quad x_{n+1} = b^n + [b + m\alpha(0)]x_n. \tag{4.1}$$

Then we have, for $n \geq 2$,

$$x_n = \begin{cases} nb^{n-1}, & \text{if } \alpha(0) = 0, \\ \dfrac{[b + m\alpha(0)]^n - b^n}{m\alpha(0)}, & \text{if } \alpha(0) \neq 0. \end{cases} \tag{4.2}$$

Proof By induction. Clearly, (4.2) is true for $n = 2$. Let us suppose that (4.2) is true for n and that $\alpha(0) = 0$. Then from (4.1), we have

$$x_{n+1} = b^n + bnb^{n-1} = (n+1)b^n.$$

Let us suppose that (4.2) is true for n and that $\alpha(0) \neq 0$. Then from (4.1), we have

$$x_{n+1} = b^n + [b + m\alpha(0)]\frac{[b + m\alpha(0)]^n - b^n}{m\alpha(0)} = \frac{[b + m\alpha(0)]^{n+1} - b^{n+1}}{m\alpha(0)}.$$

Lemma 4.2 *Let $x_n, y_n \in \mathbb{C}$ be sequences defined by the recurrence relations* (*these recurrence relations correspond to* (3.5) *with $\beta = b$ constant*)

$$x_1 = 0, \quad x_{n+1} = bx_n - m\alpha''(0)y_n, \tag{4.3}$$

$$y_1 = m, \quad y_{n+1} = mb^n + m\alpha(0)x_n + by_n. \tag{4.4}$$

Then,

(a) *If $\alpha(0)\alpha''(0) = 0$, we have $x_2 = -m^2\alpha''(0)$, $y_2 = 2mb$, and for $n \geq 3$,*

$$x_n = \frac{-m^2\alpha''(0)}{2} n(n-1)b^{n-2}, \tag{4.5}$$

$$y_n = nmb^{n-1}; \tag{4.6}$$

(b) *If $\alpha(0)\alpha''(0) \neq 0$, we have, for $n \geq 1$,*

$$x_n = \frac{1}{2\alpha(0)}(\lambda_1^n + \lambda_2^n) - \frac{b^n}{\alpha(0)}, \tag{4.7}$$

$$y_n = \frac{1}{2R}(\lambda_1^n - \lambda_2^n), \tag{4.8}$$

where $R = \sqrt{-\alpha(0)\alpha''(0)}$, $\lambda_1 = b + mR$, $\lambda_2 = b - mR$ (note that R can be a complex number and that this statement does not change if we interchange R with $-R$).

Proof (a) Suppose $\alpha(0) = 0$. Then $y_{n+1} = mb^n + by_n$, and by induction it is easy to see that $y_n = mnb^{n-1}$ for $n \geq 2$. Thus, $x_2 = -m^2\alpha''(0)$, $x_{n+1} = bx_n - m^2\alpha''(0)nb^{n-1}$ for $n \geq 2$, and next, (4.5) follows also by induction. Suppose $\alpha''(0) = 0$. Then, for all $n \geq 1$ we have $x_n = 0$; from (4.4) we have

$$y_1 = m, \quad y_{n+1} = mb^n + by_n,$$

and (4.6) follows, for all $n \geq 2$, by induction.

(b) Let us suppose $\alpha(0)\alpha''(0) \neq 0$. We will prove (4.7) and (4.8) by induction. It is obvious that (4.7) and (4.8) are true for $n = 1$. Now, suppose that (4.7) and (4.8) are true for a certain n. Then from (4.3), we have

$$\begin{aligned} x_{n+1} &= \frac{b}{2\alpha(0)}(\lambda_1^n + \lambda_2^n) - \frac{m\alpha''(0)}{2R}(\lambda_1^n - \lambda_2^n) \\ &= \Big(\frac{b}{2\alpha(0)} - \frac{m\alpha''(0)}{2R}\Big)\lambda_1^n + \Big(\frac{b}{2\alpha(0)} + \frac{m\alpha''(0)}{2R}\Big)\lambda_2^n - \frac{b^{n+1}}{\alpha(0)} \\ &= \frac{b+mR}{2\alpha(0)}\lambda_1^n + \frac{b-mR}{2\alpha(0)}\lambda_2^n - \frac{b^{n+1}}{\alpha(0)} = \frac{\lambda_1^n}{2\alpha(0)} + \frac{\lambda_2^n}{2\alpha(0)} - \frac{b^{n+1}}{\alpha(0)} \\ &= \frac{1}{2\alpha(0)}(\lambda_1^{n+1} + \lambda_2^{n+1}) - \frac{b^{n+1}}{\alpha(0)}. \end{aligned}$$

Also from (4.4), we have

$$\begin{aligned} y_{n+1} &= mb^n + \frac{m}{2}(\lambda_1^n + \lambda_2^n) - mb^n + \frac{b}{2R}(\lambda_1^n - \lambda_2^n) \\ &= \Big(\frac{m}{2} + \frac{b}{2R}\Big)\lambda_1^n + \Big(\frac{m}{2} - \frac{b}{2R}\Big)\lambda_2^n \\ &= \frac{b+mR}{2R}\lambda_1^n - \frac{b-mR}{2R}\lambda_2^n = \frac{1}{2R}(\lambda_1^{n+1} - \lambda_2^{n+1}). \end{aligned}$$

Theorem 4.1 *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function and let $T = b + m\delta$, with $b, m \in \mathbb{C}$. Then*

$$\phi \circ T = \begin{cases} \phi(b) + m\phi'(b)\delta, & \text{if } \alpha(0) = 0, \\ \phi(b) + \dfrac{\phi[b + m\alpha(0)] - \phi(b)}{\alpha(0)}\delta, & \text{if } \alpha(0) \neq 0. \end{cases}$$

Proof Put $\phi(u) = a_0 + a_1 u + a_2 u^2 + \cdots$, where $a_n = \frac{\phi^{(n)}(0)}{n!}$. We have, from Theorem 3.1,

$$\phi \circ T = a_0 + \sum_{n=1}^{\infty} a_n (b + m\delta)^n = a_0 + \sum_{n=1}^{\infty} a_n (b^n + m x_n \delta) = \phi(b) + m\Big(\sum_{n=1}^{\infty} a_n x_n\Big)\delta,$$

where x_n is the sequence defined by (4.1) (note that $\sum\limits_{n=1}^{\infty} a_n b^n$ is a convergent series and $\sum\limits_{n=1}^{\infty} a_n x_n$ is also convergent as the calculations below show). Then, if $\alpha(0) = 0$, by applying Lemma 4.1, we have

$$\begin{aligned}\phi \circ T &= \phi(b) + m\Big(a_1 x_1 + \sum_{n=2}^{\infty} a_n x_n\Big)\delta \\ &= \phi(b) + m\Big(a_1 + \sum_{n=2}^{\infty} a_n n b^{n-1}\Big)\delta \\ &= \phi(b) + m\phi'(b)\delta.\end{aligned}$$

If $\alpha(0) \neq 0$, we have, also by Lemma 4.1,

$$\begin{aligned}\phi \circ T &= \phi(b) + m\Big(a_1 x_1 + \sum_{n=2}^{\infty} a_n x_n\Big)\delta \\ &= \phi(b) + m\Big(a_1 + \sum_{n=2}^{\infty} a_n \frac{[b + m\alpha(0)]^n - b^n}{m\alpha(0)}\Big)\delta \\ &= \phi(b) + m\Big[a_1 + \frac{1}{m\alpha(0)}\Big(\sum_{n=2}^{\infty} a_n [b + m\alpha(0)]^n - \sum_{n=2}^{\infty} a_n b^n\Big)\Big]\delta \\ &= \phi(b) + +m\Big[a_1 + \frac{1}{m\alpha(0)}[\phi(b + m\alpha(0)) - a_1(b + m\alpha(0)) - a_0 - \phi(b) + a_1 b + a_0]\Big]\delta \\ &= \phi(b) + \frac{\phi[b + m\alpha(0)] - \phi(b)}{\alpha(0)}\delta.\end{aligned}$$

Theorem 4.2 *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function and let $T = b + m(D\delta)$, with $b, m \in \mathbb{C}$. Then*

$$\phi \circ T = \begin{cases} \phi(b) - \dfrac{m^2\alpha''(0)}{2}\phi''(b)\delta + m\phi'(b)(D\delta), & \text{if } \alpha(0)\alpha''(0) = 0, \\ \phi(b) + \dfrac{\phi(\lambda_1) + \phi(\lambda_2) - 2\phi(b)}{2\alpha(0)}\delta + \dfrac{\phi(\lambda_1) - \phi(\lambda_2)}{2R}(D\delta), & \text{if } \alpha(0)\alpha''(0) \neq 0, \end{cases}$$

where $R = \sqrt{-\alpha(0)\alpha''(0)}$, $\lambda_1 = b + mR$ and $\lambda_2 = b - mR$ (note that R can be a complex number and that this statement does not change if we interchange R with $-R$).

Proof Put $\phi(u) = a_0 + a_1 u + a_2 u^2 + \cdots$, where $a_n = \frac{\phi^{(n)}(0)}{n!}$. We have, from Theorem 3.2,

$$\begin{aligned}\phi \circ T &= a_0 + \sum_{n=1}^{\infty} a_n [b + m(D\delta)]^n = a_0 + \sum_{n=1}^{\infty} a_n [b^n + x_n\delta + y_n(D\delta)] \\ &= \phi(b) + \Big(\sum_{n=1}^{\infty} a_n x_n\Big)\delta + \Big(\sum_{n=1}^{\infty} a_n y_n\Big)(D\delta), \qquad (4.9)\end{aligned}$$

where the sequences x_n and y_n are defined by (4.3) and (4.4) (note that $\sum_{n=1}^{\infty} a_n b^n$ is a convergent series and $\sum_{n=1}^{\infty} a_n x_n$, $\sum_{n=1}^{\infty} a_n y_n$ are also convergent series, as a consequence of the calculations below). Then, if $\alpha(0)\alpha''(0) = 0$, we have from Lemma 4.2 that

$$\sum_{n=1}^{\infty} a_n x_n = -m^2\alpha''(0)a_2 - \frac{m^2\alpha''(0)}{2}\sum_{n=3}^{\infty} a_n n(n-1)b^{n-2} = -\frac{m^2\alpha''(0)}{2}\phi''(b),$$

$$\begin{aligned}\sum_{n=1}^{\infty} a_n y_n &= ma_1 + 2mba_2 + \sum_{n=3}^{\infty} a_n y_n \\ &= ma_1 + 2mba_2 + \sum_{n=3}^{\infty} nmb^{n-1}a_n = m\phi'(b).\end{aligned}$$

If $\alpha(0)\alpha''(0) \neq 0$, we have, also from Lemma 4.2,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n x_n &= \sum_{n=1}^{\infty}\Big(\frac{a_n}{2\alpha(0)}(\lambda_1^n + \lambda_2^n) - \frac{a_n b^n}{\alpha(0)}\Big) \\ &= \frac{1}{2\alpha(0)}\sum_{n=1}^{\infty} a_n\lambda_1^n + \frac{1}{2\alpha(0)}\sum_{n=1}^{\infty} a_n\lambda_2^n - \frac{1}{\alpha(0)}\sum_{n=1}^{\infty} a_n b^n \\ &= \frac{\phi(\lambda_1) + \phi(\lambda_2) - 2\phi(b)}{2\alpha(0)},\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} a_n y_n &= \sum_{n=1}^{\infty}\frac{a_n}{2R}(\lambda_1^n - \lambda_2^n) = \frac{1}{2R}\sum_{n=1}^{\infty} a_n\lambda_1^n - \frac{1}{2R}\sum_{n=1}^{\infty} a_n\lambda_2^n \\ &= \frac{\phi(\lambda_1) - \phi(\lambda_2)}{2R}.\end{aligned}$$

Thus, both series converge and the statement follows immediately from (4.9).

5 The Concept of α-Solution

Let I be an interval of $\mathbb{R}$ with more than one point and let $\mathcal{F}(I)$ be the space of continuously differentiable maps $\widetilde{u} : I \to \mathcal{D}'$ in the sense of the usual topology of $\mathcal{D}'$. For $t \in I$, the notation $[\widetilde{u}(t)](x)$ is sometimes used to emphasize that the distribution $\widetilde{u}(t)$ acts on functions $\xi \in \mathcal{D}$ which depend on x.

Definition 5.1 *The map $\widetilde{u} \in \mathcal{F}(I)$ is said to be an α-solution to* (1.1) *if and only if there exists an α such that for all $t \in I$,*

(a) *$\phi \circ \widetilde{u}(t)$ and $\psi \circ \widetilde{u}(t)$ are well-defined distributions;*

(b)

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}t}(t) + D[\phi \circ \widetilde{u}(t)] = \psi \circ \widetilde{u}(t). \tag{5.1}$$

This definition sees (1.1) as an evolution equation. We have the following results.

Theorem 5.1 *If u is a classical solution to* (1.1) *on $\mathbb{R}\times I$, then for any α, the map $\widetilde{u} \in \mathcal{F}(I)$ defined by $[\widetilde{u}(t)](x) = u(x,t)$ is an α-solution to* (1.1).

Remark 5.1 By a classical solution, we mean a continuously differentiable complex function $(x,t) \to u(x,t)$ which satisfies (1.1) on $\mathbb{R}\times I$.

Theorem 5.2 *If* $u : \mathbb{R}\times I \to \mathbb{C}$ *is a* C^1*-function and* $\widetilde{u} \in \mathcal{F}(I)$ *defined by* $[\widetilde{u}(t)](x) = u(x,t)$ *is an* α*-solution to* (1.1)*, then* u *is a classical solution to* (1.1)*.*

For the proof, it is enough to observe that a C^1-function $u(x,t)$ can be read as a continuously differentiable function $\widetilde{u} \in \mathcal{F}(I)$ defined by $[\widetilde{u}(t)](x) = u(x,t)$ and use the consistency of the α-products with the classical products.

Let $\Sigma(I)$ be the space of functions $u : \mathbb{R}\times I \to \mathbb{C}$ such that $\widetilde{u} : I \to \mathcal{D}'$, defined by $[\widetilde{u}(t)](x) = u(x,t)$ in $\mathcal{F}(I)$, and for each $t \in I$, $u(x,t) \in L^1_{\text{loc}}(\mathbb{R})$. The natural injection $u \mapsto \widetilde{u}$ of $\Sigma(I)$ into $\mathcal{F}(I)$ allows us to identify any function in $\Sigma(I)$ with a certain map in $\mathcal{F}(I)$. Since $C^1(\mathbb{R}\times I) \subset \Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R}\times I) \subset \Sigma(I) \subset \mathcal{F}(I).$$

Consequently, Definition 5.1 affords a consistent extension of the concept of a classical solution to (1.1).

6 The Propagation of a Wave Profile $T \in \mathcal{D}'$

For the sake of simplicity we introduce the following definition.

Definition 6.1 *Let* $\gamma : I \to \mathbb{R}$ *be a* C^1*-function. We say that* $T \in \mathcal{D}'$ α*-propagates with the movement* $\gamma(t)$*, according to* (1.1)*, if and only if the map* $\widetilde{u} \in \mathcal{F}(I)$ *defined by* $\widetilde{u}(t) = \tau_{\gamma(t)}T$ *is an* α*-solution to* (1.1)*.*

Theorem 6.1 *Let* $\gamma : I \to \mathbb{R}$ *be a* C^1*-function and let* $T \in \mathcal{D}'$ *be a nonconstant distribution. Then* T α*-propagates, according to* (1.1)*, with the movement* $\gamma(t)$ *if and only if the following conditions are verified:*

(a) $\phi \circ T$ *and* $\psi \circ T$ *are well-defined distributions*;
(b) $\gamma'(t) = c$ *is a constant function*;
(c) $D(\phi \circ T) = c(DT) + \psi \circ T$*.*

Proof Suppose that T α-propagates with the movement $\gamma(t)$. Then, by Definition 5.1, $\phi \circ (\tau_{\gamma(t)}T)$ and $\psi \circ (\tau_{\gamma(t)}T)$ are well-defined distributions for all $t \in I$. Since $\tau_{\gamma(t)}(\phi \circ T) = \phi \circ (\tau_{\gamma(t)}T)$ and $\tau_{\gamma(t)}(\psi \circ T) = \psi \circ (\tau_{\gamma(t)}T)$, we have, for all $t \in I$,

$$\phi \circ T = \tau_{-\gamma(t)}(\phi \circ \tau_{\gamma(t)}T) \quad \text{and} \quad \psi \circ T = \tau_{-\gamma(t)}(\psi \circ \tau_{\gamma(t)}T), \tag{6.1}$$

and (a) follows. From Definitions 6.1 and 5.1, we also have, for all $t \in I$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tau_{\gamma(t)}T) + D[\phi \circ (\tau_{\gamma(t)}T)] = \psi \circ (\tau_{\gamma(t)}T),$$

and so (see [18, p. 648])

$$(\tau_{\gamma(t)}DT)(-\gamma'(t)) + D[\phi \circ (\tau_{\gamma(t)}T)] = \psi \circ (\tau_{\gamma(t)}T). \tag{6.2}$$

By applying the operator $\tau_{-\gamma(t)}$ to both sides of this equation, we obtain, for each $t \in I$,

$$(DT)(-\gamma'(t)) + D[\tau_{-\gamma(t)}(\phi \circ (\tau_{\gamma(t)}T))] = \tau_{-\gamma(t)}[\psi \circ (\tau_{\gamma(t)}T)]. \tag{6.3}$$

By using (6.1), we have

$$(DT)(-\gamma'(t)) + D(\phi \circ T) = \psi \circ T.$$

Thus,

$$(DT)\gamma'(t) = D(\phi \circ T) - \psi \circ T \tag{6.4}$$

for each $t \in I$ and since the right-hand side of this equality does not depend on t, condition (b) follows. Setting $\gamma'(t) = c$, condition (c) is proved. Conversely, if (a), (b) and (c) are satisfied, (6.4) is satisfied and also (6.3) and (6.2), from which follows that T α-propagates with the movement $\gamma(t)$.

We will use the following result.

Lemma 6.1 *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire nonconstant function, I be an interval of $\mathbb{R}$ with more than one point, and $\beta : I \to \mathbb{C}$ a continuous function. Suppose also that $\phi \circ \beta$, seen as a distribution defined on the interior of I, satisfies $D(\phi \circ \beta) = 0$. Then β is a constant function.*

Proof From $D(\phi \circ \beta) = 0$, we conclude that $\phi \circ \beta$, as a distribution defined on the interior of I, is constant. Since $\phi \circ \beta$ is continuous, it follows that $\phi \circ \beta$ is a constant function. Let $z_0 = (\phi \circ \beta)(t)$ for all $t \in I$. Then $\phi(\beta(t)) - z_0 = 0$ for all $t \in I$, which means that $\beta(t)$ is a zero of the entire function $w(z) = \phi(z) - z_0$ for all $t \in I$. On the other hand, we know that any zero of an entire and nonconstant function is an isolated point. Then, since β is continuous and takes values in the set of the zeros of w, we conclude that β is a constant function.

As a consequence, we prove the following result.

Theorem 6.2 *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function with $\phi'' \neq 0$. Then, the continuous wave $T = \beta$ α-propagates according to the conservation law* (1.2)*, $u_t + [\phi(u)]_x = 0$, if and only if β is a constant function.*

Proof Suppose that the nonconstant wave $T = \beta$ α-propagates according to (1.2). Then, by Theorem 6.1, we conclude that this happens if and only if $\gamma'(t) = c$ is a constant function and $D(\phi \circ \beta) = c(D\beta)$. Meanwhile, we can write this equation in the form

$$D[(\phi - c\mathbf{1}) \circ \beta] = 0,$$

where $\mathbf{1} : \mathbb{C} \to \mathbb{C}$ is the identical function. Since $\phi - c\mathbf{1}$ is an entire nonconstant function, by Lemma 6.1 we conclude that β is a constant function.

Remark 6.1 If $\phi'' = 0$, then $\phi' = a$ is a constant function and the referred equation turns out to be $u_t + au_x = 0$. Then, by Theorem 6.1, any wave $T \in \mathcal{D}'$ α-propagates (for any α) according to this equation: the constant states with an arbitrary speed, and the nonconstant states with the speed a.

This explains why (as we said in the introduction), if we ask for travelling waves of (1.2), when $\phi'' \neq 0$, we have to seek them among distributions which are not continuous functions. In the sequel, we will examine the propagation of highly singular wave profiles $\beta + m\delta$ and $\beta + m(D\delta)$ in models ruled by the law $u_t + [\phi(u)]_x = 0$.

7 Propagation of Wave Profiles $\beta+m\delta$ and $\beta+m(D\delta)$ for the Equation $u_t + [\phi(u)]_x = 0$

Let us consider the equation (1.2) where $\phi : \mathbb{C} \to \mathbb{C}$ is an entire function taking real values on the real axis and such that $\phi'' \neq 0$. Suppose also that $\gamma : \mathbb{R} \to \mathbb{R}$ is a C^1-function.

Theorem 7.1 *Let $T = \beta + m\delta$, with $\beta : \mathbb{R} \to \mathbb{R}$ continuous and $m \in \mathbb{R}\backslash\{0\}$. Then, the wave T α-propagates, according to* (1.2)*, with the movement $\gamma(t)$, if and only if the following conditions are satisfied:*

(a) *$\beta = b$ is a constant function*;

(b) *the wave speed $\gamma'(t)$ is a constant function given by*

$$\gamma'(t) = \begin{cases} \phi'(b), & \text{if } \alpha(0) = 0, \\ \dfrac{\phi[b + m\alpha(0)] - \phi(b)}{m\alpha(0)}, & \text{if } \alpha(0) \neq 0. \end{cases}$$

Proof Suppose that T α-propagates according to (1.2). Then, by Theorem 6.1, we have

(c) $\phi \circ T$ is a well-defined distribution;

(d) $\gamma'(t) = c$ is a constant function;

(e) $D(\phi \circ T) = c(DT)$.

Putting $\phi(u) = a_0 + a_1 u + a_2 u^2 + \cdots$, for the sequence $a_n = \frac{\phi^{(n)}(0)}{n!}$, applying Theorem 3.1 and taking into account that $\phi \circ T$ is well-defined, we have

$$\phi \circ T = a_0 + \sum_{n=1}^{\infty} a_n(\beta + m\delta)^n = a_0 + \sum_{n=1}^{\infty} a_n(\beta^n + m x_n \delta) = \phi \circ \beta + m\Big(\sum_{n=1}^{\infty} a_n x_n\Big)\delta.$$

Then, from (e) we have

$$D(\phi \circ \beta) + m\Big(\sum_{n=1}^{\infty} a_n x_n\Big)(D\delta) = c(D\beta) + cm(D\delta). \tag{7.1}$$

Restricting this equality to $]0, +\infty[$, we obtain $D(\phi \circ \beta) - c(D\beta) = 0$ on $]0, +\infty[$. Thus, we can write $D[(\phi - c\mathbf{1})\circ\beta] = 0$ on $]0, +\infty[$, and by Lemma 6.1, β is a constant function on $]0, +\infty[$. By restriction of the same equality to $]-\infty, 0[$, we conclude that β is a constant function on $]-\infty, 0[$. Since β is continuous, we conclude that $\beta = b$ is a constant function on $\mathbb{R}$ and (a) follows. Thus, by (7.1), we have $c = \sum\limits_{n=1}^{\infty} a_n x_n$, with x_n defined now by (3.3) with $\beta = b$ constant, i.e., x_n is given by (4.1). Therefore, from Lemma 4.1, we have that if $\alpha(0) = 0$,

$$c = \sum_{n=1}^{\infty} a_n x_n = a_1 x_1 + \sum_{n=2}^{\infty} a_n x_n = a_1 + \sum_{n=2}^{\infty} a_n n b^{n-1} = \phi'(b),$$

and if $\alpha(0) \neq 0$,

$$c = \sum_{n=1}^{\infty} a_n x_n = a_1 x_1 + \sum_{n=2}^{\infty} a_n x_n = a_1 + \sum_{n=2}^{\infty} a_n \frac{[b + m\alpha(0)]^n - b^n}{m\alpha(0)}$$
$$= a_1 + \frac{1}{m\alpha(0)} \sum_{n=2}^{\infty} a_n [b + m\alpha(0)]^n - \frac{1}{m\alpha(0)} \sum_{n=2}^{\infty} a_n b^n$$
$$= a_1 + \frac{1}{m\alpha(0)} [\phi(b + m\alpha(0)) - a_1(b + m\alpha(0)) - a_0]$$
$$- \frac{1}{m\alpha(0)} [\phi(b) - a_1 b - a_0] = \frac{\phi[b + m\alpha(0)] - \phi(b)}{m\alpha(0)},$$

so that (b) follows from (d). Following the reasoning backwards the converse statement is obtained.

As an example, consider Burgers conservative equation (1.3). Then for each α, the wave $T = \beta + m\delta$ α-propagates if and only if $\beta = b$ is a constant function and the wave speed is $\gamma'(t) = b + \frac{m\alpha(0)}{2}$ (see [21, Section 6] for a physical interpretation of this indetermination, in the case $b = 0$).

Theorem 7.2 *Let $T = \beta + m(D\delta)$ with $\beta : \mathbb{R} \to \mathbb{R}$ continuous and $m \in \mathbb{R}\backslash\{0\}$. Then, the wave T α-propagates, according to* (1.2), *with the movement $\gamma(t)$, if and only if, $\beta = b$ is a constant function and one of the following conditions is satisfied:*

(a) $\alpha(0)\alpha''(0) = 0$, $\alpha''(0)\phi''(b) = 0$ *and the wave speed is*

$$\gamma'(t) = \phi'(b);$$

(b) $\alpha(0)\alpha''(0) \neq 0$, $\phi(b + mR) + \phi(b - mR) = 2\phi(b)$ *and the wave speed is*

$$\gamma'(t) = \frac{\phi(b + mR) - \phi(b)}{mR},$$

where $R = \sqrt{-\alpha(0)\alpha''(0)}$ (note that R can be a complex number and that this statement does not change if we interchange R with $-R$).

Proof Suppose that T α-propagates according to (1.2). Then, by Theorem 6.1, we have

(c) $\phi \circ T$ is a well-defined distribution;

(d) $\gamma'(t) = c$ is a constant function;

(e) $D(\phi \circ T) = c(DT)$.

Putting $\phi(u) = a_0 + a_1 u + a_2 u^2 + \cdots$, for the sequence $a_n = \frac{\phi^{(n)}(0)}{n!}$, we have, by applying Theorem 3.2,

$$\phi \circ T = a_0 + \sum_{n=1}^{\infty} a_n [\beta + m(D\delta)]^n = a_0 + \sum_{n=1}^{\infty} [a_n \beta^n + a_n x_n \delta + a_n y_n (D\delta)]$$
$$= \phi \circ \beta + \sum_{n=1}^{\infty} [a_n x_n \delta + a_n y_n (D\delta)], \tag{7.2}$$

where x_n, y_n are sequences defined by (3.5) and the series are convergent because $\phi \circ T$ is a well-defined distribution. Thus we have for all $\varphi \in \mathcal{D}$,

$$\langle \phi \circ T, \varphi \rangle = \langle \phi \circ \beta, \varphi \rangle + \sum_{n=1}^{\infty} [a_n x_n \varphi(0) - a_n y_n \varphi'(0)].$$

Choosing $\varphi \in \mathcal{D}$ such that $\varphi(0) = 1$ and $\varphi'(0) = 0$, we conclude that $\sum_{n=1}^{\infty} a_n x_n$ is a convergent series. Choosing $\varphi \in \mathcal{D}$ such that $\varphi(0) = 0$ and $\varphi'(0) = -1$, we conclude that $\sum_{n=1}^{\infty} a_n y_n$ is also a convergent series. Then from (7.2), we can write

$$\phi \circ T = \phi \circ \beta + \Big(\sum_{n=1}^{\infty} a_n x_n \Big) \delta + \Big(\sum_{n=1}^{\infty} a_n y_n \Big) (D\delta),$$

and from (e), we have

$$D(\phi \circ \beta) + \Big(\sum_{n=1}^{\infty} a_n x_n \Big) (D\delta) + \Big(\sum_{n=1}^{\infty} a_n y_n \Big) (D^2\delta) = c(D\beta) + cm(D^2\delta),$$

which is equivalent to

$$D(\phi \circ \beta) - c(D\beta) = -\Big(\sum_{n=1}^{\infty} a_n x_n \Big) (D\delta) + \Big(cm - \sum_{n=1}^{\infty} a_n y_n \Big) (D^2\delta). \tag{7.3}$$

By restriction of this equality to $]0, +\infty[$ and $]-\infty, 0[$, we conclude, exactly as in the proof of Theorem 7.1, that $\beta = b$ is a constant function. Then, x_n, y_n are afforded by (4.3) and (4.4), which correspond to the relations (3.5) with $\beta = b$ constant.

Now, if $\alpha(0)\alpha''(0) = 0$, we have, by (4.5),

$$\begin{aligned} \sum_{n=1}^{\infty} a_n x_n &= a_1 x_1 + a_2 x_2 + \sum_{n=3}^{\infty} a_n x_n \\ &= -m^2 \alpha''(0) a_2 - \frac{m^2 \alpha''(0)}{2} \sum_{n=3}^{\infty} a_n n(n-1) b^{n-2} \\ &= -m^2 \alpha''(0) a_2 - \frac{m^2 \alpha''(0)}{2} [\phi''(b) - 2a_2] \\ &= \frac{-m^2 \alpha''(0)}{2} \phi''(b), \end{aligned}$$

and also, by (4.6),

$$\sum_{n=1}^{\infty} a_n y_n = a_1 y_1 + a_2 y_2 + \sum_{n=3}^{\infty} a_n y_n = m a_1 + 2mb a_2 + m \sum_{n=3}^{\infty} a_n n b^{n-1} = m\phi'(b).$$

If $\alpha(0)\alpha''(0) \neq 0$, we have, by (4.7),

$$\begin{aligned}\sum_{n=1}^{\infty} a_n x_n &= \frac{1}{2\alpha(0)} \sum_{n=1}^{\infty} a_n(\lambda_1^n + \lambda_2^n) - \frac{1}{\alpha(0)} \sum_{n=1}^{\infty} a_n b^n \\ &= \frac{1}{2\alpha(0)} \sum_{n=1}^{\infty} a_n \lambda_1^n + \frac{1}{2\alpha(0)} \sum_{n=1}^{\infty} a_n \lambda_2^n - \frac{1}{\alpha(0)} \sum_{n=1}^{\infty} a_n b^n \\ &= \frac{1}{2\alpha(0)}[\phi(\lambda_1) - a_0] + \frac{1}{2\alpha(0)}[\phi(\lambda_2) - a_0] - \frac{1}{\alpha(0)}[\phi(b) - a_0] \\ &= \frac{\phi(\lambda_1) + \phi(\lambda_2) - 2\phi(b)}{2\alpha(0)},\end{aligned}$$

and also, by (4.8),

$$\begin{aligned}\sum_{n=1}^{\infty} a_n y_n &= \frac{1}{2R} \sum_{n=1}^{\infty} a_n(\lambda_1^n - \lambda_2^n) \\ &= \frac{1}{2R}[\phi(\lambda_1) - a_0] - \frac{1}{2R}[\phi(\lambda_2) - a_0] \\ &= \frac{\phi(\lambda_1) - \phi(\lambda_2)}{2R}.\end{aligned}$$

Thus, from (7.3), we have that if $\alpha(0)\alpha''(0) = 0$,

$$0 = \frac{m^2\alpha''(0)}{2}\phi''(b)(D\delta) + [cm - m\phi'(b)](D^2\delta),$$

and (a) follows. Also from (7.3), we have that if $\alpha(0)\alpha''(0) \neq 0$,

$$0 = -\frac{\phi(\lambda_1) + \phi(\lambda_2) - 2\phi(b)}{2\alpha(0)}(D\delta) + \Big[cm - \frac{\phi(\lambda_1) - \phi(\lambda_2)}{2R}\Big](D^2\delta),$$

and (b) follows because $\phi(\lambda_1) - \phi(\lambda_2) = 2\phi(\lambda_1) - 2\phi(b)$. For the converse it is sufficient to follow the reasoning backwards.

For instance, for Burgers conservative equation (1.3), it is easy to see that the condition (b) of Theorem 7.2 is impossible to be satisfied, so that $T = \beta + m(D\delta)$ α-propagates according to this equation if and only if $\alpha''(0) = 0$, $\beta = b$ is a constant function and the wave speed is $\gamma'(t) = b$. As an immediate consequence of Theorem 7.2, we have the following corollary.

Corollary 7.1 *For any α such that $\alpha''(0) = 0$, and any $b, m \in \mathbb{R}$ with $m \neq 0$, the wave $T = b + m(D\delta)$ α-propagates, according to* (1.2)*, with the speed $\gamma'(t) = \phi'(b)$.*

Consider now the equation $u_t + (u^4)_x = 0$. It is worth to note that the profile $T = b + m(D\delta)$ can emerge with the speed $\neq \phi'(b)$. According to Theorem 7.2, when $\alpha(0)\alpha''(0) = 0$, the α-propagation of the wave $T = \beta + m(D\delta)$ is possible if and only if $\beta = b$ is a constant function and the wave speed $\gamma'(t) = 4b^3$ (if $\alpha''(0) \neq 0$, then $b = 0$ and the wave $T = m(D\delta)$ has a speed zero, i.e., it is a stationary wave). When $\alpha(0)\alpha''(0) \neq 0$, the condition $(b + mR)^4 + (b - mR)^4 = 2b^4$ is equivalent to $mR = \pm \mathrm{i}b\sqrt{6}$. Thus, if $\alpha(0)\alpha''(0) < 0$, the α-propagation is impossible. However, for $\alpha(0)\alpha''(0) > 0$, the α-propagation is possible if and only if the wave speed is

$$\gamma'(t) = \frac{(b + \mathrm{i}b\sqrt{6})^4 - b^4}{\mathrm{i}b\sqrt{6}} = -20b^3.$$

So in the case $\alpha(0)\alpha''(0) > 0$, the wave profile $T = b + m(D\delta)$ can also emerge with the speed $-20b^3$.

8 Propagation of Wave Profiles $b+m\delta$ and $b+m(D\delta)$ for the Equation $u_t + [\phi(u)]_x = \psi(u)$

Now, we consider the equation (1.1), with $\phi, \psi : \mathbb{C} \to \mathbb{C}$ being entire functions that take real values on the real axis. In the following, $\gamma : \mathbb{R} \to \mathbb{R}$ is a C^1-function, $b, m \in \mathbb{R}$ and $m \neq 0$.

Theorem 8.1 *The wave $T = b + m\delta$ α-propagates, according to* (1.1)*, with the movement $\gamma(t)$ if and only if $\psi(b) = 0$ and one of the following conditions is satisfied:*

(a) $\alpha(0) = 0$, $\psi'(b) = 0$ *and the wave speed is* $\gamma'(t) = \phi'(b)$;

(b) $\alpha(0) \neq 0$, $\psi[b + m\alpha(0)] = 0$ *and the wave speed is* $\gamma'(t) = \frac{\phi[b+m\alpha(0)]-\phi(b)}{m\alpha(0)}$.

Theorem 8.2 *The wave $T = b+m(D\delta)$ α-propagates, according to* (1.1)*, with the movement $\gamma(t)$ if and only if $\psi(b) = 0$. Moreover, when $\alpha(0)\alpha''(0) = 0$, the following three conditions are satisfied*:

(a) $\alpha''(0)\psi''(b) = 0$;

(b) $\frac{m\alpha''(0)}{2}\phi''(b) + \psi'(b) = 0$;

(c) $\gamma'(t) = \phi'(b)$.

When $\alpha(0)\alpha''(0) \neq 0$, the following three conditions are satisfied:

(d) $\psi(\lambda_1) + \psi(\lambda_2) = 0$;

(e) $\phi(\lambda_1) + \phi(\lambda_2) = 2\phi(b) + \frac{2\alpha(0)}{R}\psi(\lambda_1)$;

(f) $\gamma'(t) = \frac{\phi(\lambda_1)-\phi(\lambda_2)}{2mR}$,

where $R = \sqrt{-\alpha(0)\alpha'(0)}$, $\lambda_1 = b + mR$ and $\lambda_2 = b - mR$.

For the proofs, it is sufficient to use the same methods we have applied in Section 7 (here the proofs are easier).

As an example, let us apply Theorem 8.1 to the diffusionless Burgers-Fischer equation

$$u_t + a\Big(\frac{u^2}{2}\Big)_x = ru\Big(1 - \frac{u}{k}\Big), \tag{8.1}$$

where $a > 0$, $r > 0$ and $k > 0$. This equation models an advective process (without diffusion) with a logistic nonlinear reaction. Since $\phi(u) = \frac{au^2}{2}$ and $\psi(u) = ru(1 - \frac{u}{k})$, we have $\psi(b) = 0$ if and only if $b = 0$ or $b = k$. But in both cases, condition (a) of Theorem 8.1 is impossible to satisfy. Let us apply condition (b) of Theorem 8.1. In the case $b = 0$, we have $\psi[m\alpha(0)] = rm\alpha(0)(1 - \frac{m\alpha(0)}{k})$, and so taking $\alpha(0) = \frac{k}{m}$, we have $\psi[m\alpha(0)] = 0$. Thus, in this case, the α-propagation is possible with the wave speed

$$\gamma'(t) = \frac{\phi[b + \alpha(0)] - \phi(b)}{m\alpha(0)} = \frac{\phi(k) - \phi(0)}{k} = \frac{ak}{2}.$$

Suppose now $b = k$. We have $\psi[k + m\alpha(0)] = r[k + m\alpha(0)][1 - \frac{k+m\alpha(0)}{k}]$. Since $\alpha(0) \neq 0$, the unique possibility to satisfy the equation $\psi[k + m\alpha(0)] = 0$ is to take $\alpha(0) = \frac{-k}{m}$. Thus, the

α-propagation is also possible with the wave speed

$$\gamma'(t) = \frac{\phi[b+\alpha(0)] - \phi(b)}{m\alpha(0)} = \frac{\phi(k-k) - \phi(k)}{-k} = \frac{ak}{2}.$$

As a consequence, for (8.1), the emergence of wave profiles $T = m\delta$ and $T = k + m\delta$, both with a wave speed $\gamma'(t) = \frac{ak}{2}$, is possible. Wave profiles $T = b + m(D\delta)$ are not possible to emerge in models ruled by equation (8.1), as we can easily verify by applying Theorem 8.2. Other results about this equation can be seen in [12–13].

As another example, let us consider the advection-reaction equation with stiff reaction, introduced by Leveque and Yee [11],

$$u_t + u_x = \mu u(1-u)\Big(u - \frac{1}{2}\Big), \tag{8.2}$$

where $\mu \neq 0$. In the context of Theorem 8.2, we have $\phi(u) = u$, $\psi(u) = \mu u(1-u)(u - \frac{1}{2})$, and wave profiles $T = b + m(D\delta)$, with $b, m \in \mathbb{R}$, and $m \neq 0$ are admissible only in the cases $b = 0$, $b = 1$ and $b = \frac{1}{2}$. Meanwhile, the compatibility conditions (a), (b), (d), (e) are impossible for $b = 0$ and $b = 1$. For $b = \frac{1}{2}$, condition (b) is impossible to satisfy, but (d) and (e) are satisfied for all α such that $\alpha(0)\alpha''(0) = -\frac{1}{4m^2}$, with the wave speed $\gamma'(t) = 1$. Consequently, the emergence of the profile $T = b + m(D\delta)$, in a model ruled by equation (8.2), is possible only for $b = \frac{1}{2}$, with the speed $\gamma'(t) = 1$. It is interesting to note (by applying Theorem 8.1) that, wave profiles $T = m\delta$, $T = 1 + m\delta$ and $T = \frac{1}{2} + m\delta$ are possible to be observed, just with a speed 1. For (8.2), and within our framework, we have also proved in [22] that there exist six travelling waves (all of them with a speed $\gamma'(t) = 1$) with profiles of the form $T = c_1 + (c_2 - c_1)H$, where $c_1, c_2 \in \mathbb{R}$ and $c_1 \neq c_2$: $(c_1, c_2) = (0,1), (1,0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (1, \frac{1}{2}), (\frac{1}{2}, 1)$.

Acknowledgment The author expresses his thanks to Professor Vaz Ferreira of Bologna University for the helpful discussions and critical comments.

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