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Zero Dissipation Limit to Rarefaction Waves for the 1-D Compressible Navier-Stokes Equations*

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Abstract The zero dissipation limit for the one-dimensional Navier-Stokes equations of compressible, isentropic gases in the case that the corresponding Euler equations have rarefaction wave solutions is investigated in this paper. In a paper (Comm. Pure Appl. Math., **46**, 1993, 621–665) by Z. P. Xin, the author constructed a sequence of solutions to one-dimensional Navier-Stokes isentropic equations converging to the rarefaction wave as the viscosity tends to zero. Furthermore, he obtained that the convergence rate is $\varepsilon^{\frac{1}{4}}|\ln\varepsilon|$. In this paper, Xin's convergence rate is improved to $\varepsilon^{\frac{1}{3}}|\ln\varepsilon|^2$ by different scaling arguments. The new scaling has various applications in related problems.

Keywords Compressible Navier-Stokes equations, Rarefaction wave, Compressible Euler equations
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1 Introduction and Main Result

The one-dimensional Navier-Stokes equations of compressible, isentropic gases in Lagrangian coordinate read:

$$\begin{cases} v_t^\varepsilon - u_x^\varepsilon = 0, \\ u_t^\varepsilon + p(v^\varepsilon)_x = \varepsilon\Big(\dfrac{u_x^\varepsilon}{v^\varepsilon}\Big)_x, \quad x \in \mathbb{R}^1,\ t > 0, \end{cases} \tag{1.1}$$

where $u^\varepsilon, v^\varepsilon$ and $p(v^\varepsilon)$ denote the fluid velocity, the specific volume, and the pressure of gases respectively, $\varepsilon > 0$ is the viscosity coefficient, and the pressure $p(v^\varepsilon)$ is assumed to be a smooth function of $v^\varepsilon > 0$ satisfying

$$p'(v^\varepsilon) < 0 < p''(v^\varepsilon) \quad \text{for } v^\varepsilon > 0. \tag{1.2}$$

Formally, as viscosity vanishes, the solutions to the compressible Navier-Stokes equations converge to those of the corresponding inviscid Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0. \end{cases} \tag{1.3}$$

This limit has been rigorously verified by many people when the Euler equations (1.3) have smooth solutions. However, the Euler equations (1.3) usually do not have smooth solutions. For example, it is well-known that the compressible Euler equations have three basic wave patterns

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which are two nonlinear waves, i.e. shock and rarefaction waves, and a linearly degenerate wave, contact discontinuity. All of the above basic waves are not smooth. Therefore, it is important to justify the zero dissipation limit for the basic wave patterns. Hoff and Liu [1] studied the zero dissipation limit problem for the system (1.1) in the case that the inviscid flow is a single shock wave, and they showed that the solutions to the Navier-Stokes equation (1.1) with shock data exist and converge to the inviscid shocks as the viscosity vanishes, uniformly away from the shocks. In [8], Z. Xin studied the zero dissipation limit in the rarefaction wave case, and moreover, he obtained that the convergence rate toward the rarefaction wave is $\varepsilon^{\frac{1}{4}}|\ln\varepsilon|$. In this paper, we improve Xin's convergence rate to $\varepsilon^{\frac{1}{3}}|\ln\varepsilon|^2$ by different scaling arguments. We now precisely state our main theorems. Without loss of generality, we only study the 1-rarefaction wave. First of all, we consider the Euler equations (1.3) with the Riemann initial data

$$(u,\upsilon)|_{t=0}=\begin{cases}(u_-,\upsilon_-), & x<0,\\ (u_+,\upsilon_+), & x>0,\end{cases} \tag{1.4}$$

in which $u_\pm, \upsilon_\pm$ (>0) are given constants. The end states $u_\pm, \upsilon_\pm$ in (1.4) satisfy

$$u_+ + \int_{\upsilon_-}^{\upsilon_+} \lambda_1(\upsilon)\mathrm{d}\upsilon = u_-, \quad \lambda_1(\upsilon_+) > \lambda_1(\upsilon_-), \tag{1.5}$$

so that the left state (u_-,υ_-) can be connected to (u_+,υ_+) by 1-rarefaction wave. The 1-centered rarefaction wave of (1.3) is the self-similar solution $(u^r,\upsilon^r)(\frac{x}{t})$, defined by

$$\lambda_1(\upsilon^r(\xi)) = \begin{cases}\lambda_1(\upsilon_-), & \xi<\lambda_1(\upsilon_-),\\ \xi, & \lambda_1(\upsilon_-)\le\xi\le\lambda_1(\upsilon_+),\\ \lambda_1(\upsilon_+), & \xi>\lambda_1(\upsilon_+),\end{cases} \quad \xi=\frac{x}{t}, \tag{1.6}$$

$$u^r(\xi) = u_- - \int_{\upsilon_-}^{\upsilon^r(\xi)} \lambda_1(s)\mathrm{d}s. \tag{1.7}$$

In [8], the author first approximated the centered 1-rarefaction wave $(u^r,\upsilon^r)(\frac{x}{t})$ by a smooth rarefaction wave $(u^r_\varepsilon,\upsilon^r_\varepsilon)(x,t)$, depending on the viscosity ε, so that $(u^r_\varepsilon,\upsilon^r_\varepsilon)$ converges to (u^r,υ^r) at a rate in an appropriate topology as $\varepsilon\to0$. Next, he put a small perturbation on the smooth rarefaction wave $(u^r_\varepsilon,\upsilon^r_\varepsilon)(x,t)$, and chose the scaling $y=\varepsilon^{-\frac{3}{4}}x,\ \tau=\varepsilon^{-\frac{3}{4}}t$ for the perturbation. After scaling, the perturbation were estimated by an elementary energy method on two time scales. Finally, the author proved the following theorem.

Theorem 1.1 (see [8]) *Let $(u^r,\upsilon^r)(\frac{x}{t})$ be the centered 1-rarefaction wave defined by (1.6)–(1.7), which connects two constant states $(u_\pm,\upsilon_\pm)$ satisfying (1.5) with $\upsilon_\pm>0$. Then there exists a positive constant ε_0, such that for each ε, $0<\varepsilon<\varepsilon_0$, we can construct a global smooth solution $(u^\varepsilon,\upsilon^\varepsilon)(x,t)$ to (1.1) with the following properties:*

(i)

$$\begin{aligned}&(u^\varepsilon-u^r,\upsilon^\varepsilon-\upsilon^r)\in C^0(0,+\infty;L_2),\\ &(u^\varepsilon_x,\upsilon^\varepsilon_x)\in C^0(0,+\infty;L_2),\\ &u^\varepsilon_{xx}\in L^2(0,+\infty;L_2);\end{aligned} \tag{1.8}$$

(ii) *As viscosity $\varepsilon\to0$, $(u^\varepsilon,\upsilon^\varepsilon)(x,t)$ converges to $(u^r,\upsilon^r)(\frac{x}{t})$ pointwise except at $(0,0)$. Furthermore, for any given positive constant h, there is a constant $c(h)>0$, independent of ε,* *so that*

$$\sup_{t\ge h}\Big\|(u^\varepsilon,\upsilon^\varepsilon)(\,\cdot\,,t)-(u^r,\upsilon^r)\Big(\frac{\cdot}{t}\Big)\Big\|_{L_\infty}\le c(h)\varepsilon^{\frac14}|\ln\varepsilon|. \tag{1.9}$$

Remark 1.1 Xin's method has been applied to many systems, for instance, non-isentropic Navier-Stokes system and Boltzmann equation (see [9]). For the non-isentropic Navier-Stokes system, the authors obtained the convergence rate $\varepsilon^{\frac14}|\ln\varepsilon|$. But for the Boltzmann equation, the convergence rate becomes slower, only $\varepsilon^{\frac15}|\ln\varepsilon|$.

In this paper, we use a similar argument but a different scaling $y=\varepsilon^{-\frac23}x,\ \tau=\varepsilon^{-\frac23}t$, and improve Xin's convergence rate to $\varepsilon^{\frac13}|\ln\varepsilon|^2$. More precisely, we have the following result.

Theorem 1.2 *Let $(u^r,\upsilon^r)(\frac{x}{t})$ be the centered 1-rarefaction wave defined by (1.6)–(1.7), which connects two constant states $(u_\pm,\upsilon_\pm)$ satisfying (1.5) with $\upsilon_\pm>0$. Then there exists a positive constant ε_0, such that for each ε satisfying $0<\varepsilon<\varepsilon_0$, we can construct a global smooth solution $(u^\varepsilon,\upsilon^\varepsilon)(x,t)$ to (1.1) with the following properties:*

(1)

$$(u^\varepsilon-u^r,\upsilon^\varepsilon-\upsilon^r),(u^\varepsilon_x,\upsilon^\varepsilon_x)\in C^0(0,+\infty;L_2),\quad u^\varepsilon_{xx}\in L^2(0,+\infty;L_2); \tag{1.10}$$

(2) *As viscosity $\varepsilon\to0$, $(u^\varepsilon,\upsilon^\varepsilon)(x,t)$ converges to $(u^r,\upsilon^r)(\frac{x}{t})$ pointwise except at $(0,0)$. Furthermore, for any given positive constant h, there is a constant $c(h)>0$, independent of ε, such that*

$$\sup_{t\ge h}\Big\|(u^\varepsilon,\upsilon^\varepsilon)(\,\cdot\,,t)-(u^r,\upsilon^r)\Big(\frac{\cdot}{t}\Big)\Big\|_{L_\infty}\le c(h)\varepsilon^{\frac13}|\ln\varepsilon|^2. \tag{1.11}$$

The proof of Theorem 1.2 is more simplified than that of Theorem 1.1 in [8], this is because we fully use the second property in Lemma 2.2 which was proved by F. Huang, M. Li and Y. Wang [2]. Using this property, we do not need two time scales, which is one of the main steps of Xin's proof.

The main novelty of our scaling $y=\varepsilon^{-\frac23}x,\ \tau=\varepsilon^{-\frac23}t$ is that it can improve the low order estimate, and also can make the high order estimate match the low order estimate to get the same convergence rate. We expect that this scaling argument can be applied to general systems such as non-isentropic Navier-Stokes system, Boltzmann equation, radiative hydrodynamic system and other related systems to get a better convergence rate.

Throughout this paper, $\|\cdot\|_l, l=0,1,2,\cdots$, denote the usual Sobolev norms for H^l, $\|\cdot\|\equiv\|\cdot\|_{L_2}$. For simplicity, we also write C as generic positive constants which are independent of time t and viscosity ε unless otherwise stated.

2 Rarefaction Waves

In this section, we establish some necessary estimates on the rarefaction wave of the Euler equations based on the inviscid Burgers equation. Since the rarefaction wave is only Lipschitz continuous, we shall construct a smooth approximation for the rarefaction wave in the following. Consider the Riemann problem for the inviscid Burgers equation:

$$\begin{cases} w_t+ww_x=0,\\ w(x,0)=\begin{cases} w_-, & x<0,\\ w_+, & x>0.\end{cases}\end{cases} \tag{2.1}$$

If $w_- < w_+$, then (2.1) has the centered rarefaction wave solution $w^r(x,t) = w^r(\frac{x}{t})$ given by

$$w^r(x,t) = \begin{cases} w_-, & \dfrac{x}{t} \le w_-, \\ \dfrac{x}{t}, & w_- \le \dfrac{x}{t} \le w_+, \\ w_+, & \dfrac{x}{t} \ge w_+. \end{cases} \tag{2.2}$$

To construct a smooth rarefaction wave solution to the Burgers equation which approximates the centered rarefaction wave $w^r(\frac{x}{t})$, we set for each $\delta > 0$,

$$w_\delta(x) = w\Big(\frac{x}{\delta}\Big) =: \frac{(w_+ + w_-)}{2} + \frac{(w_+ - w_-)}{2}\tanh\Big(\frac{x}{\delta}\Big), \tag{2.3}$$

and solve the following initial value problem:

$$\begin{cases} w_t + ww_x = 0, \\ w(x,0) = w_\delta(x). \end{cases} \tag{2.4}$$

Denote the solution to (2.4) by $w_\delta^r(x,t)$. We can show that the smooth rarefaction wave $w_\delta^r(x,t)$ approaches the centered rarefaction ware $w^r(\frac{x}{t})$ as δ goes to zero.

Lemma 2.1 *The problem* (2.4) *has a unique smooth global solution* $w_\delta^r(x,t)$ *for each* $\delta > 0$ *such that*

(1) $w_- < w_\delta^r(x,t) < w_+,\ \partial_x w_\delta^r(x,t) > 0$ *for* $x \in \mathbb{R}^1,\ t \ge 0,\ \delta > 0$;

(2) *The following estimates hold for all* $t > 0,\ \delta > 0$ *and* $p \in [1,\infty]$:

$$\|\partial_x w_\delta^r(x,t)\|_{L^p} \le C(w_+ - w_-)^{\frac{1}{p}}(\delta + t)^{-1+\frac{1}{p}},$$
$$\|\partial_x^2 w_\delta^r(x,t)\|_{L^p} \le C\delta^{-1+\frac{1}{p}}(\delta + t)^{-1};$$

(3) *There exists a constant* $\delta_0 \in (0,1)$, *such that for* $\delta \in (0,\delta_0],\ t > 0$,

$$\|w_\delta^r(\cdot\,,t) - w^r(\cdot\,,t)\|_{L^\infty} \le C\delta t^{-1}(\ln(1+t) + |\ln\delta|).$$

The first and the third properties of Lemma 2.1 can be found in [8], and the second one was proved in [2]. Although the second property is equivalent to the corresponding one in [8], it is more convenient to use so that we do not need two time scales, which is one of the main steps of Xin's proof (see [8]).

We now turn to rarefaction waves for the Euler equations (1.3). Here and in what follows, the constant states $(u_\pm, \upsilon_\pm)$ are fixed so that (u_+, υ_+) lies on the 1-rarefaction wave curve through (u_-, υ_-) (see (1.5) or [7]). Set $w_- = \lambda_1(\upsilon_-)$ and $w_+ = \lambda_1(\upsilon_+)$ in (2.1)–(2.4). It is easy to check that the 1-rarefaction wave $(u^r, \upsilon^r)(x,t)$ to the Riemann problem (1.3)–(1.5) is given by

$$\lambda_1(\upsilon^r(x,t)) = w^r(x,t), \tag{2.5}$$

$$u^r(x,t) + \int_{\upsilon_-}^{\upsilon^r(x,t)} \lambda_1(\upsilon)\mathrm{d}\upsilon = u_-. \tag{2.6}$$

Now for each positive constant δ, we define a pair of smooth functions $(u_\delta^r, \upsilon_\delta^r)$ by

$$\lambda_1(\upsilon_\delta^r) = w_\delta^r(x,t), \tag{2.7}$$

$$u_\delta^r(x,t) + \int_{\upsilon_-}^{\upsilon_\delta^r(x,t)} \lambda_1(\upsilon)\mathrm{d}\upsilon = u_-, \tag{2.8}$$

where $w_\delta^r(x,t)$ is defined in (2.4). Then $(u_\delta^r, \upsilon_\delta^r)$ is a smooth 1-rarefaction wave of (1.3). It follows from Lemma 2.1.

Lemma 2.2 $(u_\delta^r, \upsilon_\delta^r)$*, as described above, is a smooth solution to the Euler equations* (1.3) *satisfying:*

(1) $\partial_x u_\delta^r \geq 0$, $\forall x \in \mathbb{R},\ t \geq 0$;

(2) *The following estimates hold for all* $t > 0,\ \delta > 0$ *and* $p \in [1, \infty]$:

$$\|(\partial_x u_\delta^r(\,\cdot\,,t), \partial_x \upsilon_\delta^r(\,\cdot\,,t))\|_{L^p} \leq C(w_+ - w_-)^{\frac{1}{p}}(\delta + t)^{-1+\frac{1}{p}},$$
$$\|(\partial_{xx} u_\delta^r(\,\cdot\,,t), \partial_{xx} \upsilon_\delta^r(\,\cdot\,,t))\|_{L^p} \leq C\delta^{-1+\frac{1}{p}}(\delta + t)^{-1};$$

(3) *There exists a constant* $\delta_0 \in (0,1)$*, such that for* $\delta \in (0, \delta_0],\ t > 0,$

$$\|(u^r - u_\delta^r, \upsilon^r - \upsilon_\delta^r)\|_{L^\infty} \leq C\delta t^{-1}(\ln(1+t) + |\ln \delta|).$$

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.2. Suppose that $(u^\varepsilon, \upsilon^\varepsilon)(x,t)$ is the solution to (1.1) with the following initial data:

$$(u^\varepsilon, \upsilon^\varepsilon)(x,0) =: (u_0^\varepsilon, \upsilon_0^\varepsilon)(x) = (u_\delta^r, \upsilon_\delta^r)(x,0). \tag{3.1}$$

Define

$$(\psi, \phi)(y,\tau) = (u^\varepsilon, \upsilon^\varepsilon)(x,t) - (u_\delta^r, \upsilon_\delta^r)(x,t), \tag{3.2}$$

where

$$y = \varepsilon^{-\frac{2}{3}}x, \quad \tau = \varepsilon^{-\frac{2}{3}}t. \tag{3.3}$$

Substituting (3.1)–(3.3) into (1.1) and noting that $(u_\delta^r, \upsilon_\delta^r)(x,t)$ is a smooth solution to (1.3), we obtain

$$\begin{cases} \phi_\tau - \psi_y = 0, \\ \psi_\tau + (p'(\upsilon_\delta^r)\phi)_y + Q_y = \varepsilon^{\frac{1}{3}}\Big(\dfrac{\psi_y}{\upsilon}\Big)_y + \varepsilon^{\frac{1}{3}}\Big(\dfrac{(u_\delta^r)_y}{\upsilon}\Big)_y, \end{cases} \tag{3.4}$$

$$(\psi, \phi)(y,0) =: (\psi_0, \phi_0)(y) = 0, \tag{3.5}$$

where $Q = p(\upsilon) - p(\upsilon_\delta^r) - p'(\upsilon_\delta^r)\phi$.

We seek a global (in time) and bounded (in L_∞ norm) solution (ψ, ϕ) to the problem (3.4)–(3.5). To this end, we set the solution space for (3.4)–(3.5) to be

$$X(0,T) = \left\{ (\psi,\phi) \,\middle|\, \begin{array}{l} (\psi,\phi) \in C^0(0,T;H^1), \psi_y \in L^2(0,T;H^1), \phi_y \in L^2(0,T;L^2) \\ \text{and } \upsilon_\delta^r + \phi \geq \frac{1}{2}\upsilon_- \end{array} \right\} \tag{3.6}$$

with $0 < T \le +\infty$.

Proposition 3.1 *There exists a positive constant ε_0 which is independent of ε, δ, such that if $0 < \varepsilon \le \varepsilon_0$, $\varepsilon^{\frac{1}{3}}|\ln\varepsilon| \le \delta$, then the initial value problem* (3.4)–(3.5) *has a unique global solution $(\psi, \phi) \in X(0, +\infty)$ satisfying*

$$\begin{aligned}&\sup_{0\le\tau\le+\infty}\|(\psi,\phi)(\tau)\|_1^2 + \int_0^{+\infty}\int_{\mathbb{R}} \upsilon_{\delta\tau}^r(\phi^2+\phi_y^2)\mathrm{d}y\mathrm{d}\tau \\ &+ \varepsilon^{\frac{1}{3}}\int_0^{+\infty}(\|\phi_y(\tau)\|^2 + \|\psi_y(\tau)\|_1^2)\mathrm{d}\tau \le \frac{1}{4}\varepsilon^{\frac{2}{3}}.\end{aligned} \tag{3.7}$$

Once we have Proposition 3.1, we can take $\delta = \varepsilon^{\frac{1}{3}}|\ln\varepsilon|$. Then (3.7) implies

$$\sup_{0\le\tau\le+\infty}\|(\psi,\phi)(\tau)\|_{L_\infty} \le \frac{1}{2}\varepsilon^{\frac{1}{3}} \tag{3.8}$$

by the Sobolev's inequality. Thus the proof of Theorem 1.1 is completed.

Since the proof for the local existence of the solution to (3.4)–(3.5) is standard, the details are omitted. To prove Proposition 3.1, the crucial step is to close the following a priori assumption:

$$N(0,\tau_1) =: \sup_{0\le\tau\le\tau_1}\|(\psi,\phi)(\tau)\|_{H^1} \le \varepsilon^{\frac{1}{3}}. \tag{3.9}$$

Proposition 3.2 (A Priori Estimate) *Suppose that the problem* (3.4)–(3.5) *has a solution $(\psi, \phi) \in X(0, \tau_1)$ for some $\tau_1 > 0$. Then there exists a positive constant ε_1, independent of ε, δ and τ_1, such that if*

$$0 < \varepsilon \le \varepsilon_1, \quad \varepsilon^{\frac{1}{3}}|\ln\varepsilon| \le \delta \quad \text{and} \quad N(0,\tau_1) \le \varepsilon^{\frac{1}{3}}, \tag{3.10}$$

then

$$\begin{aligned}&\sup_{0\le\tau\le\tau_1}\|(\psi,\phi)(\tau)\|_1^2 + \int_0^{\tau}\int_{\mathbb{R}} \upsilon_{\delta\tau}^r(\phi^2+\phi_y^2)\mathrm{d}y\mathrm{d}\tau \\ &+ \varepsilon^{\frac{1}{3}}\int_0^{\tau}(\|\phi_y(\tau)\|^2 + \|\psi_y(\tau)\|_1^2)\mathrm{d}\tau \le C\varepsilon^{\frac{5}{6}}\delta^{-\frac{1}{2}} \le \frac{1}{4}\varepsilon^{\frac{2}{3}}.\end{aligned} \tag{3.11}$$

Due to the a priori assumption (3.9), the Sobolev's inequality and smallness of ε, we always have

$$\upsilon_\delta^r + \phi \ge \frac{1}{2}\upsilon_-. \tag{3.12}$$

Lemma 3.1 *Under the assumption of Proposition* 3.2*, there exists a positive constant C, such that for $0 \le \tau \le \tau_1$,*

$$\sup_{0\le\tau\le\tau_1}\|(\psi,\phi)(\tau)\|^2 + \int_0^{\tau}\int_{\mathbb{R}} \upsilon_{\delta\tau}^r\phi^2\mathrm{d}y\mathrm{d}\tau + \varepsilon^{\frac{1}{3}}\int_0^{\tau}\|\psi_y(\tau)\|^2\mathrm{d}\tau \le C\varepsilon^{\frac{5}{6}}\delta^{-\frac{1}{2}}. \tag{3.13}$$

Proof We multiply the first and second equations in (3.4) by $p(v_\delta^r)-p(v)$ and ψ, respectively, sum them, and integrate the result with respect to y over $\mathbb{R}$ to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}}\Big\{\frac{1}{2}\psi^2+p(v_\delta^r)\phi-\int_{v_\delta^r}^{v}p(s)\mathrm{d}s\Big\}\mathrm{d}y+C\int_{\mathbb{R}}v_{\delta\tau}^r\phi^2\mathrm{d}y+C\varepsilon^{\frac{1}{3}}\|\psi_y(\tau)\|^2\\&\le\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}}\psi\Big(\frac{u_{\delta y}^r}{v}\Big)_y\mathrm{d}y.\end{aligned}\tag{3.14}$$

Then, integrating with (3.14) over $[0,\tau]$, and noting that $v_{\delta\tau}^r=u_{\delta y}^r\ge 0$, we get

$$\begin{aligned}&\|(\psi,\phi)(\tau)\|^2+\int_0^\tau\int_{\mathbb{R}}v_{\delta\tau}^r\phi^2\mathrm{d}y\mathrm{d}\tau+\varepsilon^{\frac{1}{3}}\int_0^\tau\|\psi_y(\tau)\|^2\mathrm{d}\tau\\&\le C\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}\psi\Big(\frac{u_{\delta y}^r}{v}\Big)_y\mathrm{d}y\mathrm{d}\tau.\end{aligned}\tag{3.15}$$

We compute

$$\begin{aligned}\Big|\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}\psi\Big(\frac{u_{\delta y}^r}{v}\Big)_y\mathrm{d}y\mathrm{d}\tau\Big|&=\Big|\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}\psi\Big(\frac{u_{\delta y}^r}{v_\delta^r}\Big)_y\mathrm{d}y\mathrm{d}\tau+\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}\frac{u_{\delta y}^r\psi_y\phi}{vv_\delta^r}\mathrm{d}y\mathrm{d}\tau\Big|\\&\le C\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}|\psi|(|u_{\delta yy}^r|+|u_{\delta y}^r|^2)\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}|u_{\delta y}^r\psi_y\phi|\mathrm{d}y\mathrm{d}\tau\\&\le C\varepsilon^{\frac{1}{3}}\int_0^\tau\|\psi\|^{\frac{1}{2}}\|\psi_y\|^{\frac{1}{2}}(\|u_{\delta yy}^r\|_{L^1}+\|u_{\delta y}^r\|^2)\mathrm{d}\tau\\&\quad+\eta\varepsilon^{\frac{1}{3}}\int_0^\tau\|\psi_y(\tau)\|^2\mathrm{d}\tau+C_\eta\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}(v_{\delta\tau}^r)^2\phi^2\mathrm{d}y\mathrm{d}\tau\\&\le 2\eta\varepsilon^{\frac{1}{3}}\int_0^\tau\|\psi_y(\tau)\|^2\mathrm{d}\tau+C_\eta\varepsilon^{\frac{1}{3}}\int_0^\tau\|\psi\|^{\frac{2}{3}}(\|u_{\delta yy}^r\|_{L^1}^{\frac{4}{3}}+\|u_{\delta y}^r\|^{\frac{8}{3}})\mathrm{d}\tau\\&\quad+C_\eta\varepsilon\delta^{-1}\int_0^\tau\int_{\mathbb{R}}v_{\delta\tau}^r\phi^2\mathrm{d}y\mathrm{d}\tau,\end{aligned}\tag{3.16}$$

where η is a small constant. Now we estimate the second term of (3.16) on the right-hand side by Lemma 2.1

$$\begin{aligned}&\varepsilon^{\frac{1}{3}}\int_0^\tau\|\psi\|^{\frac{2}{3}}(\|u_{\delta yy}^r\|_{L^1(y)}^{\frac{4}{3}}+\|u_{\delta y}^r\|_{L^2(y)}^{\frac{8}{3}})\mathrm{d}\tau\\&\le\eta\sup_{0\le\tau\le\tau_1}\|\psi\|^2+C_\eta\varepsilon^{\frac{1}{2}}\Big(\varepsilon^{\frac{8}{9}}\int_0^\tau(\|u_{\delta xx}^r\|_{L^1(x)}^{\frac{4}{3}}+\|u_{\delta x}^r\|_{L^2(x)}^{\frac{8}{3}})\mathrm{d}\tau\Big)^{\frac{3}{2}}\\&\le\eta\sup_{0\le\tau\le\tau_1}\|\psi\|^2+C_\eta\varepsilon^{\frac{5}{6}}\delta^{-\frac{1}{2}}.\end{aligned}\tag{3.17}$$

Combining (3.16) and (3.17), using a priori assumption (3.9) and letting η,ε be sufficiently small, we complete the proof of Lemma 3.1.

We now turn to estimating the derivatives of (ψ,ϕ).

Lemma 3.2 *Under the assumption of Proposition* 3.2*, there exists a positive constant* C*, such that for* $0\le\tau\le\tau_1$*,*

$$\begin{aligned}&\sup_{0\le\tau\le\tau_1}\|(\psi_y,\phi_y)(\tau)\|^2+\int_0^\tau\int_{\mathbb{R}}v_{\delta\tau}^r\phi_y^2\mathrm{d}y\mathrm{d}\tau\\&+\varepsilon^{\frac{1}{3}}\int_0^\tau(\|\phi_y(\tau)\|^2+\|\psi_{yy}(\tau)\|^2)\mathrm{d}\tau\le C\varepsilon^{\frac{5}{6}}\delta^{-\frac{1}{2}}.\end{aligned}\tag{3.18}$$

Proof We multiply the second equation of (3.4) by ψ_{yy}, and integrate the result equation over $\mathbb{R}$ to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\{\int_{\mathbb{R}}\frac{1}{2}\psi_y^2\mathrm{d}y-\int_{\mathbb{R}}\frac{1}{2}p'(\upsilon_\delta^r)\phi_y^2\mathrm{d}y\Big\}+C\int_{\mathbb{R}}p''(\upsilon_\delta^r)\upsilon_{\delta\tau}^r\phi_y^2\mathrm{d}y+C\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2\\ \le\,& C\int_{\mathbb{R}}p''(\upsilon_\delta^r)\upsilon_{\delta y}^r\phi\psi_{yy}\mathrm{d}y+C\int_{\mathbb{R}}Q_y\psi_{yy}\mathrm{d}y\\ &+\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}}\frac{\psi_y(\phi_y+\upsilon_{\delta y}^r)}{\upsilon^2}\psi_{yy}\mathrm{d}y-\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}}\Big(\frac{u_{\delta y}^r}{\upsilon}\Big)_y\psi_{yy}\mathrm{d}y. \end{aligned}\tag{3.19}$$

We estimate each term on the right-hand side. First, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}}p''(\upsilon_\delta^r)\upsilon_{\delta y}^r\phi\psi_{yy}\mathrm{d}y\Big|&\le\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C_\eta\varepsilon^{-\frac{1}{3}}\int_{\mathbb{R}}(\upsilon_{\delta y}^r)^2\phi^2\mathrm{d}y\\ &\le\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C_\eta\varepsilon^{\frac{1}{3}}\delta^{-1}\int_{\mathbb{R}}\upsilon_{\delta\tau}^r\phi^2\mathrm{d}y,\end{aligned}\tag{3.20}$$

where we have used Lemma 2.1 and the Hölder's inequality. By (3.10), we get

$$\begin{aligned}\Big|\int_{\mathbb{R}}Q_y\psi_{yy}\mathrm{d}y\Big|&\le\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C_\eta\varepsilon^{-\frac{1}{3}}\int_{\mathbb{R}}(\phi^2\phi_y^2+(\upsilon_{\delta y}^r)^2\phi^2)dy\\ &\le\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C_\eta\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}}\phi_y^2\mathrm{d}y+C_\eta\varepsilon^{\frac{1}{3}}\delta^{-1}\int_{\mathbb{R}}\upsilon_{\delta\tau}^r\phi^2\mathrm{d}y,\end{aligned}\tag{3.21}$$

$$\begin{aligned}\Big|\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}}\frac{\psi_y(\phi_y+\upsilon_{\delta y}^r)}{\upsilon^2}\psi_{yy}\mathrm{d}y\Big|&\le\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C_\eta\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}}((\upsilon_{\delta y}^r)^2\psi_y^2+\psi_y^2\phi_y^2)\mathrm{d}y\\ &\le\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C_\eta\varepsilon^{\frac{5}{3}}\delta^{-2}\int_{\mathbb{R}}\psi_y^2\mathrm{d}y+C_\eta\varepsilon^{\frac{1}{3}}\|\psi_y\|\|\psi_{yy}\|\|\phi_y\|^2\\ &\le2\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C_\eta\varepsilon\int_{\mathbb{R}}\psi_y^2\mathrm{d}y,\end{aligned}\tag{3.22}$$

where we have used $\varepsilon^{\frac{1}{3}}|\ln\varepsilon|\le\delta$. Finally, we estimate the last term on the right-hand side of (3.19) by Lemma 2.1, i.e.,

$$\begin{aligned}\Big|\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}}\Big(\frac{u_{\delta y}^r}{\upsilon}\Big)_y\psi_{yy}\mathrm{d}y\Big|&\le\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C_\eta\varepsilon^{\frac{1}{3}}(\|u_{\delta yy}^r\|^2+\|u_{\delta y}^r\|_{L^4}^4)+C\varepsilon^{\frac{1}{3}}\int_{\mathbb{R}}(u_{\delta y}^r)^2\phi_y^2\mathrm{d}y\\ &\le\eta\varepsilon^{\frac{1}{3}}\|\psi_{yy}(\tau)\|^2+C\varepsilon\delta^{-1}\int_{\mathbb{R}}\upsilon_{\delta\tau}^r\phi_y^2\mathrm{d}y\\ &\quad+C_\eta\varepsilon^{\frac{7}{3}}[\delta^{-1}(\delta+t)^{-2}+(\delta+t)^{-3}].\end{aligned}\tag{3.23}$$

From (3.20)–(3.23), integrating (3.19) with respect to τ and letting η,ε be sufficiently small, we get

$$\begin{aligned}&\|(\psi_y,\phi_y)(\tau)\|^2+\int_0^\tau\int_{\mathbb{R}}\upsilon_{\delta\tau}^r\phi_y^2\mathrm{d}y\mathrm{d}\tau+\varepsilon^{\frac{1}{3}}\int_0^\tau\|\psi_{yy}(\tau)\|^2\mathrm{d}\tau\\ \le\,& C\int_0^\tau\int_{\mathbb{R}}\upsilon_{\delta\tau}^r\phi^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}\phi_y^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{\frac{1}{3}}\int_0^\tau\int_{\mathbb{R}}\psi_y^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{\frac{5}{3}}\delta^{-2}.\end{aligned}\tag{3.24}$$

To finish the a priori estimate (3.11), we need to control the integral $\int_0^\tau\int_{\mathbb{R}}\phi_y^2\mathrm{d}y\mathrm{d}\tau$. To this end, we multiply the second equation of (3.4) by ϕ_y and integrate it over $\mathbb{R}$ with respect to y

to obtain

$$\begin{aligned}-\int_{\mathbb{R}}(p'(v_\delta^r)\phi)_y\phi_y\mathrm{d}y\le{}&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}}\psi\phi_y\mathrm{d}y+\int_{\mathbb{R}}\psi_y^2\mathrm{d}y+\int_{\mathbb{R}}Q_y\phi_y\mathrm{d}y\\&-\varepsilon^{\frac13}\int_{\mathbb{R}}\Big(\frac{u_{\delta y}^r}{v}\Big)_y\phi_y\mathrm{d}y-\varepsilon^{\frac13}\int_{\mathbb{R}}\Big(\frac{\psi_y}{v}\Big)_y\phi_y\mathrm{d}y.\end{aligned}\tag{3.25}$$

We estimate each term in (3.25) separately as follows. First

$$\begin{aligned}-\int_{\mathbb{R}}(p'(v_\delta^r)\phi)_y\phi_y\mathrm{d}y&=-\int_{\mathbb{R}}p'(v_\delta^r)\phi_y^2\mathrm{d}y-\int_{\mathbb{R}}p''(v_\delta^r)v_{\delta y}^r\phi\phi_y\mathrm{d}y\\&\ge-\frac12\int_{\mathbb{R}}p'(v_\delta^r)\phi_y^2\mathrm{d}y-C\int_{\mathbb{R}}(v_{\delta y}^r)^2\phi^2\mathrm{d}y\\&\ge C_1\int_{\mathbb{R}}\phi_y^2\mathrm{d}y-C_2\varepsilon^{\frac23}\delta^{-1}\int_{\mathbb{R}}v_{\delta\tau}^r\phi^2\mathrm{d}y,\end{aligned}\tag{3.26}$$

where C_1, C_2 are constants. Next,

$$\begin{aligned}\Big|\varepsilon^{\frac13}\int_{\mathbb{R}}\Big(\frac{\psi_y}{v}\Big)_y\phi_y\mathrm{d}y\Big|&=\Big|\varepsilon^{\frac13}\int_{\mathbb{R}}\frac{\psi_{yy}\phi_y}{v}\mathrm{d}y-\varepsilon^{\frac13}\int_{\mathbb{R}}\frac{\psi_y\phi_y^2}{v^2}\mathrm{d}y-\varepsilon^{\frac13}\int_{\mathbb{R}}\frac{v_{\delta y}^r\psi_y\phi_y}{v^2}\mathrm{d}y\Big|\\&\le C\varepsilon^{\frac13}\int_{\mathbb{R}}\psi_{yy}^2\mathrm{d}y+C\varepsilon^{\frac13}\int_{\mathbb{R}}\phi_y^2\mathrm{d}y+C\varepsilon^{\frac13}\int_{\mathbb{R}}\psi_y^2\mathrm{d}y\\&\quad+C\varepsilon^{\frac13}\int_{\mathbb{R}}(v_{\delta y}^r)^2\phi_y^2\mathrm{d}y+C\varepsilon^{\frac13}\|\psi_y\|^{\frac12}\|\psi_{yy}\|^{\frac12}\|\phi_y\|^2\\&\le C\varepsilon^{\frac13}\int_{\mathbb{R}}\psi_{yy}^2\mathrm{d}y+C\varepsilon^{\frac13}\int_{\mathbb{R}}\phi_y^2\mathrm{d}y+C\varepsilon^{\frac13}\int_{\mathbb{R}}\psi_y^2\mathrm{d}y\\&\quad+C\varepsilon\delta^{-1}\int_{\mathbb{R}}v_{\delta\tau}^r\phi_y^2\mathrm{d}y,\end{aligned}\tag{3.27}$$

where we have used the Cauchy inequality and (3.10). The last two terms of (3.25) can be estimated as follows:

$$\begin{aligned}\Big|\int_{\mathbb{R}}Q_y\phi_y\mathrm{d}y\Big|&\le C\int_{\mathbb{R}}|\phi||\phi_y|^2\mathrm{d}y+C\int_{\mathbb{R}}|\phi||v_{\delta y}^r||\phi_y|\mathrm{d}y\\&\le(C\varepsilon^{\frac13}+\eta)\int_{\mathbb{R}}\phi_y^2\mathrm{d}y+C_\eta\varepsilon^{\frac23}\delta^{-1}\int_{\mathbb{R}}v_{\delta\tau}^r\phi^2\mathrm{d}y,\end{aligned}\tag{3.28}$$

$$\begin{aligned}\Big|\varepsilon^{\frac13}\int_{\mathbb{R}}\Big(\frac{u_{\delta y}^r}{v}\Big)_y\phi_y\mathrm{d}y\Big|&\le\eta\int_{\mathbb{R}}\phi_y^2\mathrm{d}y+C_\eta\varepsilon^{\frac23}(\|u_{\delta yy}^r\|^2+\|u_{\delta y}^r\|_{L^4}^4)+C\varepsilon^{\frac13}\int_{\mathbb{R}}|u_{\delta y}^r|\phi_y^2\mathrm{d}y\\&\le(\eta+C\varepsilon\delta^{-1})\int_{\mathbb{R}}\phi_y^2\mathrm{d}y+C_\eta\varepsilon^{\frac83}[\delta^{-1}(\delta+t)^{-2}+(\delta+t)^{-3}].\end{aligned}\tag{3.29}$$

Using estimates (3.26)–(3.29) in (3.25), integrating the resulting inequality with respect to τ, and letting η,ε be sufficiently small, we get

$$\begin{aligned}\int_0^\tau\|\phi_y\|^2\mathrm{d}\tau\le{}&C\|\psi\|^2+C\|\phi_y\|^2+C\int_0^\tau\|\psi_y\|^2\mathrm{d}\tau\\&+C\varepsilon^{\frac13}\int_0^\tau\|\psi_{yy}\|^2\mathrm{d}\tau+C\varepsilon\delta^{-1}\int_0^\tau\int_{\mathbb{R}}v_{\delta\tau}^r\phi_y^2\mathrm{d}y\mathrm{d}\tau\\&+C\varepsilon^{\frac23}\delta^{-1}\int_0^\tau\int_{\mathbb{R}}v_{\delta\tau}^r\phi^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon^2\delta^{-2}.\end{aligned}\tag{3.30}$$

Multiplying (3.30) by $\varepsilon^{\frac{1}{3}}$ and substituting it into (3.24) give

$$\begin{aligned}&\|(\psi_y,\phi_y)(\tau)\|^2+\int_0^\tau\int_{\mathbb{R}}\upsilon^r_{\delta\tau}\phi_y^2\mathrm{d}y\mathrm{d}\tau+\varepsilon^{\frac{1}{3}}\int_0^\tau(\|\phi_y(\tau)\|^2+\|\psi_{yy}(\tau)\|^2)\mathrm{d}\tau\\&\le C\varepsilon^{\frac{1}{3}}\|\psi\|^2+C\int_0^\tau\int_{\mathbb{R}}\upsilon^r_{\delta\tau}\phi^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{\frac{1}{3}}\int_0^\tau\|\psi_y\|^2\mathrm{d}\tau+C\varepsilon^{\frac{5}{3}}\delta^{-2},\end{aligned}\tag{3.31}$$

which together with Lemma 3.1 immediately yields the desired estimate (3.18). Thus Lemma 3.2 is proved.

Now taking $\delta=\varepsilon^{\frac{1}{3}}|\ln\varepsilon|$ and letting ε be small enough, we prove Proposition 3.2.

References

[1] David, H. and Liu, T. P., The inviscid limit for the Navier-Stokes equations of compressible, isentropic flow with shock data, *Indiana Univ. Math. J.*, **38**(4), 1989, 861–915.

[2] Huang, F. M., Li, M. J. and Wang, Y., Zero dissipation limit to rarefaction wave with vacuum for 1-D compressible Navier-Stokes equations, preprint.

[3] Jiang, S., Ni, G. X. and Wen, J. S., Vanishing viscosity limit to rarefaction waves for the Navier-Stokes equations of one-dimensional compressible heat-conducting fluids (electronic), *SIAM J. Math. Anal.*, **38**(2), 2006, 368–384.

[4] Lax, P. D., Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, Slam Reg. Conf. Ser. Appl. Math., Philadelphia, **11**, 1973.

[5] Liu, T. P. and Xin, Z. P., Nonlinear stability of rarefaction waves for compressible Navier-Stokes equation, *Comm. Math. Phys.*, **118**, 1988, 451–465.

[6] Matsumura, A. and Nishihara, K., Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.*, **3**(1) , 1986, 1–13.

[7] Smoller, J., Shock Waves and Rarefaction-Diffusion Equations, Springer-Verlag, New York, Berlin, 1983.

[8] Xin, Z. P., Zero dissipation limit to rarefaction waves for the one-dimensional Navier-Stokes equations of compressible isentropic gases, *Comm. Pure Appl. Math.*, **46**, 1993, 621–665.

[9] Xin, Z. P. and Zeng, H., Convergence to rarefaction waves for the nonlinear Boltzmann equation and compressible Navier-Stokes equations, *J. Diff. Eq.*, **249**, 2010, 827–871.