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Large-Time Behavior of Periodic Solutions to Fractal Burgers Equation with Large Initial Data*

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Abstract The asymptotic behavior of periodic solutions to fractal nonlinear Burgers equation is considered and the initial data are allowed to be arbitrarily large. The exponential decay estimates of the solutions are obtained for the power of Laplacian $\alpha \in [\frac{1}{2}, 1)$.

Keywords Fractal Burgers equation, Large-time behavior, Large initial data, Periodic solution, Exponential decay
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1 Introduction

Consider the following Burgers equation with fractional dissipation:

$$\begin{cases} u_t + (-\Delta)^\alpha u = \mathrm{div} f(u), & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

The parameter α regulates the strength of dissipation. In this paper, we consider the range $\frac{1}{2} \le \alpha < 1$, and $f_i(u) = u^2$ $(i = 1, \cdots, n)$.

The Burgers equation with $\alpha = 0$ and $\alpha = 1$ has received an extensive amount of attention since 1940s. If $\alpha = 0$, the equation is perhaps the most classical example of a PDE evolution leading to shocks. If $\alpha = 1$, it provides an important model for studying the interaction between nonlinear and dissipative phenomena. The cases $\alpha > \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\alpha < \frac{1}{2}$ are called respectively sub-critical, critical and super-critical. Recently, the fractal Burgers equation has attracted much attention from various authors. For instance, Dong et al. [1] studied the finite time singularities and global well-posedness of the 1-dimensional fractal Burgers equations, and their estimates show that if the initial data belong to $L^{\frac{1}{\gamma-1}}$, then the solution and its spatial derivatives are in the Gevrey class G^γ. They also proved that solutions with initial data in $H^{\frac{1}{2}}(\mathbb{R})$ are analytic in x for the critical case. In the last two years, researchers were more interested in the critical Burgers equation. Miao and Wu [2] obtained the global well-posedness of the critical Burgers in critical Besov spaces $\dot{B}^{\frac{1}{p}}_{p,1}$ $(p \in [1, +\infty))$ by the method of modulus of continuity and Fourier localization technique. Chan and Czubak [3] considered the N-dimensional critical Burgers equation and obtained the existence of smooth solutions given any initial data in $L^2(\mathbb{R}^N)$ by de-Giorgi's method. There are also some papers in which the spatial

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domain considered is the period box $\Omega = [0, 2\pi]^n$. For instance, Kiselev et al. [4] obtained the global existence of the periodic solution to the critical Burgers equation and also got the uniform in time control of $\|\nabla u\|_{L^\infty}$. Additionally, they mentioned the existence of solution with large initial data for $\frac{1}{2} < \alpha < 1$ without giving the proof, but they proved that the solution would blow up in finite time for $\alpha < \frac{1}{2}$. Due to the fact that the fractal Burgers equation and the dissipative quasi-geostrophic equation have the same fractional dissipation, they have some similar properties. Lately, the dissipative quasi-geostrophic equation has been intensively investigated too. For example, Kiselev et al. [5] obtained the global periodic solution to the critical 2-dimensional dissipative quasi-geostrophic equation with periodic smooth initial data $u_0(x)$. However, the global regularity of solutions or finite time blow up for dissipative quasi-geostrophic equation remains unsolved when dissipation is not strong enough (namely, $\alpha < \frac{1}{2}$). Dong [6] considered the critical and super-critical dissipative quasi-geostrophic equations in $\mathbb{R}^2$ or $\mathbb{T}^2$. He established the global well-posedness for the critical 2-dimensional quasi-geostrophic equations with periodic H^1 data. Some other recent results for the fractal Burgers equations and dissipative quasi-geostrophic equation can be found in [7–13].

In this paper, we consider the asymptotic behavior of solutions to fractal Burgers equation with the spatial domain $\Omega = [0, 2\pi]^n$ and periodic boundary condition, i.e.,

$$\begin{cases} u_t + (-\Delta)^\alpha u = \mathrm{div} f(u), & t > 0,\ x \in \Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\frac{1}{2} \le \alpha < 1$ and $f_i(u) = u^2$ $(i = 1, \cdots, n)$. Without loss of generality, we will restrict the discussion to initial data u_0 with zero mean, namely $\int_\Omega u_0(x)\mathrm{d}x = 0$.

We aim to get the exponential decay estimates of $\|u\|_{H^s}$ in this paper. Generally speaking, the exponential decay of L^2 norm would be obtained much more easily than that of H^s norm due to the fact $\int_\Omega \mathrm{div} f(u)\, u \mathrm{d}x = 0$. When we consider the decay of the derivative of u, the integration of the nonlinear term will not equal zero (i.e., $\int_\Omega \Lambda^s u \Lambda^s \mathrm{div} f(u)\mathrm{d}x \neq 0$). So we should find some ideas to make sure that the nonlinear term can be controlled by the terms on the left-hand side of the equation. In order to get it, we use different methods to deal with the critical case and the sub-critical case. In the sub-critical case, we firstly recall a maximum principle, i.e., the uniform bound with respect to u_0, in the space L^p for any $p \ge 2$, of the solution to (1.2). By the maximum principle, we can replaced $\|u\|_{L^\infty}$ by a constant, and then by the Gagliardo-Nirenberg's inequality, the nonlinear term can be controlled by the left-hand side of the equation. The use of Poincaré's inequality and the exponential decay of $\|u\|_{L^2}$ can yield the exponential decay estimates of $\|u\|_{H^s}$. In the critical case, Kiselev et al. obtained the uniform boundedness of $\|\nabla u\|_{L^\infty}$ in [4]. Making use of this result and the important commutator and the product estimates, we can also get the exponential decay estimates of solutions for $\alpha = \frac{1}{2}$, which is an improvement of the results in [4].

Additionally, although the existence of global solutions was obtained by Kiselev et al. [4], we give a different method to prove the global existence of solutions for $\frac{1}{2} < \alpha < 1$ in this paper just for the convenience of readers. To obtain the global existence, we firstly get the local solution by an iteration scheme which is based on the equation. Then by using the exponential decay estimates, we extend the local solution to be a global one. In the last part of this paper, we get the continuity of the solution with respect to t by a useful lemma which is a particular case of a general interpolation theorem of Lions and Magenes [14].

We have the following main result.

Theorem 1.1 *Assume that $\frac{1}{2} \le \alpha < 1$ and the initial data u_0 belongs to $H^s(\Omega)$, $s > \frac{n}{2}+1$. Then there exists a unique global solution $u(x,t)$ to (1.2), such that*

$$u(x,t) \in C([0,\infty), H^s(\Omega)) \cap L^2([0,\infty), H^{s+\alpha}(\Omega)).$$

Moreover, we have

$$\|u\|_{H^s(\Omega)} \le C\|u_0\|_{H^s(\Omega)}\ \mathrm{e}^{-\frac{1}{2}\lambda_1^{2\alpha}t},$$

where C is a constant and λ_1 is the first eigenvalue of Λ.

Remark 1.1 The global existence for the critical Burgers equation has already been obtained by Kiselev et al. [4].

The rest of this paper is arranged as follows. In Section 2, we present some notations and recall some important preliminary results as an introduction and preparation. In Section 3, we get the local existence of the solution directly by constructing a Cauchy sequence. In Section 4.1, we obtain a maximum principle which is an immediate consequence of positive lemma. In Section 4.2, we get the exponential decay estimates for the sub-critical case $\frac{1}{2} < \alpha < 1$. In Section 5, we obtain the exponential decay for the critical case $\alpha = \frac{1}{2}$. In Section 6, we extend the local solution to be a global one, and present the continuity of solution with respect to t.

2 Notations and Preliminaries

We now recall the notations used throughout the paper. The Fourier transform $\widehat{f}$ of $f(x)$ on Ω is defined as

$$\widehat{f}(k) = \frac{1}{(2\pi)^n}\int_\Omega f(x)\ \mathrm{e}^{-\mathrm{i}k\cdot x}\mathrm{d}x.$$

We denote the square root of the Laplacian $(-\Delta)^{\frac{1}{2}}$ by Λ, and obviously,

$$\widehat{\Lambda f}(k) = |k|\ \widehat{f}(k).$$

More generally, $\Lambda^\beta f$ for $\beta \in \mathbb{R}$ can be identified with the Fourier series

$$\sum_{k\in\mathbb{Z}^n} |k|^\beta\ \widehat{f}(k)\ \mathrm{e}^{-\mathrm{i}k\cdot x}.$$

$L^p(\Omega)$ denotes the space of the pth-power integrable functions named by

$$\|f\|_{L^p} = \Big(\int_\Omega |f(x)|^p\mathrm{d}x\Big)^{\frac{1}{p}}, \quad \|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

For any f on Ω and $s \in \mathbb{R}$, we define

$$\|f\|_{H^s} = \|\Lambda^s f\|_{L^2} = \Big(\sum_{k\in\mathbb{Z}^n} |k|^{2s}\ |\widehat{f}(k)|^2\Big)^{\frac{1}{2}},$$

and H^s denotes the Sobolev space of all f for which $\|f\|_{H^s}$ is finite. For $1 \le p \le \infty$ and $s \in \mathbb{R}$, the space $H^{s,p}(\Omega)$ is a subspace of $L^p(\Omega)$, consisting of all f which can be written in the form $f = \Lambda^{-s}g,\ g \in L^p(\Omega)$, and the $H^{s,p}$ norm of f is defined to be the L^p norm of g, i.e.,

$$\|f\|_{H^{s,p}} = \|\Lambda^s f\|_{L^p} = \Big(\sum_{k\in\mathbb{Z}^n} (|k|^s|\widehat{f}(k)|)^p\Big)^{\frac{1}{p}}.$$

3 Local Existence

In this section, we construct a convergent sequence to get the local solution.

Construct a sequence $\{u^m(x,t)\}$, where $u^m(x,t)$ is the solution to the following linear problem:

$$\begin{cases} u_t^m + (-\Delta)^\alpha u^m = \mathrm{div} f(u^{m-1}), \\ u^m(0,x) = u_0(x) \end{cases} \tag{3.1}$$

for $m \geq 1$ and $u^0(x,t) = 0,\ u_0(x) \in H^s(\Omega)$.

We now try to prove that the sequence is convergent in a space and the limit then is the solution to the problem (1.2).

First we introduce a set of functions as follows. For a given integer $s > \frac{n}{2} + 1$ (here n is the spatial dimension)

$$X = \{u(x,t) \mid \|u\|_X \leq E\},$$

the norm $\|u\|_X$ is defined as

$$\|u\|_X = \sup_{0 \leq t \leq T_0} \|\Lambda^s u\|_{L^2(\Omega)} + \Big(\int_0^{T_0} \|\Lambda^{s+\alpha} u\|_{L^2}^2 \mathrm{d}t\Big)^{\frac{1}{2}},$$

where $E = C_0\ \|u_0\|_{H^s}$, C_0 is a constant larger than 2, and T_0 will be determined later.

The matric in X is induced by the norm $\|u\|_X$,

$$\rho(u,v) = \|u - v\|_X, \quad \forall u, v \in X.$$

It is obvious that X is a non-empty and complete metric space with respect to this metric.

Lemma 3.1 *There exists some constant T_0 sufficiently small, such that $\{u^m(x,t)\}$ belongs to X.*

Proof We prove the lemma by the induction method.

For $m = 1$, we have

$$u_t^1 + (-\Delta)^\alpha u^1 = 0. \tag{3.2}$$

Multiplying (3.2) with $\Lambda^{2s} u^1$ and taking the inner product in L^2, one can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u^1\|_{L^2}^2 + \|\Lambda^{s+\alpha} u^1\|_{L^2}^2 = 0.$$

So

$$\|u^1\|_{H^s}^2 + 2\int_0^t \|u^1\|_{H^{s+\alpha}}^2 \mathrm{d}s = \|u_0\|_{H^s}^2.$$

For any $T_0 > 0$, we have $u^1 \in X$.

We assume that there exists a T_0 sufficiently small, such that when $j \leq m,\ u^j(x,t) \in X$. Now we consider $u^{m+1}(x,t)$. From (3.1), we get

$$u_t^{m+1} + (-\Delta)^\alpha u^{m+1} = \mathrm{div} f(u^m). \tag{3.3}$$

Multiplying (3.3) with $\Lambda^{2s}u^{m+1}$ and taking the inner product in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u^{m+1}\|_{L^2}^2+\|\Lambda^{s+\alpha}u^{m+1}\|_{L^2}^2=(\Lambda^{2s}u^{m+1},\mathrm{div}f(u^m)). \tag{3.4}$$

For the term on the right-hand side, we have

$$|(\Lambda^{2s}u^{m+1},\mathrm{div}f(u^m))|\le\|\Lambda^{s+\alpha}u^{m+1}\|_{L^2}\|\Lambda^{s+1-\alpha}f(u^m)\|_{L^2}. \tag{3.5}$$

To proceed with the proof, we need the calculus inequality

$$\|\Lambda^\gamma(FG)\|_l\le C(\|\Lambda^\gamma F\|_p\|G\|_q+\|F\|_q\|\Lambda^\gamma G\|_p), \tag{3.6}$$

where $\gamma>0$, $1<l\le p\le\infty$ and $\frac{1}{l}=\frac{1}{p}+\frac{1}{q}$.

Estimates (3.5)–(3.6) immediately yield

$$|(\Lambda^{2s}u^{m+1},\mathrm{div}f(u^m))|\le C\|\Lambda^{s+\alpha}u^{m+1}\|_{L^2}\|u^m\|_{L^\infty}\|\Lambda^{s+1-\alpha}u^m\|_{L^2}. \tag{3.7}$$

By Gagliardo-Nirenberg's inequality, when r_1,r_2 satisfy

$$\begin{cases}\dfrac{1}{r_1}+\dfrac{1}{r_2}=1,\\ \dfrac{s}{r_1}+\dfrac{s+\alpha}{r_2}=s+1-\alpha,\end{cases}$$

we have (obviously, $\frac{1}{2}<\alpha<1$ is a sufficient condition, such that $1<r_1,r_2<\infty$)

$$\|u^m\|_{H^{s+1-\alpha}}\le\|u^m\|_{H^s}^{\frac{1}{r_1}}\|u^m\|_{H^{s+\alpha}}^{\frac{1}{r_2}}. \tag{3.8}$$

By (3.5), (3.7)–(3.8), and via Young's inequality, we get

$$\begin{aligned}|(\Lambda^{2s}u^{m+1},\mathrm{div}f(u^m))|&\le C\|\Lambda^{s+\alpha}u^{m+1}\|_{L^2}\|u^m\|_{L^\infty}\|\Lambda^s u^m\|_{L^2}^{\frac{1}{r_1}}\|\Lambda^{s+\alpha}u^m\|_{L^2}^{\frac{1}{r_2}}\\&\le\frac{1}{2}\|\Lambda^{s+\alpha}u^{m+1}\|_{L^2}^2+C\|u^m\|_{L^\infty}^2\|\Lambda^s u^m\|_{L^2}^{\frac{2}{r_1}}\|\Lambda^{s+\alpha}u^m\|_{L^2}^{\frac{2}{r_2}}.\end{aligned}$$

From (3.4) and the above inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u^{m+1}\|_{L^2}^2+\|\Lambda^{s+\alpha}u^{m+1}\|_{L^2}^2\le C\|u^m\|_{L^\infty}^2\|\Lambda^s u^m\|_{L^2}^{\frac{2}{r_1}}\|\Lambda^{s+\alpha}u^m\|_{L^2}^{\frac{2}{r_2}}.$$

Thus,

$$\|u^{m+1}\|_{H^s}^2+\int_0^t\|\Lambda^{s+\alpha}u^{m+1}\|_{L^2}^2\mathrm{d}s\le C\int_0^t\|u^m\|_{L^\infty}^2\|\Lambda^s u^m\|_{L^2}^{\frac{2}{r_1}}\|\Lambda^{s+\alpha}u^m\|_{L^2}^{\frac{2}{r_2}}\,\mathrm{d}s.$$

Since $s>\frac{n}{2}+1$, by Sobolev inequality, we have $\|u^m\|_{L^\infty}\le C\|u^m\|_{H^s}$. Noticing also that $u^m\in X$, i.e., $\|u^m\|_X\le E$, and via Hölder inequality, we get

$$\begin{aligned}&\sup_{0\le t\le T_0}\|u^{m+1}\|_{H^s}^2+\int_0^{T_0}\|u^{m+1}\|_{H^{s+\alpha}}^2\mathrm{d}s\\&\le C\int_0^{T_0}\|u^m\|_{L^\infty}^2\|\Lambda^s u^m\|_{L^2}^{\frac{2}{r_1}}\|\Lambda^{s+\alpha}u^m\|_{L^2}^{\frac{2}{r_2}}\mathrm{d}s\\&\le CE^2\int_0^{T_0}\|\Lambda^s u^m\|_{L^2}^{\frac{2}{r_1}}\|\Lambda^{s+\alpha}u^m\|_{L^2}^{\frac{2}{r_2}}\mathrm{d}s\\&\le CE^2\Big(\int_0^{T_0}\|\Lambda^s u^m\|_{L^2}^{\frac{2}{r_1}\cdot r_1}\mathrm{d}s\Big)^{\frac{1}{r_1}}\Big(\int_0^{T_0}\|\Lambda^{s+\alpha}u^m\|_{L^2}^{\frac{2}{r_2}\cdot r_2}\mathrm{d}s\Big)^{\frac{1}{r_2}}\\&\le CE^2\ E^{\frac{2}{r_1}}\ T_0^{\frac{1}{r_1}}\ E^{\frac{2}{r_2}}=CE^4T_0^{\frac{1}{r_1}}.\end{aligned}$$

Then, by the above inequality, we have

$$\|u^{m+1}\|_X = \sup_{0\le t\le T_0} \|u^{m+1}\|_{H^s} + \Big(\int_0^{T_0} \|u^{m+1}\|^2_{H^{s+\alpha}} \mathrm{d}s\Big)^{\frac{1}{2}} \le CE^2\ T_0^{\frac{1}{2r_1}}. \tag{3.9}$$

We choose T_0 sufficiently small, such that $CE^2 T_0^{\frac{1}{2r_1}} \le E$. Then $u^{m+1}(x,t)$ belongs to X.

From the above results, we have $\{u^m(x,t)\} \in X$.

Lemma 3.2 *For T_0 mentioned in Lemma* 3.1, *$\{u^m(x,t)\}$ constructed by* (3.1) *is a Cauchy sequence in X.*

Proof We only need to prove $\|u^{m+1}-u^m\|_X \le \lambda\ \|u^m - u^{m-1}\|_X$, where $0<\lambda<1$.

Consider $u^{m+1}-u^m$. From (3.1), it satisfies

$$u_t^m + (-\Delta)^\alpha u^m = \mathrm{div} f(u^{m-1}),$$
$$u_t^{m+1} + (-\Delta)^\alpha u^{m+1} = \mathrm{div} f(u^m).$$

Thus

$$(u^{m+1}-u^m)_t + (-\Delta)^\alpha (u^{m+1}-u^m) = \mathrm{div}(f(u^m)-f(u^{m-1})). \tag{3.10}$$

Multiplying (3.10) with $\Lambda^{2s}(u^{m+1}-u^m)$ and taking the inner product in L^2, we have

$$\begin{aligned}&((u^{m+1}-u^m)_t,\ \Lambda^{2s}(u^{m+1}-u^m)) + (\Lambda^{2\alpha}(u^{m+1}-u^m),\ \Lambda^{2s}(u^{m+1}-u^m))\\ =\ &(\Lambda^{2s}(u^{m+1}-u^m), \mathrm{div}(f(u^m)-f(u^{m-1}))).\end{aligned}$$

For the term on the right-hand side, we have

$$\begin{aligned}&|(\Lambda^{2s}(u^{m+1}-u^m),\ \mathrm{div}(f(u^m)-f(u^{m-1})))|\\ \le\ &2\Big|\int_\Omega \Lambda^{2s}(u^{m+1}-u^m)\ u^m(u^m_{x_1}+\cdots+u^m_{x_n}-u^{m-1}_{x_1}-\cdots-u^{m-1}_{x_n})\mathrm{d}x\Big|\\ &+2\Big|\int_\Omega \Lambda^{2s}(u^{m+1}-u^m)(u^m-u^{m-1})(u^{m-1}_{x_1}+\cdots+u^{m-1}_{x_n})\mathrm{d}x\Big|\\ \le\ &2\|\Lambda^{s+\alpha}(u^{m+1}-u^m)\|_{L^2}\|\Lambda^{s-\alpha}(u^m(u^m_{x_1}+\cdots+u^m_{x_n}-u^{m-1}_{x_1}-\cdots-u^{m-1}_{x_n}))\|_{L^2}\\ &+2\|\Lambda^{s+\alpha}(u^{m+1}-u^m)\|_{L^2}\|\Lambda^{s-\alpha}((u^m-u^{m-1})(u^{m-1}_{x_1}+\cdots+u^{m-1}_{x_n}))\|_{L^2}.\end{aligned}$$

By (3.6) and Gagliardo-Nirenberg's inequality, when r_1, r_2 satisfy

$$\begin{cases}\dfrac{1}{r_1}+\dfrac{1}{r_2}=1,\\ \dfrac{s}{r_1}+\dfrac{s+\alpha}{r_2}=s+1-\alpha,\end{cases}$$

we have

$$\begin{aligned}&|(\Lambda^{2s}(u^{m+1}-u^m),\ \mathrm{div}(f(u^m)-f(u^{m-1})))|\\ \le\ &\frac{1}{2}\|\Lambda^{s+\alpha}(u^{m+1}-u^m)\|^2_{L^2} + C\|\Lambda^{s-\alpha}u^m\|^2_{L^2}\|\nabla(u^m-u^{m-1})\|^2_{L^\infty}\\ &+C\|u^m\|^2_{L^\infty}\|\Lambda^s(u^m-u^{m-1})\|^{\frac{2}{r_1}}_{L^2}\|\Lambda^{s+\alpha}(u^m-u^{m-1})\|^{\frac{2}{r_2}}_{L^2}\\ &+C\|\Lambda^{s-\alpha}(u^m-u^{m-1})\|^2_{L^2}\|\nabla u^{m-1}\|^2_{L^\infty} + C\|u^m-u^{m-1}\|^2_{L^\infty}\|\Lambda^s u^{m-1}\|^{\frac{2}{r_1}}_{L^2}\|\Lambda^{s+\alpha}u^{m-1}\|^{\frac{2}{r_2}}_{L^2}.\end{aligned}$$

Thus

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s(u^{m+1}-u^m)\|_{L^2}^2+\|\Lambda^{s+\alpha}(u^{m+1}-u^m)\|_{L^2}^2\\
\le{}& C\|\Lambda^{s-\alpha}u^m\|_{L^2}^2\|\nabla(u^m-u^{m-1})\|_{L^\infty}^2\\
&+C\|u^m\|_{L^\infty}^2\|\Lambda^s(u^m-u^{m-1})\|_{L^2}^{\frac{2}{r_1}}\|\Lambda^{s+\alpha}(u^m-u^{m-1})\|_{L^2}^{\frac{2}{r_2}}\\
&+C\|\Lambda^{s-\alpha}(u^m-u^{m-1})\|_{L^2}^2\|\nabla u^{m-1}\|_{L^\infty}^2+C\|u^m-u^{m-1}\|_{L^\infty}^2\|\Lambda^s u^{m-1}\|_{L^2}^{\frac{2}{r_1}}\|\Lambda^{s+\alpha}u^{m-1}\|_{L^2}^{\frac{2}{r_2}}.
\end{aligned}
$$

Since $\{u^m\}\in X$, via Hölder inequality, in a similar way to the previous method, we get

$$\|u^{m+1}-u^m\|_X^2\le C(E^2T_0+E^2T_0^{\frac{1}{r_1}})\|u^m-u^{m-1}\|_X^2.$$

Choose T_0 sufficiently small, such that $C(E^2T_0+E^2T_0^{\frac{1}{r_1}})<\frac{1}{4}$. Therefore

$$\|u^{m+1}-u^m\|_X\le\frac{1}{2}\ \|u^m-u^{m-1}\|_X.$$

So $\{u^m\}$ is a Cauchy sequence in X.

Since X is a complete metric space, from Lemmas 3.1–3.2, there exists a $u(x,t)\in X$ which satisfies equation (1.2). Thus, the local existence is proved.

So we have the following theorem.

Theorem 3.1 *Assume that* $\frac{1}{2}<\alpha<1$, *and the initial data* u_0 *belongs to* $H^s(\Omega)$, $s>\frac{n}{2}+1$. *Then when* T_0 *is small enough, there exists a local solution* $u(x,t)$ *to* (1.2), *such that*

$$u(x,t)\in L^\infty([0,T_0),H^s(\Omega))\cap L^2([0,T_0),H^{s+\alpha}(\Omega)).$$

4 Decay Estimates for the Sub-critical Case

4.1 Maximum principle

In this section, we present a useful maximum principle, which plays an important role in the proof of the decay estimates for $u(x,t)$. The key to the maximum principle is the positivity lemma, the original version of which was firstly presented by Resnick [15] as follows. See also Cordoba et al. [12] for a proof of the lemma.

Lemma 4.1 (Positivity Lemma) *Suppose* $\alpha\in[0,2]$, $u,\Lambda^\alpha u\in L^p$ *with* $p\in(1,+\infty)$. *Then*

$$\int_\Omega |u|^{p-2}u\Lambda^\alpha u\mathrm{d}x\ge 0.$$

The immediate consequence of Lemma 4.1 is the following maximum principle.

Lemma 4.2 (Maximum Principle) *Let* u *be a smooth function on* Ω *satisfying* $u_t+\Lambda^\alpha u=\mathrm{div}f(u)$ *with* $0\le\alpha\le 2$. *Then for* $1\le p\le\infty$, *we have*

$$\|u\|_{L^p}\le\|u_0\|_{L^p}. \tag{4.1}$$

This lemma can be proved in the same way as a corresponding result for the quasi-geostrophic equations (see [12, 15] for more details). Here, we only give a rough proof of the lemma, and the details are omitted for brevity.

Proof of Lemma 4.2

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^p\mathrm{d}x = p\int_\Omega |u|^{p-2}u[-\Lambda^\alpha u+\mathrm{div}f(u)]\mathrm{d}x \le -p\int_\Omega |u|^{p-2}u\Lambda^\alpha u\mathrm{d}x \le 0,$$

where we have used the fact $f_i(u)=u^2$ $(i=1,\cdots,n)$ and the positivity lemma.

4.2 Decay estimates for the sub-critical case $\frac{1}{2}<\alpha<1$

The goal of this section is to get the decay estimate of $u(x,t)$ in the space of $H^s(\Omega)$ when $\frac{1}{2}<\alpha<1$.

Firstly, we are going to give the decay estimate of $\|u\|_{L^2}$. Consider

$$u_t+(-\Delta)^\alpha u=\mathrm{div}f(u). \tag{4.2}$$

Multiplying (4.2) with u and taking the inner product in L^2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2+\|\Lambda^\alpha u\|_{L^2}^2=0.$$

Denote by λ_1 the first eigenvalue of Λ. Since u is mean zero, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2+2\lambda_1^{2\alpha}\|u\|_{L^2}^2\le 0.$$

By integration, we have

$$\|u\|_{L^2}^2\le \mathrm{e}^{-2\lambda_1^{2\alpha}t}\|u_0\|_{L^2}^2. \tag{4.3}$$

Now, we come to obtain the decay estimate of $u(x,t)$ in the $H^s(\Omega)$ space. Take the $L^2(\Omega)$ inner product of equation (4.2) with $\Lambda^{2s}u$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2+\|u\|_{H^{s+\alpha}}^2=(\Lambda^{2s}u,\mathrm{div}f(u)). \tag{4.4}$$

For the term on the right-hand side, we have

$$|(\Lambda^{2s}u,\mathrm{div}f(u))|\le\|\Lambda^{s+\alpha}u\|_{L^2}\|\Lambda^{s-\alpha+1}f(u)\|_{L^2}. \tag{4.5}$$

By (3.6) and (4.5), we have

$$\begin{aligned}|(\Lambda^{2s}u,\mathrm{div}f(u))| &\le \|u\|_{H^{s+\alpha}}\|f(u)\|_{H^{s-\alpha+1}}\\ &\le C\|u\|_{H^{s+\alpha}}\|u\|_{L^\infty}\|u\|_{H^{s-\alpha+1}}\\ &\le C\|u\|_{H^{s+\alpha}}\|u\|_{H^{s-\alpha+1}}.\end{aligned} \tag{4.6}$$

Here, we replace $\|u\|_{L^\infty}$ by a constant, which is related only to u_0 according to Lemma 4.2.

To deal with the term $\|u\|_{H^{s-\alpha+1}}$, we use the following Gagliardo-Nirenberg's inequality:

$$\|u\|_{H^{s-\alpha+1}}\le C\|u\|_{H^0}^{\frac{1}{r_1}}\|u\|_{H^{s+\alpha}}^{\frac{1}{r_2}}=C\|u\|_{L^2}^{\frac{1}{r_1}}\|u\|_{H^{s+\alpha}}^{\frac{1}{r_2}}, \tag{4.7}$$

where r_1, r_2 satisfy

$$\begin{cases} \dfrac{1}{r_1} + \dfrac{1}{r_2} = 1, \\ \dfrac{0}{r_1} + \dfrac{s+\alpha}{r_2} = s - \alpha + 1, \end{cases}$$

i.e.,

$$\begin{cases} \dfrac{1}{r_1} = \dfrac{2\alpha - 1}{s+\alpha}, \\ \dfrac{1}{r_2} = \dfrac{s-\alpha+1}{s+\alpha}. \end{cases}$$

Obviously, $\frac{1}{2} < \alpha < 1$ is a sufficient condition, such that $1 < r_1, r_2 < \infty$.

(4.6)–(4.7) imply

$$|(\Lambda^{2s}u, \mathrm{div} f(u))| \le C\|u\|_{H^{s+\alpha}}^{1+\frac{1}{r_2}} \|u\|_{L^2}^{\frac{1}{r_1}}.$$

By Young's inequality, the above inequality shows that

$$|(\Lambda^{2s}u, \mathrm{div} f(u))| \le \frac{1}{2}\|u\|_{H^{s+\alpha}}^2 + C\|u\|_{L^2}^2. \tag{4.8}$$

Estimates (4.4) and (4.8) immediately yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 + \|u\|_{H^{s+\alpha}}^2 \le C\|u\|_{L^2}^2.$$

Since $\lambda_1^\alpha \|u\|_{H^s} \le \|u\|_{H^{s+\alpha}}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 + \lambda_1^{2\alpha}\|u\|_{H^s}^2 \le C\|u\|_{L^2}^2.$$

Then, by integration, we have

$$\mathrm{e}^{\lambda_1^{2\alpha}t}\|u\|_{H^s}^2 - \|u_0\|_{H^s}^2 \le C\int_0^t \mathrm{e}^{\lambda_1^{2\alpha}\tau}\|u\|_{L^2}^2(\tau)\mathrm{d}\tau. \tag{4.9}$$

By (4.3) and (4.9), we have

$$\mathrm{e}^{\lambda_1^{2\alpha}t}\|u\|_{H^s}^2 - \|u_0\|_{H^s}^2 \le C\int_0^t \mathrm{e}^{\lambda_1^{2\alpha}\tau} \cdot \mathrm{e}^{-2\lambda_1^{2\alpha}\tau}\|u_0\|_{L^2}^2\mathrm{d}\tau = C\|u_0\|_{L^2}^2 \int_0^t \mathrm{e}^{-\lambda_1^{2\alpha}\tau}\mathrm{d}\tau. \tag{4.10}$$

The inequality (4.10) immediately yields

$$\mathrm{e}^{\lambda_1^{2\alpha}t}\|u\|_{H^s}^2 - \|u_0\|_{H^s}^2 \le C\|u_0\|_{L^2}^2 \frac{1 - \mathrm{e}^{-\lambda_1^{2\alpha}t}}{\lambda_1^{2\alpha}}.$$

Thus,

$$\|u\|_{H^s}^2 \le \mathrm{e}^{-\lambda_1^{2\alpha}t}\|u_0\|_{H^s}^2 + \frac{C\|u_0\|_{L^2}^2}{\lambda_1^{2\alpha}}(\mathrm{e}^{-\lambda_1^{2\alpha}t} - \mathrm{e}^{-2\lambda_1^{2\alpha}t}) \le C\|u_0\|_{H^s}^2\ \mathrm{e}^{-\lambda_1^{2\alpha}t}.$$

Now, we get the exponential decay of the solution

$$\|u\|_{H^s}^2 \le C\|u_0\|_{H^s}^2\ \mathrm{e}^{-\lambda_1^{2\alpha}t},$$

where λ_1 is the first eigenvalue of Λ.

5 Decay Estimates for the Critical Case $\alpha = \frac{1}{2}$

In this section, we obtain the exponential decay of the solution to the critical case $\alpha = \frac{1}{2}$. The initial data $u_0(x) \in H^s$ and $s > \frac{n}{2} + 1$. We firstly recall the following useful lemma for the critical case, which was obtained in [4, Corollary 3.5].

Lemma 5.1 *Assume that the initial data $u_0(x)$ satisfy $\|\nabla u_0\|_{L^\infty} < \infty$. Then for every time t, the solution $u(x,t)$ to the critical Burgers equation satisfies*

$$\|\nabla u\|_{L^\infty} \le \|\nabla u_0\|_{L^\infty} \exp(C\|u_0\|_{L^\infty}). \tag{5.1}$$

Obviously, $s > \frac{n}{2} + 1$ and $u_0 \in H^s$ are sufficient conditions, such that $\|\nabla u_0\|_{L^\infty} < \infty$. Now we are going to prove the decay estimates of $u(x,t)$. Consider

$$u_t + (-\Delta)^{\frac{1}{2}} u = \mathrm{div} f(u). \tag{5.2}$$

Take the $L^2(\Omega)$ inner product of equation (5.2) with u to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} u\|_{L^2}^2 = 0.$$

Denote by λ_1 the first eigenvalue of Λ. Since u is mean zero, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + 2\lambda_1\|u\|_{L^2}^2 \le 0.$$

By integration, we get

$$\|u\|_{L^2}^2 \le \mathrm{e}^{-2\lambda_1 t}\|u_0\|_{L^2}^2. \tag{5.3}$$

To complete this section, we shall state the commutator estimates needed in the coming proof.

Lemma 5.2 *Suppose that $s > 0$ and $p \in (1, +\infty)$. If $f, g \in \mathcal{S}$, the Schwarz class, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|g\|_{H^{s-1,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{L^{p_4}}) \tag{5.4}$$

with $p_2, p_3 \in (1, +\infty)$, such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Remark 5.1 It is clear that the lemma is true whenever the corresponding right-hand side is finite. The above lemma was proved by Kenig et al. [16] with the homogeneous $H^{s,p}$ spaces being replaced by non-homogeneous ones and Λ being replaced by $(1-\Delta)^{\frac{1}{2}}$. This lemma can be proved still valid for Λ by making use of a dilation argument of Kato [17].

Now, we are ready to obtain the decay estimates of $\|u\|_{H^s}$. Take the $L^2(\Omega)$ inner product of equation (5.2) with $\Lambda^{2s}u$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^{s+\frac{1}{2}} u\|_{L^2}^2 = (\Lambda^{2s}u, \mathrm{div} f(u)). \tag{5.5}$$

For the term on the right-hand side, we have

$$\begin{aligned} |(\Lambda^{2s}u, \mathrm{div} f(u))| &= 2|(\Lambda^s u, \Lambda^s(u\ (u_{x_1} + \cdots + u_{x_n})))| \\ &\le 2|(\Lambda^s u, \Lambda^s(u\ (u_{x_1} + \cdots + u_{x_n})) - u\ \Lambda^s(u_{x_1} + \cdots + u_{x_n}))| \\ &\quad + 2|(\Lambda^s u, u\ \Lambda^s(u_{x_1} + \cdots + u_{x_n}))| \equiv \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{5.6}$$

Now we treat two terms on the right-hand side of (5.6) separately as follows. We first estimate I_2. Notice that

$$\Big|\int_\Omega \Lambda^s u\ u\ \Lambda^s u_{x_j} \mathrm{d}x\Big| = \Big|\int_\Omega \Lambda^s u\ \Lambda^s u_{x_j}\ u\ \mathrm{d}x\Big| = \frac{1}{2}\Big|\int_\Omega \frac{\partial}{\partial x_j}(\Lambda^s u)^2\ u\ \mathrm{d}x\Big| = \frac{1}{2}\Big|\int_\Omega (\Lambda^s u)^2 u_{x_j} \mathrm{d}x\Big|.$$

So

$$\begin{aligned}\mathrm{I}_2 &= 2\Big|\int_\Omega \Lambda^s u\ u\ \Lambda^s(u_{x_1}+\cdots+u_{x_n})\mathrm{d}x\Big| \\ &= \Big|\int_\Omega (\Lambda^s u)^2(u_{x_1}+\cdots+u_{x_n})\mathrm{d}x\Big| \\ &\le \|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2. \end{aligned} \tag{5.7}$$

Now, we use Lemma 5.2 to deal with I_1.

$$\begin{aligned}\mathrm{I}_1 &= 2|(\Lambda^s u, \Lambda^s(u\ (u_{x_1}+\cdots+u_{x_n})) - u\ \Lambda^s(u_{x_1}+\cdots+u_{x_n}))| \\ &\le 2\|\Lambda^s u\|_{L^2}\|\Lambda^s(u(u_{x_1}+\cdots+u_{x_n}) - u\ \Lambda^s(u_{x_1}+\cdots+u_{x_n}))\|_{L^2} \\ &\le C\|\Lambda^s u\|_{L^2}(\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2} + \|\Lambda^s u\|_{L^2}\|\nabla u\|_{L^\infty}) \\ &\le C\|\Lambda^s u\|_{L^2}^2\|\nabla u\|_{L^\infty}. \end{aligned} \tag{5.8}$$

By (5.7)–(5.8), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^{s+\frac{1}{2}}u\|_{L^2}^2 \le C\|\Lambda^s u\|_{L^2}^2\|\nabla u\|_{L^\infty} \le C\|\Lambda^s u\|_{L^2}^2.$$

Here, we replace $\|\nabla u\|_{L^\infty}$ by a constant according to Lemma 5.1.

We have the following Gagliardo-Nirenberg's inequality:

$$\|u\|_{H^s} \le \|u\|_{H^0}^{\frac{1}{r_1}}\ \|u\|_{H^{s+\frac{1}{2}}}^{\frac{1}{r_2}}, \tag{5.9}$$

where r_1, r_2 satisfy

$$\begin{cases}\dfrac{1}{r_1} + \dfrac{1}{r_2} = 1, \\ \dfrac{0}{r_1} + \dfrac{s+\frac{1}{2}}{r_2} = s,\end{cases}$$

i.e.,

$$\begin{cases}\dfrac{1}{r_1} = \dfrac{\frac{1}{2}}{s+\frac{1}{2}}, \\ \dfrac{1}{r_2} = \dfrac{s}{s+\frac{1}{2}}.\end{cases}$$

Then, by (5.9), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^{s+\frac{1}{2}}u\|_{L^2}^2 \le C\|u\|_{L^2}^{\frac{1}{s+\frac{1}{2}}}\ \|u\|_{H^{s+\frac{1}{2}}}^{\frac{2s}{s+\frac{1}{2}}}. \tag{5.10}$$

By Young's inequality, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 + \|u\|_{H^{s+\frac{1}{2}}}^2 \le \frac{1}{2}\|u\|_{H^{s+\frac{1}{2}}}^2 + C\|u\|_{L^2}^2.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2+\|u\|_{H^{s+\frac{1}{2}}}^2\le C\|u\|_{L^2}^2.$$

Since $\lambda_1^{\frac{1}{2}}\|u\|_{H^s}\le\|u\|_{H^{s+\frac{1}{2}}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2+\lambda_1\|u\|_{H^s}^2\le C\|u\|_{L^2}^2.$$

Noticing also (5.3), we have

$$\mathrm{e}^{\lambda_1 t}\|u\|_{H^s}^2-\|u_0\|_{H^s}^2\le C\int_0^t\mathrm{e}^{\lambda_1\tau}\mathrm{e}^{-2\lambda_1\tau}\|u_0\|_{L^2}^2\mathrm{d}\tau\le C\|u_0\|_{L^2}^2\frac{1-\mathrm{e}^{-\lambda_1 t}}{\lambda_1}.$$

So

$$\begin{aligned}\|u\|_{H^s}^2&\le\mathrm{e}^{-\lambda_1 t}\|u_0\|_{H^s}^2+\frac{C\|u_0\|_{L^2}^2}{\lambda_1}(\mathrm{e}^{-\lambda_1 t}-\mathrm{e}^{-2\lambda_1 t})\\&\le\mathrm{e}^{-\lambda_1 t}\|u_0\|_{H^s}^2+\frac{C\|u_0\|_{L^2}^2}{\lambda_1}\mathrm{e}^{-\lambda_1 t}\\&\le C\|u_0\|_{H^s}^2\ \mathrm{e}^{-\lambda_1 t}.\end{aligned}$$

The exponential decay estimates of solutions to (1.2) are proved.

Observe that as a byproduct, we have the following corollary.

Corollary 5.1 *Assume that $\alpha=\frac{1}{2}$, and the initial data u_0 belongs to $H^s(\Omega)$, $s>\frac{n}{2}+1$. Then the solution $u(x,t)$ to the critical Burgers equation satisfies*

$$\|\nabla u\|_{L^\infty}\le C\|u_0\|_{H^s(\Omega)}\ \mathrm{e}^{-\lambda_1 t},$$

where λ_1 is the first eigenvalue of Λ.

6 Global Existence and Continuity with Respect to t

By the decay estimates of solution to (1.2), we can extend the local solution to be a global one. So the global solution $u(x,t)$ to the problem (1.2) is obtained, satisfying

$$u\in L^\infty(0,\infty;H^s)\cap L^2(0,\infty;H^{s+\alpha}).$$

Now we prove the continuity with respect to t.

For this purpose, we first recall the following useful lemma which is a particular case of a general interpolation theorem of Lions and Magenes [14]. A beautiful elementary proof of this lemma was given by Teman [18].

Lemma 6.1 *Let V, H, V' be three Hilbert spaces, such that*

$$V\subset H=H'\subset V',$$

where H' is the dual space of H, and V' is the dual space of V. If a function u belongs to $L^2(0,T;V)$ and its derivative u_t' belongs to $L^2(0,T;V')$, then u is almost everywhere equal to a function continuous from $[0,T]$ to H.

We have already known that $u \in L^2(0,T;H^{s+\alpha})$, i.e.,

$$\Lambda^s u \in L^2(0,T;H^{\alpha}).$$

We want to show that $u \in C(0,T;H^s)$, i.e., $\Lambda^s u \in C(0,T;L^2)$. According to Lemma 6.1, we just need to show that

$$\Lambda^s u_t \in L^2(0,T;H^{-\alpha}).$$

For any $\varphi \in H^{\alpha}$,

$$\begin{aligned}(\Lambda^s u_t, \varphi) &= (-\Lambda^s(-\Delta)^{\alpha} u, \varphi) + (\Lambda^s \mathrm{div} f(u), \varphi)\\ &= -(\Lambda^{s+2\alpha} u, \varphi) + (\Lambda^s \mathrm{div} f(u), \varphi).\end{aligned}$$

Therefore,

$$\begin{aligned}|(\Lambda^s u_t,\ \varphi)| &\le \|\Lambda^{s+\alpha} u\|_{L^2} \|\Lambda^{\alpha}\varphi\|_{L^2} + \|\Lambda^{s-\alpha} \mathrm{div} f(u)\|_{L^2} \|\Lambda^{\alpha}\varphi\|_{L^2}\\ &= (\|\Lambda^{s+\alpha} u\|_{L^2} + \|\Lambda^{s-\alpha} \mathrm{div} f(u)\|_{L^2}) \|\Lambda^{\alpha}\varphi\|_{L^2},\end{aligned}$$

that is,

$$\begin{aligned}\|\Lambda^s u_t\|_{H^{-\alpha}} &\le \|\Lambda^{s+\alpha} u\|_{L^2} + \|\Lambda^{s-\alpha} \mathrm{div} f(u)\|_{L^2}\\ &\le \|\Lambda^{s+\alpha} u\|_{L^2} + \|\Lambda^{s+1-\alpha} f(u)\|_{L^2}.\end{aligned} \tag{6.1}$$

Using (3.6), we have

$$\|\Lambda^{1+s-\alpha} f(u)\|_{L^2} \le C\|u\|_{L^\infty} \|\Lambda^{1+s-\alpha} u\|_{L^2}. \tag{6.2}$$

Since $\frac{1}{2} \le \alpha < 1$, we have $1+s-\alpha \le s+\alpha$. Then

$$\|\Lambda^{1+s-\alpha} u\|_{L^2} = \|u\|_{H^{1+s-\alpha}} \le C\|u\|_{H^{s+\alpha}} = C\|\Lambda^{s+\alpha} u\|_{L^2}. \tag{6.3}$$

By (6.1)–(6.3), we have

$$\|\Lambda^s u_t\|_{H^{-\alpha}} \le \|\Lambda^{s+\alpha} u\|_{L^2} + C\|u\|_{L^\infty} \|\Lambda^{s+\alpha} u\|_{L^2} = (1 + C\|u\|_{L^\infty}) \|\Lambda^{s+\alpha} u\|_{L^2}. \tag{6.4}$$

Notice also (4.1) and (6.4). Therefore,

$$\int_0^T \|\Lambda^s u_t\|_{H^{-\alpha}}^2 < \infty.$$

This completes the proof of continuity with respect to t. Therefore, Theorem 1.1 is proved.

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