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A Characterization of Topologically Transitive Attributes for a Class of Dynamical Systems*

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Abstract In this work, by virtue of the properties of weakly almost periodic points of a dynamical system (X, T) with at least two points, the authors prove that, if the measure center $M(T)$ of T is the whole space, that is, $M(T) = X$, then the following statements are equivalent:

(1) (X, T) is ergodic mixing; (2) (X, T) is topologically double ergodic;
(3) (X, T) is weak mixing; (4) (X, T) is extremely scattering;
(5) (X, T) is strong scattering; (6) $(X \times X, T \times T)$ is strong scattering;
(7) $(X \times X, T \times T)$ is extremely scattering;
(8) For any subset S of $\mathbb{N}$ with upper density 1, there is a c-dense F_σ-chaotic set with respect to S.

As an application, the authors show that, for the sub-shift σ_A of finite type determined by a $k \times k$-$(0, 1)$ matrix A, σ_A is strong mixing if and only if σ_A is totally transitive.

Keywords Weakly almost periodic point, Measure center, Topologically transitive attribute, Chaotic set
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1 Introduction

It is well-known that the central problem of a dynamical system is the asymptotic behavior or topological structure of the orbits of points. Is the orbit of every point of equal importance? From the viewpoint of pure topology, we know that only the orbits of the points possessing certain recurrence are important. Classically, the recurrence has four layers: periodic points, almost periodic points, recurrent points, and non-wandering points. The latter possesses the mildest recurrence and thus only the orbits generated by the non-wandering points are important and it suffices to study only the non-wandering set. In this sense, we may say that all important dynamical behaviors (including the structure of the orbits of points) of a dynamical system concentrate upon its non-wandering set. But the ergodic theory reveals that all important behaviors of a dynamical system take place on a full measure set. Of course, for such a set, the smaller one is better (in the sense of set inclusion). In order to look for the smallest full measure set, Zhou [1–2] introduced the notions of weakly almost periodic point and measure

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center, proved that the closure of all weakly almost periodic points is just the measure center, and pointed out that almost all the dynamical behaviors of a dynamical system are concentrated on its measure center in the framework of ergodic theory, which makes the intrinsic properties of a dynamical system much clearer not only in the sense of topological structure but also in the sense of ergodic theory.

On the other hand, we know that some topologically transitive attributes such as strong mixing, weak mixing and topologically transitive are usually used to describe different complexities of a dynamical system. It is clear that a strong mixing system is weak mixing and a weak mixing system is topologically transitive. And there are examples showing that a topologically transitive system may not be a weak mixing system, and a weak mixing system may not be a strong mixing system (see [3]). In order to describe the complexities of a dynamical system more accurately, Huang and Ye [3–6] (see also the references therein) introduced several new topologically transitive attributes between strong mixing and topologically transitive, such as extremely scattering, strong scattering, scattering and weak scattering, and gave some equivalent descriptions of them by means of weak disjointness.

Now, a natural problem arises: under what conditions, these topologically transitive attributes (or part of them) mentioned above are equivalent?

Let (X,T) denote a dynamical system with at least two points, namely, (X,d) is a compact metric space, and T is a continuous surjective map from X to itself. In the present paper, we prove, if $M(T)=X$, then the following statements are equivalent:

(1) (X,T) is ergodic mixing;

(2) (X,T) is topologically double ergodic;

(3) (X,T) is weak mixing;

(4) (X,T) is extremely scattering;

(5) (X,T) is strong scattering;

(6) $(X\times X, T\times T)$ is strong scattering;

(7) $(X\times X, T\times T)$ is extremely scattering;

(8) For any subset S of $\mathbb{N}$ (the set of positive integers) with upper density 1, there is a c-dense F_σ-chaotic set with respect to S.

As an application, we prove that, for the sub-shift σ_A of finite type determined by a $k\times k$-$(0,1)$ matrix A, σ_A is strong mixing if and only if σ_A is totally transitive.

This paper is organized as follows. In Section 2, some basic concepts and notations are given. In Section 3, we give some lemmas which play key roles in the proofs of our main results. The main results and their proofs are given in Section 4.

2 Basic Concepts and Notations

Let (X,T) denote a dynamical system. For a point x in X, let $\omega(x,T)$ denote the ω-limit set of x under T. We use $P(T)$, $A(T)$, $R(T)$ and $\Omega(T)$ to denote the sets of the periodic points, almost periodic points, recurrent points and non-wandering points, respectively. By the definitions, it is clear that

$$P(T) \subseteq A(T) \subseteq R(T) \subseteq \Omega(T). \tag{2.1}$$

For more details, refer to [7–9].

A point x in X is called a weakly almost periodic point, if for any $\varepsilon > 0$, there exists an $N > 0$, such that

$$\sharp(\{r \mid T^r(x) \in V(x, \varepsilon), 0 \leq r < nN\}) \geq n \tag{2.2}$$

for all $n \geq 0$, where $\sharp(\cdot)$ denotes the cardinality. Denote by $W(T)$ the sets of weakly almost periodic points. Then,

$$P(T) \subseteq A(T) \subseteq W(T) \subseteq R(T) \subseteq \Omega(T). \tag{2.3}$$

$x \in X$ is called a minimal point of T if for any open set U containing x,

$$N(x, U) = \{n \in \mathbb{N} \mid T^n(x) \in U\} \tag{2.4}$$

is syndetic, i.e., with bounded gaps. $x \in X$ is called a regular minimal point of T, if for any open set V containing x, there is a $k \in \mathbb{N}$, such that for any $j \in \mathbb{N}$, $T^{kj}(x) \in V$. Denote by $\mathrm{RG}(T)$ the set of regular minimal points of T. Clearly, a periodic point is a regular minimal point, and a regular minimal point is a minimal point.

A closed invariant set M of X is called the measure center of T if $\mu(M) = 1$ for any $\mu \in M(X)$, where $M(X)$ denotes the set of invariant measures of T, and no proper subset of M has these properties. We denote the measure center of T by $M(T)$. Then, the restriction of T to $M(T)$, $T|_{M(T)}$, is the smallest subsystem which maintains all the important dynamical properties of the system in the framework of ergodic theory.

Zhou [2] proved that

$$\overline{W(T)} = M(T), \tag{2.5}$$

which reveals the structure of the measure center and connects the recurrence of the orbits and the measure center. Thus, it makes the intrinsic properties of a dynamical system much clearer not only in the sense of topological structures of orbits but also in the sense of ergodic theory (see [1–2] for more details).

Let $\mathbb{Z}^+$ be the set of nonnegative integers, and $\mathcal{P}$ be the set consisting of all the subsets of $\mathbb{Z}^+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is called a family if $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_1 \in \mathcal{F}$ implies $\mathcal{F}_2 \in \mathcal{F}$. Assume that $\mathcal{F}$ is a family, and if $\mathcal{F} \neq \mathcal{P}$ and $\mathcal{F} \neq \emptyset$, we call $\mathcal{F}$ a proper family. The dual family of $\mathcal{F}$ is defined as

$$k\mathcal{F} = \{F \in \mathcal{F} : \forall F_1 \in \mathcal{F}, F \cap F_1 \neq \emptyset\}.$$

$\mathcal{F}$ is called a translative invariant family, if for any $i \in \mathbb{Z}^+$, $F \in \mathcal{F} \Leftrightarrow g^{-i}(F) \in \mathcal{F}$, where $g^{-i}(F) = \{j \in \mathbb{Z}^+ : i + j \in F\}$. Let

$$\tau\mathcal{F} = \{F \in \mathcal{F} : \text{for any } \ i_1, i_2, \cdots, i_k \in \mathbb{Z}^+, \ g^{-i_1}(F) \cap g^{-i_2}(F) \cap \cdots \cap g^{-i_k}(F) \in \mathcal{F}\}.$$

Assume that $\mathcal{B}$ is the family consisting of all the infinite subsets of $\mathbb{Z}^+$. Then the dual family $k\tau\mathcal{B}$ of $\tau\mathcal{B}$ is called a syndetic family (refer to [3, 10] for more details).

Let $\mathcal{F}$ be a family. Recall that

(1) (X,T) is $\mathcal{F}$-transitive, if for any nonempty open subsets U,V of X,

$$N(U,V)=\{n\in\mathbb{N}\mid T^{-n}(U)\cap V\neq\emptyset\}\in\mathcal{F}. \tag{2.6}$$

(2) (X,T) is $\mathcal{F}$-mixing, if $(X\times X,T\times T)$ is $\mathcal{F}$-transitive; (X,T) is called ergodic mixing if $(X\times X,T\times T)$ is $k\tau\mathcal{B}$-mixing.

(3) (X,T) is called topologically ergodic, if for any nonempty open subsets U,V of X,

$$\limsup_{n\to\infty}\frac{1}{n}\,[\sharp(N(U,V)\cap\{1,2,\cdots,n\})]>0. \tag{2.7}$$

$S\subset\mathbb{N}$ is called a positive upper density set or a positive lower density set, if

$$\limsup_{n\to\infty}\frac{1}{n}\,[\sharp(S\cap\{1,2,\cdots,n\})]>0 \tag{2.8}$$

or

$$\liminf_{n\to\infty}\frac{1}{n}\,[\sharp(S\cap\{1,2,\cdots,n\})]>0. \tag{2.9}$$

Hence, if for any nonempty open subsets U,V of X, $N(U,V)$ is a positive upper density set, then (X,T) is topologically ergodic.

(4) (X,T) is topologically double ergodic, if $(X\times X,T\times T)$ is topologically ergodic.

(5) (X,T) is $\mathcal{F}$-central, if for any nonempty open subset U of X, $N(U,U)\in\mathcal{F}$.

We recall that (X,T) is transitive, if for each pair of nonempty open subsets U and V of X, $N(U,V)$ is infinite; (X,T) is totally topologically transitive (totally transitive for short), if for any $n\in\mathbb{N}$, (X,T^n) is transitive; (X,T) is weak mixing, if $(X\times X,T\times T)$ is transitive; (X,T) is strong mixing, if for each pair of nonempty open subsets U and V of X, there exists an $N>0$, such that $T^n(V)\cap U\neq\emptyset$ for $n>N$.

A dynamical system is called an E-system, if it is transitive and there is an invariant measure μ with full support, i.e., $\mathrm{supp}(\mu)=X$. It is well-known that a minimal system is an E-system and an E-system is topologically ergodic. Two dynamical systems are called weakly disjoint, if their product is transitive. Call a system

(1) extremely scattering, if it is weakly disjoint from all topologically ergodic systems;

(2) strong scattering, if it is weakly disjoint from all E-systems;

(3) scattering, if it is weakly disjoint from all minimal systems;

(4) weak scattering, if it is weakly disjoint from all minimal equicontinuous systems.

So, from their definitions, we can obtain

$$\text{extremely scattering}\Rightarrow\text{strong scattering}\Rightarrow\text{scattering}\Rightarrow\text{weak scattering}.$$

Moreover, there are examples showing that the converse does not hold (see [3]).

Let (X,T) be a dynamical system. Suppose $\{m_i\}$ is an increasing sequence of positive integers. A set C is called a chaotic set (resp. finitely chaotic set) of T with respect to $\{m_i\}$, if for any set (resp. finite set) $A \subset C$ and any continuous map $F : A \to X$, there exists a subsequence $\{r_i\}$ of $\{m_i\}$, such that

$$\lim_{i\to\infty} T^{r_i}(x) = F(x)$$

for all $x \in A$.

We say that (X, T) is chaotic (resp. finitely chaotic) with respect to $\{m_i\}$, if there exists an uncountable chaotic set (resp. finitely chaotic set) with respect to $\{m_i\}$ (see [11]).

Now, we introduce some basic notations of symbolic dynamical system.

Set $K = \{0, 1, \cdots, k-1\}$ $(k \geq 2)$ with the discrete topology. The one-sided symbolic space generated by K is denoted as

$$\Sigma_k = \{x = (x_0, x_1, \cdots) \mid x_i \in S,\ \forall i \geq 0\}. \tag{2.10}$$

Under the product topology, Σ_k is a compact metric space with the second axiom of countability. Define a metric d which is compatible with the product topology on Σ_k as follows: $\forall x = (x_0, x_1, \cdots),\ y = (y_0, y_1, \cdots) \in \Sigma_k$,

$$d(x, y) = \begin{cases} 0, & x = y, \\ \dfrac{1}{k^N}, & x \neq y,\ N = \min\{n : x_n \neq y_n\}. \end{cases}$$

Let $\sigma : \Sigma_k \to \Sigma_k$ denote the shift map, namely, $\sigma(x_0x_1x_2\cdots) = (x_1x_2\cdots)$ for any $x = (x_0x_1x_2\cdots) \in \Sigma_k$.

Suppose that $A = (a_{ij})_{0\leq i,j\leq k-1}$ is a $k\times k$-$(0,1)$ matrix and suppose its every row, as well as every column, has at least one element. Such a matrix is called irreducible, if for any i, j, there is some $n > 0$, such that $a_{ij}^{(n)} > 0$, where $a_{ij}^{(n)}$ is the (i, j)th-element of A^n. $A = (a_{ij})_{0\leq i,j\leq k-1}$ is called aperiodic, if there exists an $n > 0$, such that $a_{ij}^{(n)} > 0$ for all i, j.

Put

$$\Sigma_A = \{x = (x_0, x_1, \cdots, x_n, \cdots) \in \Sigma_k, a_{x_i x_{i+1}} = 1,\ \forall i \geq 0\}. \tag{2.11}$$

Then Σ_A is a compact subset of Σ_k and Σ_A is called the set of finite type determined by A. Write $\sigma_A = \sigma|_{\Sigma_A} : \Sigma_A \to \Sigma_A$ as the sub-shift map yielded by A (see [8–9]).

3 Some Lemmas

In this section, we present some lemmas which play a key role in the proofs of our main results.

Lemma 3.1 (see [2]) *Let (X, T) be a dynamical system, $x \in R(T)$. The following statements are equivalent*:

(1) $x \in W(T)$;

(2) $\underline{P}_x(V(x, \varepsilon)) > 0$, $\forall \varepsilon > 0$;

(3) $x \in C_x = S_m$, $\forall m \in M_x$;

(4) $S_m = \omega(x, T)$, $\forall m \in M_x$,

where

$$\underline{P}_x(V(x,\varepsilon)) = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{V(x,\varepsilon)}(T^i(x)) \tag{3.1}$$

and $\chi(\cdot)$ denotes the characteristic function.

Remark 3.1 For the concepts used in Lemma 3.1, we refer to [1–2].

Lemma 3.2 (see [8]) *Let (X,T) be a dynamical system, E be an open subset of X and F a closed subset of X. $m, m_i \in M(X)$, $i = 1, 2, \cdots$, m_i converges to m under the weak$*$ topology. Then*

(1) $\liminf_{i\to\infty} m_i(E) \geq m(E)$;

(2) $\limsup_{i\to\infty} m_i(F) \leq m(F)$.

Lemma 3.3 *Let (X,T) be a dynamical system, then $T(\mathrm{RG}(T)) \subset \mathrm{RG}(T)$, and for any $n \in \mathbb{N}$, $\mathrm{RG}(T^n) = \mathrm{RG}(T)$.*

Proof Since the proof is simple, we omit it.

Lemma 3.4 (see [3]) *Let (X,T) be a dynamical system. Then (X,T) is ergodic mixing if and only if (X,T) is weak mixing and $k\tau\mathcal{B}$-transitive.*

Lemma 3.5 (see [6]) *Let (X,T) be a dynamical system. Then*

(1) *For a topologically ergodic system, extremely scattering implies weak mixing;*

(2) *For an E-system, strong scattering implies weak mixing;*

(3) *For a minimal system, weak scattering implies weak mixing;*

(4) *If (X,T) is weak mixing and (Y,S) is weak mixing and topologically ergodic, then $T\times S$ is weak mixing. If (X,T) is topologically ergodic and (Y,S) is weak mixing and topologically ergodic, then $T\times S$ is topologically ergodic. Thus, if both (X,T) and (Y,S) are weak mixing and topologically ergodic, then $T\times S$ is weak mixing and topologically ergodic;*

(5) *If (X,T) is extremely scattering and (Y,S) is extremely scattering which is also topologically ergodic, then $T\times S$ is extremely scattering;*

(6) *If (X,T) is strong scattering and (Y,S) is strong scattering which is also an E-system, then $T\times S$ is strong scattering;*

(7) *If (X,T) is scattering and (Y,S) is scattering with dense set of minimal points, then $T\times S$ is scattering.*

Lemma 3.6 (see [3]) *If S is a positive upper density set of $\mathbb{N}$, then*

$$S - S = \{l-k \mid l,k\in S\} \in k\tau\mathcal{B}.$$

Lemma 3.7 (see [7]) *Let (X,T) be a dynamical system. If $\mathcal{F}$ is a proper translation invariant family, then (X,T) is $\mathcal{F}$-transitive if and only if (X,T) is $\mathcal{F}$-central and topologically transitive.*

Lemma 3.8 *Let (X, T) be a dynamical system. If T is topologically transitive and $M(T) = X$, then (X, T) is $k\tau\mathcal{B}$-transitive.*

Proof Suppose that V is a nonempty open subset of X. Then $W(T) \cap V \neq \emptyset$. Hence there is an $x \in W(T) \cap V$. From the definition of weakly almost periodic point and Lemma 3.1, we get that

$$N_T(x, V) = \{i : T^i(x) \in V\} \tag{3.2}$$

is a positive lower density set. Suppose $n_1, n_2 \in N_T(x, V)$, $n_1 < n_2$. Then $T^{n_1}(x) \in V$, $T^{n_2}(x) \in V$. Set $y = T^{n_1}(x)$. Then $x \in T^{-n_1}(y)$ and $y \in V$. Therefore,

$$T^{n_2-n_1}(V) \cap V \supset T^{n_2-n_1}(y) \cap V \supset T^{n_2}(x) \cap V \neq \emptyset, \tag{3.3}$$

which implies that $n_2 - n_1 \in N_T(V, V)$, namely, $N_T(x, V) - N_T(x, V) \subset N_T(V, V)$. By Lemma 3.6, we get $N_T(x, V) - N_T(x, V) \in k\tau\mathcal{B}$, which implies T is $k\tau\mathcal{B}$-central. Thus Lemma 3.7 yields that T is $k\tau\mathcal{B}$-transitive.

Lemma 3.9 *Let (X, T) be a dynamical system. If T is topologically transitive and $M(T) = X$, then (X, T) is an E-system.*

Proof Since X is a compact metric space, X has a countable basis, say $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$. Then, for any $n \geq 1$, there is some ergodic measure μ_n of (X, T), such that $U_n \cap \operatorname{supp}(\mu_n) \neq \emptyset$ and hence $\mu_n(U_n) > 0$. This implies that for the generalized convex combination

$$\mu = \sum_n 2^{-n} \mu_n, \tag{3.4}$$

it has full support. This completes the proof of Lemma 3.9.

Remark 3.2 After the submission of this paper, we unexpectedly found the results similar to Lemmas 3.8–3.9 in [12]. But the proof methods are greatly different from those used in [12].

Corollary 3.1 *Let (X, T) be a dynamical system. If T is topologically transitive and $M(T) = X$, then (X, T) is topologically ergodic.*

Proof Since an E-system is topologically ergodic, from Lemma 3.9, we obtain that (X, T) is topologically ergodic.

Remark 3.3 For more details of the involved concepts to be used in Lemmas 3.3–3.9, we refer to [1–3, 8–10].

4 Main Results and Proofs

Theorem 4.1 *Let (X, T) denote a dynamical system. If $M(T) = X$, then the following statements are equivalent:*

(1) *(X, T) is ergodic mixing*;

(2) (X, T) *is topologically double ergodic*;

(3) (X, T) *is weak mixing*;

(4) (X, T) *is extremely scattering*;

(5) (X, T) *is strong scattering*;

(6) $(X \times X, T \times T)$ *is strong scattering*;

(7) $(X \times X, T \times T)$ *is extremely scattering*;

(8) *For any subset S of $\mathbb{N}$ with upper density 1, there is a c-dense F_σ-chaotic set with respect to S.*

Proof Firstly, we prove (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6)$\Leftrightarrow$ (7). If one of the seven conditions is satisfied, then T is topologically transitive. From the given conditions and Corollary 3.1, we know that (X, T) is topologically ergodic. Hence, from Lemma 3.5(4), it holds that (1) $\Rightarrow$ (2). And it is clear that(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5).

Now, we prove (5) $\Rightarrow$ (6). Assume that (X, T) is strong scattering. Then (X, T) is topologically transitive. By Lemma 3.9, we get (X, T) is an E-system. Thus, from Lemma 3.5(2), we obtain that $(X \times X, T \times T)$ is strong scattering.

Now, we show (6) $\Rightarrow$ (1). Since $(X \times X, T \times T)$ is strong scattering, $(X \times X, T \times T)$ is topologically transitive. And it is easy to prove that $M(T \times T) = X \times X$ if $M(T) = X$. Thus the conclusion of Lemma 3.8 yields that $T \times T$ is $k\tau\mathcal{B}$-transitive, that is, (X, T) is ergodic mixing.

Clearly, (7) implies (4). On the other hand, by Corollary 3.1, we know that (X, T) is topologically ergodic. Therefore, if (X, T) is extremely scattering, by Lemma 3.5(5), we get that $(X \times X, T \times T)$ is extremely scattering. So (7) $\Leftrightarrow$ (4).

From [13, Theorem 3], we get that (7) is equivalent to (8).

Thus, we prove (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6)$\Leftrightarrow$ (7) $\Leftrightarrow$ (8).

Remark 4.1 After the submission of this paper, we also found a result similar to that of Theorem 4.1 in [12]. But we want to emphasize that the result of Theorem 4.1 is more profound and our proof method is different from that used in [12].

To introduce the following result conveniently, we introduce a notation firstly. Suppose that (X, T) is a dynamical system. For any $m \in \mathbb{N}$, let

$$(X^m, T^m) = (\overbrace{X \times \cdots \times X}^{m}, \overbrace{T \times \cdots \times T}^{m}).$$

Theorem 4.2 *Suppose that (X, T) is a totally transitive system and $\overline{\mathrm{RG}(T)} = X$. Then for any $n, m \in \mathbb{N}$, $(X^m, (T^n)^m)$ is topologically double ergodic.*

Proof For any $n \in \mathbb{N}$, from Lemma 3.3 and the given assumptions, we get $\overline{\mathrm{RG}(T)} = \overline{\mathrm{RG}(T^n)} = X$, so (X, T^n) is an M-system, namely, (X, T^n) is topologically transitive and the set of minimal points is dense in X. From the proof of Lemma 3.9, we know that there exists an invariant measure with a full support, which implies that T^n is topologically ergodic.

Since (X,T) is topologically transitive, for any nonempty open sets U, V, there exists an $l\in\mathbb{N}$, such that $T^{-l}(V)\cap U\neq\emptyset$. Set $W=T^{-l}(V)\cap U$, W is a nonempty open set. Therefore, there exists an $x\in W\cap RG(T)$. From the definition of regular minimal point, there is a $k\in\mathbb{N}$, such that $T^{kj}(x)\in W$ for any $j\in\mathbb{N}$. Thus, $T^{kj+l}(x)\in V$ and $T^{kj+l}(x)\in T^{kj+l}(U)$. That is $V\cap T^{kj+l}(U)\neq\emptyset$. Noting that T^k is topologically ergodic, we get that there exists a positive upper density set J of $\mathbb{N}$, such that for any $s\in J$, $T^{ks}(U)\cap T^{-l}(U)\neq\emptyset$, which yields $T^{ks+l}(U)\cap U\neq\emptyset$. Set $\widetilde{J}=kJ+l=\{ks+l: s\in J\}$. Then $\widetilde{J}$ is a positive upper density set too. From the derivation above, we have $T^t(U)\cap U\neq\emptyset$ and $T^t(U)\cap V\neq\emptyset$ for any $t\in\widetilde{J}$. From [10, Lemma 1], we get that (X,T) is topologically double ergodic.

Since (X,T) is totally transitive, for any $i\in\mathbb{N}$, (X^i,T^i) is also totally transitive. By the above proof, we can also get that (X^i,T^i) is topologically double ergodic. By [14, Lemma 2], for any $m\in\mathbb{N}$, $(X^m,(T^n)^m)$ is topologically double ergodic.

Theorem 4.3 *Assume that $\sigma_A=\sigma|_{\Sigma_A}:\Sigma_A\to\Sigma_A$ is the sub-shift map of finite type yielded by a $k\times k$-$(0,1)$ matrix A. Then σ_A is strong mixing if and only if σ_A is totally transitive.*

Proof It is sufficient to prove that (Σ_A,σ_A) being totally transitive implies that (Σ_A,σ_A) is strong mixing.

If (Σ_A,σ_A) is totally transitive, then (Σ_A,σ_A) is topologically transitive, which implies that A is irreducible. So the set of periodic points of σ_A is dense in Σ_A. Of course, the set of regular minimal points of σ_A is dense in Σ_A. Then Theorem 4.2 gives that (Σ_A,σ_A) is topologically double ergodic, which implies (Σ_A,σ_A) is weak mixing. It is well-known that (Σ_A,σ_A) is weak mixing if and only if (Σ_A,σ_A) is strong mixing, so we prove that (Σ_A,σ_A) being totally transitive implies that (Σ_A,σ_A) is strong mixing.

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