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Asymptotic Distribution of the Jump Change-Point Estimator*

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Abstract The asymptotic distribution of the change-point estimator in a jump change-point model is considered. For the jump change-point model $X_i = a + \theta I\{[n\tau_0] < i \leq n\} + \varepsilon_i$, where ε_i $(i = 1, \cdots, n)$ are independent identically distributed random variables with $E\varepsilon_i = 0$ and $\mathrm{Var}(\varepsilon_i) < \infty$, with the help of the slip window method, the asymptotic distribution of the jump change-point estimator $\widehat{\tau}$ is studied under the condition of the local alternative hypothesis.

Keywords Change-point, Local alternative hypothesis, Asymptotic distribution
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1 Introduction

The change-point analysis is widely used in many fields including quality control, economics and finance and so on (see [1–6, 8, 21, 24]). There is rich literature on the change-point analysis under the assumption that random variables in consideration are independent. Among others, Chernoff and Zacks [9] proposed a test statistic for detecting a mean shift in normal distributions; Ramanayake and Gupta [19] used a likelihood ratio test based method to tackle change-point problems in exponential distributions; Daniel and Hartigan [11], Rudoy et al [20] studied the change-point problems by using Bayesian approach; Csörgő and Horváth [10], Huskova [15] discussed the nonparametric methods used in the change-point analysis; Chen [7] dealt with the inference on jump shift and Miao [16] considered the inference on the slope change-point and so on.

About the study of asymptotic distribution of the change-point estimator, there is little literature, since it is involved into many theories of large samples. However, it is very important to the depth of the change-point theory and its application. In the local alternative hypothesis, Bai [2] and Bai and Perron [3] proposed the estimation of change-points in a linear regression model by minimizing the sum of squared residuals and obtained its limiting distribution. The

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kernel-type estimation method is used to estimate the time of change in the mean in a sequence of independent observations in [13]. At the same time, the asymptotic distribution of the change-point estimator is derived when the size of the change is small. For the series of the independent and exponentially distributed random variables, Fotopoulos and Jandhyala [12] gave the exact expression for the asymptotic distribution of the maximum likelihood estimate of the change-point. The modified information criterion (MIC) is applied to detect the estimation of multiple change-points and their limiting distribution in [17]. Perron and Qu [18] studied the limiting distribution of the estimates of the structure change-point for both fixed and shrinking magnitude of change in the linear regression model.

In [7], the problem of testing and estimation about the change-point has been studied. Consider the model with at most one jump change in [7]:

$$X_i = \begin{cases} a + \varepsilon_i, & 1 \le i \le [n\tau_0], \\ a + \theta + \varepsilon_i, & [n\tau_0] < i \le n, \end{cases} \tag{1.1}$$

where a, θ and $\tau_0 \in (0,1)$ are unknown parameters, $[a]$ denotes the integer part of a number a, and θ is called the jump at the change-point τ_0. $\varepsilon_1, \cdots, \varepsilon_n$ are independent identically distributed random variables with $E(\varepsilon_i) = 0$ and $\mathrm{Var}(\varepsilon_i) = \sigma^2 < \infty$. In this paper, based on [7], we will discuss the asymptotic distribution of the change-point estimator by the slide window method in the local alternative hypothesis further.

For convenience, throughout the paper, we let

$$Y_m = \frac{1}{\sqrt{2l}}\Big(\sum_{i=m+1}^{m+l} X_i - \sum_{i=m-l+1}^{m} X_i\Big), \quad X_i^* = X_i - EX_i.$$

At the same time, $c, c_1, \cdots \in (0, \infty)$ stand for positive constants whose values do not depend on n and may vary from formula to formula.

2 The Asymptotic Distribution of $\widehat{\tau}$

If we know in advance or by the test in [7] that there is a change, then we define the estimator of τ_0 as

$$\widehat{\tau} = \frac{\widehat{m}}{n} = \frac{1}{n}\min\{m : |Y_m| = \max_{l \le j \le n-l} |Y_j|\}. \tag{2.1}$$

When the jump θ is a constant, the consistency and convergence rate of the change-point estimator $\widehat{\tau}$ have been studied in [22–23]. In this paper, we mainly discuss the asymptotic distribution of $\widehat{\tau}$ in the local alternative hypothesis, that is, the jump θ depends on the size of the sample n. Hence, we denote it by θ_n, and θ_n satisfies $\theta_n \to 0$ as $n \to \infty$. If θ is a constant independent of n, the results for the independent binomial distributed case in [14] indicate that the limiting distribution of $\widehat{\tau}$ depends on the underlying distributions in addition to θ. Consequently, confidence intervals cannot be easily constructed. Note that if θ_n is a constant or it is large relative to the variances of ε_i, the change-point estimation is usually quite precise. Hence, in practice it may be more important to construct confidence intervals for τ_0 when the jump θ_n is small.

To derive the asymptotic distribution of $\widehat{\tau}$, we define

$$V^*(m) = Y_m^2 - Y_{m_0}^2, \quad V_n^*(s) = V^*\Big(\Big[m_0 + \frac{s}{\theta_n^2}\Big]\Big) \tag{2.2}$$

and

$$W(s) = \begin{cases} W_1(-s), & s < 0, \\ 0, & s = 0, \\ W_2(s), & s > 0, \end{cases} \tag{2.3}$$

where $\{W_1(s),\ 0 \le s < \infty\}$ and $\{W_2(s),\ 0 \le s < \infty\}$ are two independent Wiener processes with $W_1(0) = W_2(0) = 0$.

Theorem 2.1 A*ssume that* (1.1) *and* (2.1) *hold, and* $X_1, X_2, \cdots, X_n$ *are independent random variables which satisfy*

$$E|X_i|^\delta < \infty, \quad i = 1, 2, \cdots, n \quad for\ some \ \ \delta > 2; \tag{2.4}$$

$l = l_n$ *is a positive integer such that*

$$n^{\frac{2}{\delta}} \ll l \ll n, \tag{2.5}$$

where $l \ll n$ *means* $\lim\limits_{n\to\infty} \frac{l}{n} = 0$. *Further, let*

$$m_0 = [n\tau_0], \quad \widehat{m} = [n\widehat{\tau}], \quad m = [n\tau] \ with\ some\ 0 < \tau < 1. \tag{2.6}$$

When the jump θ_n *satisfies*

$$\theta_n \to 0, \quad \frac{l\theta_n^2}{\log n} \to \infty, \tag{2.7}$$

for all $M > 0$, *as* σ^2 *is known, we have*

$$V_n^*(s) \xrightarrow{D[-M,\ M]} \sqrt{6}\ \sigma W(s) - |s|, \tag{2.8}$$

$$\frac{n\theta_n^2}{6\sigma^2}(\widehat{\tau} - \tau_0) \xrightarrow{\ \ d\ \ } \arg \sup_{-\infty<s<+\infty} (W(s) - |s|), \tag{2.9}$$

where $D[-M,\ M]$ *means the weak convergence on* $[-M,\ M]$, *and* $W(s)$ *is a two-side Brown motion on* $(-\infty, +\infty)$ *defined in* (2.3).

Let

$$\widehat{X}_m = \frac{1}{m}\sum_{1\le i\le m} X_i, \quad \widetilde{X}_m = \frac{1}{n-m}\sum_{m<i\le n} X_i.$$

Then we propose the estimators for σ^2 and θ_n as follows, respectively

$$\widehat{\sigma}_n^2 = \frac{1}{n}\min_{1\le m\le n}\Big\{\sum_{1\le i\le m}(X_i - \widehat{X}_m)^2 + \sum_{m<i\le n}(X_i - \widetilde{X}_m)^2\Big\}, \tag{2.10}$$

$$\widehat{\theta}_n^2(m) = (\widetilde{X}_{m+l} - \widehat{X}_{m-l})^2. \tag{2.11}$$

Theorem 2.2 *Assume that all the conditions of Theorem* 2.1 *hold. When σ^2 is unknown, we still have*

$$\frac{n\widehat{\theta}_n^2(\widehat{m})}{6\widehat{\sigma}_n^2}(\widehat{\tau}-\tau_0) \xrightarrow{d} \arg\max_{-\infty<s<+\infty}(W(s)-|s|), \tag{2.12}$$

where $W(s)$ is a two-side Brown motion defined in (2.3)*, $\widehat{\sigma}_n^2, \widehat{\theta}_n^2(\widehat{m})$ are defined in* (2.10) *and* (2.11)*, respectively, and $\widehat{m}$ is the consistence estimator of m_0.*

It is noted that Theorems 2.1 and 2.2 can be used to construct the asymptotic confidence interval and test the existence of a change-point τ_0. At the same time, it provides some qualitative information on estimation of other parameters related to the change-point in the model.

In addition, it is easy to see that the conclusion of Theorem 2.1 (Theorem 2.2) holds only if X_i $(i=1,2,\cdots,n)$ satisfy $E|X_i|^\delta<\infty$ for some $\delta>2$, that is, the errors ε_i $(i=1,2,\cdots,n)$ only need to satisfy $E|\varepsilon_i|^\delta<\infty$. Hence, the result of Theorem 2.1 (Theorem 2.2) remains true for the weighty-trail distribution, such as t-distribution, Pareto-distribution and so on, which are widely used in economics and finance. Hence Theorem 2.1 plays a very important role in application to economics and finance and so on, such as the analysis of financial contagion.

Remark 2.1 If X_i $(i=1,2,\cdots,n)$ are independent normal random variables, then the window width l_n can be relaxed as

$$\log^2 n \ll l \ll n. \tag{2.13}$$

Remark 2.2 If X_i $(i=1,2,\cdots,n)$ are independent non-normal random variables, but the moment-generated function of X_i exists, that is,

$$E\mathrm{e}^{tX_1}<\infty \quad \text{for some } t>0, \tag{2.14}$$

then the window width l_n can also be chosen as (2.13).

To prove the above theorems, we need the following lemmas first.

Lemma 2.1 *Assume that all the conditions of Theorem* 2.1 *hold. Then we have*

$$|\widehat{\tau}-\tau_0| = O_p\Big(\frac{1}{n\ \theta_n^2}\Big). \tag{2.15}$$

The proof can be found in [23].

Lemma 2.2 *Let X_i $(i=1,\cdots,n)$ be independent random variables with $E|X_i|^\beta<\infty$ for some $\beta>2$, satisfying*

$$X_i\sim F(x), \quad i=1,\cdots,m_0; \quad X_i\sim F(x-\theta), \quad i=m_0+1,\cdots,n. \tag{2.16}$$

Then we have

$$|\widehat{\theta}_n-\theta|\to 0 \quad a.s., \quad |\widehat{\sigma}_n^2-\sigma^2|\log n\to 0 \quad in\ P, \tag{2.17}$$

where $\widehat{\sigma}_n^2, \widehat{\theta}_n^2$ are defined in (2.10) *and* (2.11)*, respectively.*

The proof can be found in [7].

Proof of Theorem 2.1 For every $M > 0$, let $d(n) = \frac{M}{\theta_n^2}$. By Lemma 2.1, it follows that $\widehat{m} \in (m_0 - d(n), m_0 + d(n))$ in P. By (2.7), we know that $d(n) \ll l$. Hence we only need to examine the behavior of $V^*(m)$ for those m in the neighborhood of m_0 such that $m = [m_0 + \frac{s}{\theta_n^2}]$, where s varies in an arbitrary bounded interval, that is, $-M \le s \le M$.

Notice that $Y_m = \frac{1}{\sqrt{2l}}\Big(\sum\limits_{i=m+1}^{m+l} X_i - \sum\limits_{i=m-l+1}^{m} X_i\Big)$, which is constructed by i.i.d. random variables series. It is easy to see that $EY_m = 0$ for $m \le m_0 - l$ or $m > m_0 + l$. As for $m_0 - l < m \le m_0$, we have $EY_m = \frac{1}{\sqrt{2l}}(l + m - m_0)\theta_n$ by simple calculation, and $EY_m = \frac{1}{\sqrt{2l}}(l - m + m_0)\theta_n$ for $m_0 < m \le m_0 + l$.

By the simple decomposition and computation, $V^*(m)$ can be rewritten as

$$\begin{aligned} V^*(m) &= [(EY_m)^2 - (EY_{m_0})^2] + [(Y_m - EY_m)^2 - (Y_{m_0} - EY_{m_0})^2] \\ &\quad + 2(EY_m - EY_{m_0})(Y_{m_0} - EY_{m_0}) + 2EY_m(Y_m - EY_m - Y_{m_0} + EY_{m_0}) \\ &\widehat{=} A_1 + A_2 + A_3 + A_4, \end{aligned} \tag{2.18}$$

where

$$A_1 = \begin{cases} -\dfrac{1}{2} l\theta_n^2, & m \le m_0 - l \text{ or } m > m_0 + l, \\ -\Big(1 - \dfrac{m_0 - m}{2l}\Big)(m_0 - m)\theta_n^2, & m_0 - l < m \le m_0, \\ -\Big(1 - \dfrac{m - m_0}{2l}\Big)(m - m_0)\theta_n^2, & m_0 < m \le m_0 + l; \end{cases} \tag{2.19}$$

$$A_2 = \frac{1}{2l}\Big(\sum_{i=m+1}^{m+l} X_i^* - \sum_{i=m-l+1}^{m} X_i^*\Big)^2 - \frac{1}{2l}\Big(\sum_{i=m_0+1}^{m_0+l} X_i^* - \sum_{i=m_0-l+1}^{m_0} X_i^*\Big)^2; \tag{2.20}$$

$$A_3 = \Big(\sum_{i=m_0+1}^{m_0+l} X_i^* - \sum_{i=m_0-l+1}^{m_0} X_i^*\Big) \cdot \begin{cases} -\theta_n, & m \le m_0 - l \text{ or } m > m_0 + l, \\ -\dfrac{(m_0 - m)}{l}\theta_n, & m_0 - l < m \le m_0, \\ -\dfrac{(m - m_0)}{l}\theta_n, & m_0 < m \le m_0 + l; \end{cases} \tag{2.21}$$

$$A_4 = \begin{cases} 0, \quad m \le m_0 - l \text{ or } m > m_0 + l, \\ \Big(1 - \dfrac{m_0 - m}{l}\Big)\theta_n\Big(2\sum\limits_{i=m+1}^{m_0} X_i^* - \sum\limits_{i=m+l+1}^{m_0+l} X_i^* - \sum\limits_{i=m-l+1}^{m_0-l} X_i^*\Big), & m_0 - l < m \le m_0, \\ \Big(1 - \dfrac{m - m_0}{l}\Big)\theta_n\Big(\sum\limits_{i=m_0+l+1}^{m+l} X_i^* + \sum\limits_{i=m_0-l+1}^{m-l} X_i^* - 2\sum\limits_{i=m_0+1}^{m} X_i^*\Big), & m_0 < m \le m_0 + l. \end{cases} \tag{2.22}$$

Now, we first consider the case of $m \le m_0$, that is, $s \le 0$. By (2.19), as $n \to \infty$, we get

$$A_1 = -(m_0 - m)\theta_n^2\Big(1 - \frac{m_0 - m}{2l}\Big) = -\Big(1 - \frac{m_0 - m}{2l}\Big)(-s) \to -|s|. \tag{2.23}$$

Since

$$A_2 = \frac{1}{2l}\Big(2\sum_{i=m+1}^{m_0} X_i^* - \sum_{i=m-l+1}^{m_0-l} X_i^* - \sum_{i=m+l+1}^{m_0+l} X_i^*\Big)$$

$$\cdot\Big(2\sum_{i=m_0+1}^{m+l}X_i^*+\sum_{i=m+l+1}^{m_0+l}X_i^*-\sum_{i=m-l+1}^{m_0-l}X_i^*-2\sum_{i=m_0-l+1}^{m}X_i^*\Big) \tag{2.24}$$

and

$$\begin{aligned}\max_{m_0-d(n)\le m\le m_0}\Big|\sum_{i=m+1}^{m_0}X_i^*\Big| &= \max_{m_0-d(n)\le m\le m_0}\Big|\sum_{i=1}^{m_0-m}X_{i+m}^*\Big|\\ &= \max_{0\le m_0-m\le d(n)}\Big|\sum_{i=1}^{m_0-m}X_{i+m}^*\Big|\\ &= \max_{1\le t\le d(n)}\Big|\sum_{i=1}^{t}X_{i+m}^*\Big|\\ &= d^{\frac12}(n)\max_{1\le t\le d(n)}\Big|\frac{1}{d^{\frac12}(n)}\sum_{i=1}^{t}X_{i+m}^*\Big|\\ &\to d^{\frac12}(n)\sup_{0\le s\le 1}|W(s)|, \end{aligned}\tag{2.25}$$

$$\begin{aligned}\max_{m_0-d(n)\le m\le m_0}\Big|\sum_{i=m_0+1}^{m+l}X_i^*\Big| &= \max_{0\le m_0-m\le d(n)}\Big|\sum_{i=1}^{l-(m_0-m)}X_{i+m_0}^*\Big|\\ &\le \max_{0\le m_0-m\le d(n)}\Big|\sum_{i=1}^{l}X_{i+m_0}^*\Big|+\max_{0\le m_0-m\le d(n)}\Big|\sum_{j=1}^{m_0-m}X_{m_0+l-j}^*\Big|\\ &\to l^{\frac12}O_P(1)+d^{\frac12}(n)\sup_{0\le s\le 1}|W(s)|\\ &= l^{\frac12}O_P(1)+l^{\frac12}\frac{M^{\frac12}}{l^{\frac12}\theta_n}\sup_{0\le s\le 1}|W(s)|\\ &= l^{\frac12}\Big(1+\frac{M^{\frac12}}{l^{\frac12}\theta_n}\Big)O_p(1)\ . \end{aligned}\tag{2.26}$$

Combining (2.24)–(2.26), we have

$$\begin{aligned}A_2 &= \frac{1}{2l}\Big(\frac{\sqrt{M}}{\theta_n}O_p(1)l^{\frac12}\Big(1+\frac{M^{\frac12}}{l^{\frac12}\theta_n}\Big)O_p(1)+\frac{M}{\theta_n^2}O_p(1)\Big)\\ &= \Big(\frac{\sqrt{M}}{l^{\frac12}\theta_n}+\frac{M}{l\theta_n^2}\Big)O_p(1). \end{aligned}\tag{2.27}$$

By (2.21), we obtain

$$A_3=-\frac{1}{l}(m_0-m)\theta_n l^{\frac12}O_P(1)=-\frac{1}{l}\ \frac{-s}{\theta_n^2}\ \theta_n\ l^{\frac12}O_p(1)=\frac{-s}{l^{\frac12}\theta_n}O_p(1). \tag{2.28}$$

It follows from the definition of $m=[m_0+\frac{s}{\theta_n^2}]$ and (2.22) that

$$\begin{aligned}A_4 &= \Big(1-\frac{m_0-m}{l}\Big)\frac{(-s)^{\frac12}}{\sqrt{m_0-m}}\Big(2\sum_{i=m+1}^{m_0}X_i^*-\sum_{i=m-l+1}^{m_0-l}X_i^*-\sum_{i=m+l+1}^{m_0+l}X_i^*\Big)\\ &\to \Big(1-\frac{m_0-m}{l}\Big)\sqrt{6}\ \sigma W_1(-s). \end{aligned}\tag{2.29}$$

Putting together (2.7), (2.23) and (2.27)–(2.29), we have

$$V_n^*(s) \xrightarrow{D[-M,0]} \sqrt{6}\ \sigma W_1(-s) - |s|. \tag{2.30}$$

Similar arguments give also

$$V_n^*(s) \xrightarrow{D[0,M]} \sqrt{6}\ \sigma W_2(s) - |s|. \tag{2.31}$$

Since $W_1(-s)$ is determined by X_m with $m \le m_0$, and $W_2(s)$ is determined by X_m with $m > m_0$, we have $W_1(-s)$ and $W_2(s)$ are independent. Combining (2.30) and (2.31), for every $M > 0$, we have that on $C[-M, M]$,

$$V_n^*(s) \xrightarrow{D[-M,M]} \sqrt{6}\ \sigma W(s) - |s|, \tag{2.32}$$

where $C[-M, M]$ denotes the space of continuous function on $[-M, M]$. This finishes the proof of (2.8).

To obtain the limiting distribution of $\widehat{\tau}$, define $C_{\max}[-M, M]$ to be the subset of $C[-M, M]$ such that each function has a unique maximum. It is straightforward to show that the argmax function is a continuous function on the set $C_{\max}[-M, M]$. By the continuous mapping theorem, we have

$$n\theta_n^2(\widehat{\tau} - \tau_0) \xrightarrow{\quad d \quad} 6\sigma^2 \arg \sup_{-M<s<M} \Big(W\Big(\frac{s}{6\sigma^2}\Big) - \frac{|s|}{6\sigma^2}\Big). \tag{2.33}$$

Replacing $\frac{|s|}{6\sigma^2}$ by s in (2.33) and letting $M \to +\infty$, we get (2.9) immediately.

The proof is completed.

Proof of Theorem 2.2 Since

$$\frac{n\widehat{\theta}_n^2}{6}\widehat{\sigma}_n^2(\widehat{\tau} - \tau_0) = \frac{\sigma^2}{\widehat{\sigma}_n^2}\ \frac{\widehat{\theta}_n^2}{\theta_n^2}\frac{n\theta_n^2}{6\sigma^2}(\widehat{\tau} - \tau_0),$$

by Lemma 3.2 and Slutsky Lemma, we get (2.12) immediately.

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