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Nonresonance and Global Existence of Nonlinear Elastic Wave Equations with External Forces*

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Abstract This paper establishes the global existence of classical solution to the system of homogeneous, isotropic hyperelasticity with time-independent external force, provided that the nonlinear term obeys a type of null condition. The authors first prove the existence and uniqueness of the stationary solution. Then they show that the solution to the dynamical system converges to the stationary solution as time goes to infinity.

Keywords Nonlinear elasticity, Global existence, Null condition, External force
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1 Introduction and Main Result

Points in $\mathbb{R}^4$ are denoted by $X = (x^0, x^1, x^2, x^3) = (t, x)$. Partial derivatives are written as $\partial_l = \frac{\partial}{\partial x^l}$, $l = 1, 2, 3$, with the abbreviations $\partial = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla)$. We study the nonlinear dynamical system of homogeneous, isotropic hyperelasticity with time-independent external force:

$$Lu^i \equiv \partial_t^2 u^i - c_2^2 \Delta u^i - (c_1^2 - c_2^2)\partial_i(\nabla \cdot u) = N^i(u, u) + h^i(x), \quad i = 1, 2, 3, \tag{1.1}$$

where $u = (u^1, u^2, u^3)$ is the displacement vector and $h = (h^1, h^2, h^3)$ is the external force. The constants c_1, c_2 satisfy that $c_1^2 > \frac{4c_2^2}{3} > 0$ and the nonlinear term of system (1.1) is expressed as follows:

$$N^i(u, v) = B_{lmn}^{ijh} \partial_l(\partial_m u^j \partial_n v^h)$$

with coefficients B_{lmn}^{ijh} satisfying the symmetric properties $B_{lmn}^{ijh} = B_{mln}^{jih} = B_{lnm}^{ihj}$. Throughout this paper, we use the summation convention that repeated indices are summed over $l, m, n, j, h = 1, 2, 3$.

When the external force $h(x)$ is equivalent to zero, it is well-known that there are two conditions necessary for the global existence of solutions to system (1.1): first, the initial data must be small; second, the nonlinear term must obey a compatible condition with respect to the corresponding linear equations, which is a type of null condition. If one of these two

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assumptions is not satisfied, the solutions may blow up at finite time. When the initial data is large, Tahvildar-Zadeh [8] provided a blow-up example for system (1.1). When the nonlinear term of system (1.1) does not satisfy the null condition that we will introduce later, John [3] proved that any spherically symmetric data can develop the singularities, no matter how small it is. Moreover, John [4] proved the almost global existence for the solution to system (1.1), provided that $h(x) \equiv 0$ and initial data is small enough (see also [5]). Zhou and Xu [9] generalized the results in [5] to the case that $h(x)$ is small in some sense. When $h(x) \equiv 0$ and the nonlinear term obeys a type of null condition, Sideris [7], Agemi [1] and Sideris [6] proved the global existence for system (1.1) with small initial data.

Now let us give the null condition introduced in [6]. For each direction $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2$, there are two families of elementary plane waves for the linear equation $Lu = 0$:

$$\mathcal{W}_1(\xi) = \{\xi\phi(\langle x, \xi\rangle - c_1 t) : \ \phi \text{ is a scalar function}\},$$
$$\mathcal{W}_2(\xi) = \{\eta\psi(\langle x, \xi\rangle - c_2 t) : \ \langle \eta, \xi\rangle = 0, \ \psi \text{ is a scalar function}\},$$

where plane waves in $\mathcal{W}_1(\xi)$ represent longitudinal waves propagating in the direction ξ with speed c_1, and plane waves in $\mathcal{W}_2(\xi)$ represent transverse waves propagating in the direction ξ with speed c_2. The null condition is that the quadratic interaction of elementary waves of each wave family only produces waves in the other family.

Definition 1.1 *If for all resonant triples* $(u, v, w) \in \mathcal{W}_\alpha(\xi) \times \mathcal{W}_\alpha(\xi) \times \mathcal{W}_\alpha(\xi)$, $\alpha = 1, 2$,

$$\langle u, N(v, w)\rangle = 0, \tag{1.2}$$

then we say the nonlinear term N *is null with respect to the linear operator* L.

By direct substitution, the null condition 1.2 is equivalent to

$$B^{ijh}_{lmn}\xi_i\xi_j\xi_h\xi_l\xi_m\xi_n = 0 \quad \text{for all } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2, \tag{1.3}$$
$$B^{ijh}_{lmn}\eta^{(1)}_i\eta^{(2)}_j\eta^{(3)}_h\xi_l\xi_m\xi_n = 0 \quad \text{for all } \xi, \eta^{(a)} \in \mathbb{S}^2 \text{ with } \langle \xi, \eta^{(a)}\rangle = 0. \tag{1.4}$$

From [6, Lemma 2.2], the null condition (1.2) holds for all isotropic materials.

The aim of this paper is to generalize the result of Sideris [6] to the case of the nonlinear elasticity with time-independent external force. Precisely, we will prove that small initial disturbance and external force give rise to global classical solution to system (1.1), when the null condition (1.3) holds. Our problem is physically relevant, since for example the external force can be the gravity. Our proofs rely on the generalized energy estimates and the techniques represent an evolution of the ideas in [2, 5–7, 9]. We first prove the existence and uniqueness of the stationary solution. Then it suffices to prove the global existence of the original solution minus stationary solution, by using the invariance of system (1.1) under translations, simultaneous rotations and changes of scaling. This proof is carried out in line with Sideris [6].

The angular momentum operators are the vector fields $\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla$ with $\wedge$ being the usual vector cross product in $\mathbb{R}^3$, and the scaling operator is defined by $S = t\partial_t + r\partial_r$, where $r = |x|$ and $\partial_r = \frac{x}{r} \cdot \nabla$. For the operator L, a central role is played by the generators of simultaneous rotations. They are given as follows:

$$\widetilde{\Omega}_i = \Omega_i I + U_i, \quad i = 1, 2, 3$$

with

$$U_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the operator L commutes with $\widetilde{\Omega}$, and $\widetilde{\Omega}N(u,v) = N(\widetilde{\Omega}u, v) + N(u, \widetilde{\Omega}v)$. Let $\widetilde{S} = S - 1$. Then $[\widetilde{S}, L] = -2L$ and $\widetilde{S}N(u,v) = N(\widetilde{S}u, v) + N(u, \widetilde{S}v) - 2N(u,v)$.

The eight vector fields $(\partial, \widetilde{\Omega}, \widetilde{S})$ will be written as $\Gamma = (\Gamma_0, \cdots, \Gamma_7)$. The commutator of any two Γ's is either 0 or in the span of Γ, and in particular, the commutator of ∂ and Γ is either 0 or in the span of ∂. By Γ^a, $a = (a_0, \cdots, a_7)$, we denote an ordered product of vector fields $\Gamma_0^{a_0} \cdots \Gamma_7^{a_7}$.

To describe the solution space, we also introduce the time-independent analog of Γ. Let $\Lambda = (\Lambda_1, \cdots, \Lambda_7) = (\nabla, \widetilde{\Omega}, r\partial_r - 1)$. Then the Λ's have similar commutation properties to the Γ's. The commutator of any two Λ's is either 0 or is in the span of Λ, and in particular, the commutator of ∇ and Λ is either 0 or in the span of ∇. By Λ^a, $a = (a_1, \cdots, a_7)$, we denote an ordered product of vector fields $\Lambda_1^{a_1} \cdots \Gamma_7^{a_7}$.

Define the time-independent space

$$H_\Lambda^k = \{f \in L^2(\mathbb{R}^3)^3 : \Lambda^a f \in L^2(\mathbb{R}^3)^3,\ |a| \le k\}$$

with the norm

$$\|f\|_{\Lambda,k} = \Big(\sum_{|a|\le k} \|\Lambda^a f\|_{L^2}^2\Big)^{\frac{1}{2}}, \quad k = 0, 1, 2, \cdots.$$

Let F_Λ^k be the closure of the set $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ in the norm $\|h\|_{F_\Lambda^k} = \sum\limits_{|a|\le k} \||x|\Lambda^a h\|_{L^2}$.

We use the energy norm associated to the operator L

$$E_1(u(t)) = \frac{1}{2}\int_{\mathbb{R}^3} [|\partial_t u(t,\cdot)|^2 + c_2^2|\nabla u(t,\cdot)|^2 + (c_1^2 - c_2^2)|\nabla \cdot u(t,\cdot)|^2]\mathrm{d}x,$$

and the high-order generalized energy norms

$$E_k(u(t)) = \sum_{|a|\le k-1} E_1(\Gamma^a u(t)), \quad k = 2, 3, \cdots.$$

And we will construct the solutions to system (1.1) in the space $\dot{H}_\Gamma^k(T)$, which is the closure of the set $C^\infty([0,T); C^\infty(\mathbb{R}^3, \mathbb{R}^3))$ in the norm $\sup\limits_{0\le t<T} E_k^{\frac{1}{2}}(u(t))$. Thus,

$$\dot{H}_\Gamma^k(T) \subset \Big\{u(t,x) : \partial u(t,\cdot) \in \bigcap_{j=0}^{k-1} C^j([0,T); H_\Lambda^{k-1-j})\Big\}.$$

Define the orthogonal projections onto radial and transverse directions by

$$P_1 u = \Big(\frac{x}{r} \otimes \frac{x}{r}\Big)u = \frac{x}{r}\Big\langle \frac{x}{r}, u\Big\rangle,$$
$$P_2 u = [I - P_1]u = -\frac{x}{r} \wedge \Big(\frac{x}{r} \wedge u\Big).$$

Employing the notation $\langle A\rangle = (1+|A|^2)^{\frac{1}{2}}$, we will use the weighted L^2-norms

$$M_k(u(t)) = \sum_{\alpha=1}^{2}\sum_{|a|\le k-2} \|\langle c_\alpha t - r\rangle P_\alpha \partial\nabla\Gamma^a u(t,\cdot)\|_{L^2}, \quad k=2,3,\cdots.$$

Now let us give our main result. Precisely, we obtain the following global existence theorem for the Cauchy problem of nonlinear elastic wave equations with the time-independent external force.

Theorem 1.1 *Let $k\ge 11$. Suppose that coefficients B^{ijh}_{lmn} satisfy the symmetric properties $B^{ijh}_{lmn} = B^{jih}_{mln} = B^{ihj}_{lnm}$ and the null condition* (1.3) *holds. Then the Cauchy problem for system* (1.1) *with external force $h \in F^{k-1}_\Lambda$ and initial data $(u(0,\cdot),\partial_t u(0,\cdot))$ satisfying $\partial u(0,\cdot)\in H^{k-1}_\Lambda$, admits a unique solution $u\in \dot{H}^k_\Gamma(T)$ for every positive constant T, provided that ϵ is sufficiently small and*

$$E_k^{\frac{1}{2}}(u(0)) + \sum_{|a|\le k-1} \||x|\Lambda^a h\|_{L^2} \le \epsilon. \tag{1.5}$$

The rest of this paper is organized as follows. In Section 2, we present some Sobolev-type inequalities and some estimates for the null condition. Then we prove the existence and uniqueness of the stationary solution in Section 3, and the weighted $L^2(\mathbb{R}^3)$-estimates of the original solution minus the stationary solution are bounded by the corresponding generalized energy estimates in Section 4, respectively. In Section 5, we complete the proof of the global existence theorem by the generalized energy method and the continuity argument.

2 Preliminaries

First, we introduce some Sobolev-type inequalities that will be used in the generalized energy estimates.

Lemma 2.1 *For any smooth vector function $v(t,\cdot)$ with sufficient decay in the infinity, if the norms on the right-hand side are bounded, the following inequalities hold:*

$$\langle r\rangle^{\frac{1}{2}}|\Gamma^a v(t,x)| \le C E^{\frac{1}{2}}_{|a|+2}(v(t)), \tag{2.1}$$

$$\langle r\rangle|\partial\Gamma^a v(t,x)| \le C E^{\frac{1}{2}}_{|a|+3}(v(t)), \tag{2.2}$$

$$\langle r\rangle\langle c_\alpha t - r\rangle^{\frac{1}{2}}|P_\alpha\partial\Gamma^a v(t,x)| \le C[E^{\frac{1}{2}}_{|a|+3}(v(t)) + M_{|a|+3}(v(t))], \quad \alpha=1,2, \tag{2.3}$$

$$\langle r\rangle\langle c_\alpha t - r\rangle|P_\alpha\partial\nabla\Gamma^a v(t,x)| \le C M_{|a|+4}(v(t)), \quad \alpha=1,2, \tag{2.4}$$

$$\begin{aligned}\langle r\rangle^{\frac{1}{2}}\tau(t,r)|\nabla v(t,x)| \le{}& C\sum_{|a|\ge 1}\sum_{|a|+|b|\le 3}\|\nabla^a\widetilde{\Omega}^b v\|_{L^2}\\ &+ C\sum_{|a|\ge 2}\sum_{|a|+|b|\le 3}\|\tau(t,\cdot)\nabla^a\widetilde{\Omega}^b v\|_{L^2},\end{aligned} \tag{2.5}$$

where

$$\tau(t,r) = \frac{\langle c_1 t - r\rangle\langle c_2 t - r\rangle}{\langle c_1 t - r\rangle + \langle c_2 t - r\rangle}.$$

Proof For the proofs of inequalities (2.1)–(2.5), see [6, Proposition 3.3]. And the proof of inequality (2.5) is similar to that of inequality (4.2) in [2, Lemma 4.1] with $\tau(t,r)$ in the place of $\langle c_j t - r\rangle$.

Next, we give some estimates on the null condition that will also be used in the course of the generalized energy estimates. Let $\widetilde{N}(u,v,w) = B^{ijh}_{lmn}\partial_l u^i \partial_m v^j \partial_n w^h$ and the set of nonresonant indices $\mathcal{N} = \{(\alpha,\beta,\gamma) \neq (1,1,1),(2,2,2)\}$.

Lemma 2.2 *Suppose that $u,v,w \in H^2_\Lambda$ and the null condition* (1.3) *holds. Then*

$$
\begin{aligned}
|\langle u, N(v,w)\rangle| \le \frac{C}{r}|u| \sum_{|a|\le 1} &[|\nabla\widetilde{\Omega}^a v||\nabla w| + |\nabla\widetilde{\Omega}^a w||\nabla v| \\
&+ |\nabla^2 v||\widetilde{\Omega}^a w| + |\nabla^2 w||\widetilde{\Omega}^a v|] \\
&+ C\sum_{\mathcal{N}} |P_\alpha u|[|P_\beta\nabla^2 v||P_\gamma\nabla w| + |P_\beta\nabla^2 w||P_\gamma \nabla v|].
\end{aligned}
$$

Proof See [6, Proposition 3.2].

Lemma 2.3 *Suppose that $u,v,w \in H^2_\Lambda$ and the null condition* (1.3) *holds. Then*

$$
\begin{aligned}
|\widetilde{N}(u,v,w)| \le \frac{C}{r}\sum_{|a|\le 1} &(|\nabla u||\nabla v||\widetilde{\Omega}^a w| + |\nabla u||\widetilde{\Omega}^a v||\nabla w| + |\widetilde{\Omega}^a u||\nabla v||\nabla w|) \\
&+ C\sum_{\mathcal{N}} |P_\alpha \nabla u||P_\beta\nabla v||P_\gamma\nabla w|.
\end{aligned}
$$

Proof This is a trivial modification of Lemma 2.2.

3 Existence and Uniqueness of Stationary Solution

Let $w(x) = (w^1(x), w^2(x), w^3(x))$ be the stationary solution to system (1.1). Then $w(x)$ satisfies the following coupled system of elliptic equations:

$$
Aw = N(w,w) + h(x), \quad x \in \mathbb{R}^3, \tag{3.1}
$$

where

$$
Aw = -c_2^2\Delta w - (c_1^2 - c_2^2)\nabla(\nabla\cdot w).
$$

To assure the uniqueness of solutions to the nonlinear elliptic system (3.1), we assume for any $0 \le |a| \le k-1$, $\lim\limits_{r\to\infty}\sup\limits_{\theta\in\mathbb{S}^2}|\Lambda^a w(r\theta)| = 0$.

Now let us give the existence and uniqueness theorem of the solution to system (3.1).

Lemma 3.1 *Let $k \ge 6$ and B^{ijh}_{lmn} be constants for $i,j,h,l,m,n = 1,2,3$. Then there exist positive constants M and ϵ, such that system* (3.1) *admits a unique solution $w(x)$ satisfying the uniqueness condition and the bound*

$$
\|\nabla w\|^2_{\Lambda,k-1} \le M^2 \sum_{|a|\le k-1} \||x|\Lambda^a h\|^2_{L^2}, \tag{3.2}
$$

provided that $h \in F^{k-1}_\Lambda$ and $\sum\limits_{|a|\le k-1}\||x|\Lambda^a h\|_{L^2} \le \epsilon$.

Proof We prove this lemma by the standard contraction mapping theorem and generalized energy estimates. For any $\phi=(\phi^1,\phi^2,\phi^3)\in D_M$, where

$$D_M=\Big\{\phi:\|\nabla\phi\|^2_{\Lambda,k-1}\le M^2\sum_{|a|\le k-1}\||x|\Lambda^a h\|^2_{L^2},\\ \lim_{r\to\infty}\sup_{\theta\in\mathbb{S}^2}|\Lambda^a\phi(r\theta)|=0\quad\text{for any }0\le|a|\le k-1\Big\},$$

we define a map Π: $\phi\to w$, where $w=(w^1,w^2,w^3)$ satisfies the following problem:

$$Aw=N(\phi,w)+h(x).\tag{3.3}$$

To assure the uniqueness of the solution to the linear elliptic system (3.3), we assume that $\lim\limits_{r\to\infty}\sup\limits_{\theta\in\mathbb{S}^2}|\Lambda^a w(r\theta)|=0$ for any $0\le|a|\le k-1$. By using this uniqueness condition, we can prove that system (3.3) admits a unique solution $w(x)$ satisfying $\Lambda^a w\in\dot H^1(\mathbb{R}^3,\mathbb{R}^3)$ for any $0\le|a|\le k-1$, provided that $h\in F^{k-1}_\Lambda$. This permits the integration by parts at the infinity.

The operators ∇ and $\widetilde\Omega$ commute with A, and $[r\partial_r,A]=-2A$, which implies

$$-A\Lambda^a w=-\Lambda^a Aw+\sum_{|b|\le|a|-1}C_{ab}\Lambda^b Aw\tag{3.4}$$

for any $1\le|a|\le k-1$ and with C_{ab} being constants.

Taking the $L^2(\mathbb{R}^3)^3$ inner product of equation (3.4) with $\Lambda^a w$, employing equation (3.3) and integrating by parts, we obtain the generalized energy estimates,

$$\begin{aligned}&\sum_{|a|\le k-1}(c_2^2\|\nabla\Lambda^a w\|^2_{L^2}+(c_1^2-c_2^2)\|\nabla\cdot(\Lambda^a w)\|^2_{L^2})\\ \le{}& C\sum_{\substack{|a|\le k-1\\|b|+|c|\le k-1}}\int_{\mathbb{R}^3}|\nabla\Lambda^b\phi|\cdot|\nabla\Lambda^c w|\cdot|\nabla\Lambda^a w|\mathrm{d}x\\ &+C\sum_{|a|\le k-1}\sum_{|b|\le|a|}\int_{\mathbb{R}^3}|\Lambda^b h|\cdot|\Lambda^a w|\mathrm{d}x.\end{aligned}\tag{3.5}$$

To deal with the first term of the right-hand side of inequality (3.5), we separate two cases: either $|b|\le\left[\frac{k}{2}\right]$ or $|c|\le\left[\frac{k}{2}\right]-1$. In the first case,

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla\Lambda^b\phi|\cdot|\nabla\Lambda^c w|\cdot|\nabla\Lambda^a w|\mathrm{d}x&\le\|\nabla\Lambda^b\phi\|_{L^\infty}\|\nabla\Lambda^c w\|_{L^2}\|\nabla\Lambda^a w\|_{L^2}\\ &\le C\|\nabla\phi\|_{\Lambda,|b|+2}\|\nabla w\|_{\Lambda,|c|}\|\nabla w\|_{\Lambda,|a|},\end{aligned}$$

where we use Hölder inequality and Sobolev Embedding Theorem $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$. Otherwise, similarly, we obtain

$$\int_{\mathbb{R}^3}|\nabla\Lambda^b\phi|\cdot|\nabla\Lambda^c w|\cdot|\nabla\Lambda^a w|\mathrm{d}x\le C\|\nabla\phi\|_{\Lambda,|b|}\|\nabla w\|_{\Lambda,|c|+2}\|\nabla w\|_{\Lambda,|a|}.$$

For the second term of the right-hand side of inequality (3.5), by Hölder inequality, Hardy's inequality and Young inequality, for any $\delta>0$, we have

$$\int_{\mathbb{R}^3}|\Lambda^b h|\cdot|\Lambda^a w|\mathrm{d}x\le C\||x|\Lambda^b h\|_{L^2}\|\nabla\Lambda^a w\|_{L^2}\le\delta\|\nabla\Lambda^a w\|^2_{L^2}+C(\delta)\||x|\Lambda^b h\|^2_{L^2}.$$

Let δ be sufficiently small. Recalling $k \geq 6$, we see $\left[\frac{k}{2}\right] + 2 \leq k-1$. By inequality (3.5) and the estimates obtained above, we have

$$\|\nabla w\|_{\Lambda,k-1}^2 \leq C \sum_{|a|\leq k-1} \|\nabla \Lambda^a w\|_{L^2}^2 \leq C\|\nabla \phi\|_{\Lambda,k-1}\|\nabla w\|_{\Lambda,k-1}^2 + \widetilde{C} \sum_{|a|\leq k-1} \||x|\Lambda^a h\|_{L^2}^2, \quad (3.6)$$

where $\widetilde{C}$ depends only on k.

Let $M^2 = 2\widetilde{C}$. By inequality (3.6), there exists a positive constant ϵ_1 depending only on M, such that for all $0 < \epsilon < \epsilon_1$, inequality (3.2) holds, provided that $\sum\limits_{|a|\leq k-1} \||x|\Lambda^a h\|_{L^2} \leq \epsilon$. Hence $w \in D_M$.

For any $\phi_1, \phi_2 \in D_M$, define $w_1 = \Pi\phi_1$, $w_2 = \Pi\phi_2$. Similarly, there exists a positive constant $\epsilon_0 < \epsilon_1$ depending only on M, such that Π is a strict contraction from D_M to D_M for all $0 < \epsilon < \epsilon_0$, provided that $\sum\limits_{|a|\leq k-1} \||x|\Lambda^a h\|_{L^2} \leq \epsilon$. By the standard contraction mapping theorem, there exists a unique fixed point $w \in D_M$, such that $\Pi w = w$. Therefore w solves system (3.1) and satisfies the estimate (3.2).

Next let us give two inequalities which will be used to deal with the terms involving $w(x)$ in the generalized energy estimates.

Lemma 3.2 *Assume that the norm on the right-hand side below is bounded. Then*

$$\sum_{|a|\leq k-2} \|\langle r\rangle \nabla^2 \Lambda^a w\|_{L^2} \leq C\|\nabla w\|_{\Lambda,k-1}.$$

Proof The formula $\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2} \wedge \Omega$ implies that

$$\langle r\rangle |\nabla w| \leq C \sum_{|a|\leq 1} |\Lambda^a w|.$$

By the commutation property between ∇ and Λ, and taking L^2-norm, we have

$$\|\langle r\rangle \nabla^2 w\|_{L^2} \leq C\|\nabla w\|_{\Lambda,1}. \quad (3.7)$$

Apply inequality (3.7) to $\Lambda^a w$, $|a| \leq k-2$. This completes the proof of Lemma 3.2.

Lemma 3.3 *Assume that the norm on the right-hand side below is bounded for any smooth vector function $v(t,\cdot)$ with sufficient decay in the infinity. Then*

$$\langle r\rangle^{\frac{3}{2}} |w(x)| \leq C \sum_{|a|\leq 2} \|\Lambda^a w\|_{L^2}. \quad (3.8)$$

Proof For $r \leq 1$, by the standard Sobolev Embedding Theorem: $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, inequality (3.8) holds. Then it suffices to prove the case $r \geq 1$. By using a standard Sobolev Embedding Theorem on $\mathbb{S}^2$: $H^{1,4}(\mathbb{S}^2) \hookrightarrow L^\infty(\mathbb{S}^2)$, for any $i = 1, 2, 3$, we have

$$r^{\frac{3}{2}} |w^i(x)| \leq r^{\frac{3}{2}} \sup_{\theta\in\mathbb{S}^2} |w^i(r\theta)| \leq C r^{\frac{3}{2}} \sum_{|a|\leq 1} \Big(\int_{\mathbb{S}^2} |\Omega^a w^i(r\theta)|^4 \mathrm{d}\theta\Big)^{\frac{1}{4}}. \quad (3.9)$$

By a straight calculation, we have

$$|\Omega^a w^i(r\theta)|^4 = -4\int_r^\infty (\Omega^a w^i(\rho\theta))^3 \partial_\rho \Omega^a w^i(\rho\theta) \mathrm{d}\rho. \quad (3.10)$$

By inequality (3.9) and inequality (3.10), Hölder inequality, Sobolev Embedding Theorem: $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and the formula $\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2} \wedge \Omega$, we have

$$\begin{aligned} r^{\frac{3}{2}} \sum_{i=1}^{3} |w^i(x)| &\le C \sum_{i=1}^{3} \sum_{|a|\le 1} \Big(\int_{\mathbb{S}^2} \int_r^\infty |\partial_\rho \Omega^a w^i(\rho\theta)||\Omega^a w^i(\rho\theta)|^3 \rho^6 \mathrm{d}\rho \mathrm{d}\theta \Big)^{\frac{1}{4}} \\ &\le C \sum_{i=1}^{3} \sum_{|a|\le 1} \Big(\int_{\mathbb{R}^3} |r\partial_r \Omega^a w^i||r\Omega^a w^i|^3 \mathrm{d}x \Big)^{\frac{1}{4}} \\ &\le C \sum_{|a|\le 2} \|\Lambda^a w\|_{L^2}, \end{aligned}$$

which completes the proof of Lemma 3.3.

4 Weighted L^2-Estimates

Let u be the local solution to system (1.1) and w be the solution to system (3.1). Let $v = u - w$. Hence, $v = (v^1, v^2, v^3)$ satisfies

$$\begin{cases} Lv = N(v,v) + 2N(v,w), & t > 0,\ x \in \mathbb{R}^3, \\ v(0,x) = u(0,x) - w(x), & \partial_t v(0,x) = \partial_t u(0,x), \quad x \in \mathbb{R}^3. \end{cases} \tag{4.1}$$

Then it suffices to prove the global existence for the solution to system (4.1). In this section, we will prove that the weighted L^2-norms $M_k(v(t))$ are controlled by the generalized energy norms $E_k^{\frac{1}{2}}(v(t))$ for the small solution to system (4.1). To this end, we need the following lemma.

Lemma 4.1 *Let $v \in \dot{H}^2_\Gamma(T)$. Then*

$$M_2(v(t)) \le C[E_2^{\frac{1}{2}}(v(t)) + t\|Lv(t)\|_{L^2}]. \tag{4.2}$$

Proof See [6, Lemma 3.4].

Next, we estimate the second term on the right-hand side of inequality (4.2).

Lemma 4.2 *Let $\mu \ge 2$ and $\mu' = \left[\frac{\mu-1}{2}\right] + 3$. Suppose that $v \in \dot{H}^\mu_\Gamma(T)$ is a solution to system* (4.1). *Then*

$$\begin{aligned} M_\mu(v(t)) \le{}& CE_\mu^{\frac{1}{2}}(v(t)) + CM_{\mu'}(v(t))(E_\mu^{\frac{1}{2}}(v(t)) + \|\nabla w\|_{\Lambda,\mu-1}) \\ &+ CM_\mu(v(t))(E_{\mu'}^{\frac{1}{2}}(v(t)) + \|\nabla w\|_{\Lambda,\mu'-1}) \\ &+ CE_\mu^{\frac{1}{2}}(v(t))\|\nabla w\|_{\Lambda,\mu'-1} + CE_{\mu'}^{\frac{1}{2}}(v(t))\|\nabla w\|_{\Lambda,\mu-1}. \end{aligned}$$

Proof To prove this lemma, it suffices to estimate $t\|L\Gamma^a v\|_{L^2}$ for $|a| \le \mu - 2$. By the commutation property $L\Gamma^a v = \sum\limits_{b+c=a} (N(\Gamma^b v, \Gamma^c v) + 2N(\Gamma^b v, \Lambda^c w))$, where $b + c = a$ means that $b_i + c_i = a_i$, $\forall i = 0, \cdots, 7$, we only need to estimate terms of the form $t\|\nabla^2\Gamma^b v \nabla\Gamma^c v\|_{L^2}$, $t\|\nabla^2\Gamma^b v\nabla\Lambda^c w\|_{L^2}$ and $t\|\nabla^2\Lambda^b w\nabla\Gamma^c v\|_{L^2}$ for $|b|+|c| \le |a|$. We separate it into two cases: either $|b| \le m-1$ or $|c| \le m$, where $m = \left[\frac{\mu-1}{2}\right]$.

When $|b| \le m - 1$, we have

$$t\|\nabla^2\Gamma^b v\nabla\Gamma^c v\|_{L^2} \le C\|r\nabla^2\Gamma^b v\nabla\Gamma^c v\|_{L^2} + C\sum_\alpha \|\langle c_\alpha t - r\rangle(P_\alpha\nabla^2\Gamma^b v)\nabla\Gamma^c v\|_{L^2}$$

$$\leq CE_{\mu'}^{\frac{1}{2}}(v(t))E_{\mu}^{\frac{1}{2}}(v(t)) + CM_{\mu'}(v(t))E_{\mu}^{\frac{1}{2}}(v(t)).$$

Similarly, when $|c| \leq m$, we have

$$t\|\nabla^2\Gamma^b v\nabla\Gamma^c v\|_{L^2} \leq CE_{\mu'}^{\frac{1}{2}}(v(t))E_{\mu}^{\frac{1}{2}}(v(t)) + CM_{\mu}(v(t))E_{\mu'}^{\frac{1}{2}}(v(t)).$$

In a similar fashion, we obtain

$$t\|\nabla^2\Gamma^b v\nabla\Lambda^c w\|_{L^2} \leq \begin{cases} C(M_{\mu'}(v(t)) + E_{\mu'}^{\frac{1}{2}}(v(t)))\|\nabla w\|_{\Lambda,\mu-1}, & |b| \leq m-1, \\ CM_{\mu}(v(t))\|\nabla w\|_{\Lambda,\mu'-1}, & |c| \leq m. \end{cases}$$

Now it remains to estimate $t\|\nabla^2\Lambda^b w\nabla\Gamma^c v\|_{L^2}$. For this term, we have

$$t\|\nabla^2\Lambda^b w\nabla\Gamma^c v\|_{L^2} \leq C(\|\langle r\rangle\nabla^2\Lambda^b w\nabla\Gamma^c v\|_{L^2} + \|\tau(t,r)\nabla^2\Lambda^b w\nabla\Gamma^c v\|_{L^2}). \tag{4.3}$$

By inequality (2.2) and inequality (3.8), the first term on the right-hand side of inequality (4.3) is estimated as follows:

$$\begin{aligned} \|\langle r\rangle\nabla^2\Lambda^b w\nabla\Gamma^c v\|_{L^2} &\leq C\begin{cases} \|\langle r\rangle\nabla^2\Lambda^b w\|_{L^\infty}\|\nabla\Gamma^c v\|_{L^2}, & |b| \leq m-1, \\ \|\langle r\rangle\nabla\Gamma^c v\|_{L^\infty}\|\nabla^2\Lambda^b w\|_{L^2}, & |c| \leq m \end{cases} \\ &\leq C\begin{cases} \|\nabla w\|_{\Lambda,\mu'-1}E_{\mu}^{\frac{1}{2}}(v(t)), & |b| \leq m-1, \\ E_{\mu'}^{\frac{1}{2}}(v(t))\|\nabla w\|_{\Lambda,\mu-1}, & |c| \leq m. \end{cases} \end{aligned}$$

For the second term on the right-hand side of inequality (4.3), if $|b| \leq m-1$, by Hardy's inequality and inequality (3.8),

$$\begin{aligned} \|\tau(t,r)\nabla^2\Lambda^b w\nabla\Gamma^c v\|_{L^2} &\leq C\Big\|\frac{\tau(t,r)\nabla\Gamma^c v}{r}\Big\|_{L^2}\|r\nabla^2\Lambda^b w\|_{L^\infty} \\ &\leq C(\|\nabla\Gamma^c v\|_{L^2} + \|\tau(t,r)\nabla^2\Gamma^c v\|_{L^2})\|\nabla w\|_{\Lambda,\mu'-1} \\ &\leq C(E_{\mu}^{\frac{1}{2}}(v(t)) + \sum_{\alpha}\|\tau(t,r)P_\alpha\nabla^2\Gamma^c v\|_{L^2})\|\nabla w\|_{\Lambda,\mu'-1} \\ &\leq C(E_{\mu}^{\frac{1}{2}}(v(t)) + M_{\mu}(v(t)))\|\nabla w\|_{\Lambda,\mu'-1}. \end{aligned}$$

Otherwise, if $|c| \leq m$, by inequality (2.5),

$$\begin{aligned} \|\tau(t,r)\nabla\Gamma^c v\|_{L^\infty} &\leq C\sum_{|a|\geq 1}\sum_{|a|+|b|\leq|c|+3}\|\nabla^a\Gamma^b v\|_{L^2} + C\sum_{|a|\geq 2}\sum_{|a|+|b|\leq|c|+3}\|\tau(t,\cdot)\nabla^a\Gamma^b v\|_{L^2} \\ &\leq C(E_{\mu'}^{\frac{1}{2}}(v(t)) + M_{\mu'}(v(t))), \end{aligned}$$

which implies if $|c| \leq m$,

$$\|\tau(t,r)\nabla^2\Lambda^b w\nabla\Gamma^c v\|_{L^2} \leq C(E_{\mu'}^{\frac{1}{2}}(v(t)) + M_{\mu'}(v(t)))\|\nabla w\|_{\Lambda,\mu-1}.$$

Combining Lemma 4.1 with the above estimates shows that Lemma 4.2 holds.

The following lemma completes the argument of weighted-L^2 norm estimates.

Lemma 4.3 *Let $v \in \dot{H}_\Gamma^k(T)$, $k \geq 11$ be a solution to system* (4.1) *and $\nu = k-3$. If $E_\nu^{\frac{1}{2}}(v(t))$ remains sufficiently small for $0 \leq t < T$ for any T, and $\sum_{|a|\leq k-1}\||x|\Lambda^a h\|_{L^2}$ is sufficiently small, then*

$$M_\nu(v(t)) \leq CE_\nu^{\frac{1}{2}}(v(t)), \tag{4.4}$$

$$M_k(v(t)) \leq CE_k^{\frac{1}{2}}(v(t)). \tag{4.5}$$

Proof Recalling $\nu \geq 8$, we see $\nu' = \left[\frac{\nu-1}{2}\right] + 3 \leq \nu$. Then inequality (4.4) is an immediate consequence of Lemmas 3.1 and 4.2. Recalling $k \geq 11$, we see $k' = \left[\frac{k-1}{2}\right] + 3 \leq k - 3 = \nu$. Then inequality (4.5) is an immediate consequence of Lemmas 3.1, 4.2 and inequality (4.4).

5 Energy Estimates

In this section, we will complete the proof of Theorem 1.2 by the generalized energy estimates and the continuity argument. The techniques of this proof represent an evolution of the ideas in [2, 5–7, 9]. Let $k \geq 11$ and $\nu = k - 3$. Suppose that $v(t,x)$ is a classical local solution to system (4.1). In view of the local existence result, to extend the local solution to be a solution belonging to $\dot{H}^k_\Gamma(T)$ for any $T > 0$, it suffices to give a bound to $\sup\limits_{0\leq t<T} E_\nu^{\frac{1}{2}}(v(t))$. We will prove this by the continuity argument. By Lemma 3.1, for the initial data $(u(0,x)-w(x), \partial_t u(0,x))$, if $E_k^{\frac{1}{2}}(u(0)) + \sum\limits_{|a|\leq k-1} \||x|\Lambda^a h\|_{L^2} \leq \epsilon$ for a sufficiently small ϵ, then there exists a positive constant $\widetilde{C}$, such that $E_k^{\frac{1}{2}}(v(0)) + \|\nabla w\|_{\Lambda,k-1} \leq \widetilde{C}\epsilon$. By assuming that $\sup\limits_{0\leq t<T} E_\nu^{\frac{1}{2}}(v(t)) \leq 8\widetilde{C}\epsilon$ for any $T > 0$, it suffices to prove that $\sup\limits_{0\leq t<T} E_\nu^{\frac{1}{2}}(v(t)) \leq 4\widetilde{C}\epsilon$ for any $T > 0$.

Following the energy method, we have

$$E'_\mu(v(t)) = \sum_{|a|\leq\mu-1} \int \langle \partial_t \Gamma^a v, L\Gamma^a v\rangle \mathrm{d}x \quad \text{for any } \mu = 3, 4, \cdots, k.$$

A straight calculation shows that

$$\begin{aligned}
\int \langle \partial_t \Gamma^a v, N(\Gamma^a v, v)\rangle \mathrm{d}x &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} B^{ijh}_{lmn} \int \partial_l(\Gamma^a v)^i \partial_m(\Gamma^a v)^j \partial_n v^h \mathrm{d}x \\
&\quad + \frac{1}{2} B^{ijh}_{lmn} \int \partial_l(\Gamma^a v)^i \partial_m(\Gamma^a v)^j \partial_t\partial_n v^h \mathrm{d}x, \\
\int \langle \partial_t \Gamma^a v, N(\Lambda^a w, v)\rangle \mathrm{d}x &= -\frac{\mathrm{d}}{\mathrm{d}t} B^{ijh}_{lmn} \int \partial_l(\Gamma^a v)^i \partial_m(\Lambda^a w)^j \partial_n v^h \mathrm{d}x \\
&\quad + B^{ijh}_{lmn} \int \partial_l(\Gamma^a v)^i \partial_m(\Lambda^a w)^j \partial_t\partial_n v^h \mathrm{d}x, \\
\int \langle \partial_t \Gamma^a v, N(\Gamma^a v, w)\rangle \mathrm{d}x &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} B^{ijh}_{lmn} \int \partial_l(\Gamma^a v)^i \partial_m(\Gamma^a v)^j \partial_n w^h \mathrm{d}x.
\end{aligned}$$

Let

$$\begin{aligned}
\widetilde{E}_\mu(v(t)) = E_\mu(v(t)) &+ \sum_{|a|\leq\mu-1} B^{ijh}_{lmn} \int \partial_l(\Gamma^a v)^i (\partial_m(\Gamma^a v)^j + 2\partial_m(\Lambda^a w)^j) \partial_n v^h \mathrm{d}x \\
&+ \sum_{|a|\leq\mu-1} B^{ijh}_{lmn} \int \partial_l(\Gamma^a v)^i \partial_m(\Gamma^a v)^j \partial_n w^h \mathrm{d}x.
\end{aligned}$$

The perturbation is bounded by

$$(\|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^\infty}) E_\mu(v(t)) + \|\nabla v\|_{L^\infty} \|\nabla w\|_{\Lambda,\mu-1} E_\mu^{\frac{1}{2}}(v(t)).$$

By the Sobolev Embedding Theorem, norms $\|\nabla v\|_{L^\infty}$ and $\|\nabla w\|_{L^\infty}$ are controlled by $E_3^{\frac{1}{2}}(v(t))$ and $\|\nabla w\|_{\Lambda,2}$, respectively. Thus, if ϵ is small enough, for any $\mu \geq 3$,

$$\frac{1}{2} E_\mu(v(t)) \leq \widetilde{E}_\mu(v(t)) \leq 2E_\mu(v(t)).$$

By the commutation property $L\Gamma^a v = \sum_{b+c=a}(N(\Gamma^b v,\Gamma^c v)+2N(\Gamma^b v,\Lambda^c w))$, we have

$$\begin{aligned}\widetilde{E}'_\mu(v(t)) &= \sum_{|a|\le\mu-1} B^{ijh}_{lmn}\int \partial_l(\Gamma^a v)^i(\partial_m(\Gamma^a v)^j+2\partial_m(\Lambda^a w)^j)\partial_t\partial_n v^h \mathrm{d}x\\ &\quad+\sum_{|a|\le\mu-1}\sum_{\substack{b+c=a\\ b,c\ne a}}\int\langle\partial_t\Gamma^a v, N(\Gamma^b v,\Gamma^c v)+2N(\Gamma^b v,\Lambda^c w)\rangle\mathrm{d}x\\ &= \mathrm{V}+\mathrm{VI}. \end{aligned}\tag{5.1}$$

5.1 Higher-order energy estimates

For the first series of estimates, we take $\mu = k$ in equality (5.1).

By inequalities (2.2), (2.4) and Lemma 4.3, we have

$$\begin{aligned}\|\partial\nabla v\|_{L^\infty} &\le C(1+t)^{-1}\Big(\|\langle r\rangle\partial\nabla v\|_{L^\infty}+\sum_\alpha\|\langle c_\alpha t-r\rangle P_\alpha\partial\nabla v\|_{L^\infty}\Big)\\ &\le C(1+t)^{-1}E_\nu^{\frac12}(v(t)),\end{aligned}$$

which implies V$\le C(1+t)^{-1}(\|\nabla w\|_{\Lambda,k-1}+E_\nu^{\frac12}(v(t)))E_k(v(t))$.

By Hölder inequality, IV is estimated as follows:

$$\mathrm{IV}\le C\sum_{\substack{|a|\le k-1\\ |b|+|c|\le|a|\\ |b|,|c|\ne|a|}}\|\partial\Gamma^a v\|_{L^2}(\|\nabla^2\Gamma^b v\nabla\Gamma^c v\|_{L^2}+\|\nabla^2\Gamma^b v\nabla\Lambda^c w\|_{L^2}+\|\nabla\Gamma^b v\nabla^2\Lambda^c w\|_{L^2}).$$

We see simply but crucially $\left[\frac{k}{2}\right]+3\le k-3=\nu$ for $k\ge 11$. In a similar fashion of the weighted L^2-norm estimates in Lemma 4.2, we obtain

$$\mathrm{IV}\le C(1+t)^{-1}(\|\nabla w\|_{\Lambda,k-1}+E_\nu^{\frac12}(v(t)))E_k(v(t)),$$

where we have used Lemma 4.3. Thus we obtain the higher-order energy estimates

$$\widetilde{E}'_k(v(t))\le C(1+t)^{-1}(\|\nabla w\|_{\Lambda,k-1}+E_\nu^{\frac12}(v(t)))\widetilde{E}_k(v(t)).\tag{5.2}$$

5.2 Lower-order energy estimates

Now let us do the lower-order energy estimates, where we need to exploit the null condition (1.3). Let $\mu=\nu$ in equality (5.1). Then

$$\begin{aligned}\widetilde{E}'_\nu(v(t)) &= \sum_{|a|\le\nu-1}\int(\widetilde{N}(\Gamma^a v,\Gamma^a v,\partial_t v)+2\widetilde{N}(\Gamma^a v,\Lambda^a w,\partial_t v))\mathrm{d}x\\ &\quad+\sum_{|a|\le\nu-1}\sum_{\substack{b+c=a\\ b,c\ne a}}\int\langle\partial_t\Gamma^a v,N(\Gamma^b v,\Gamma^c v)+2N(\Gamma^b v,\Lambda^c w)\rangle\mathrm{d}x\\ &=\sum_{|a|\le\nu-1}\int\psi(\widetilde{N}(\Gamma^a v,\Gamma^a v,\partial_t v)+2\widetilde{N}(\Gamma^a v,\Lambda^a w,\partial_t v))\mathrm{d}x\\ &\quad+\sum_{|a|\le\nu-1}\int(1-\psi)(\widetilde{N}(\Gamma^a v,\Gamma^a v,\partial_t v)+2\widetilde{N}(\Gamma^a v,\Lambda^a w,\partial_t v))\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{|a|\le\nu-1\\ b+c=a\\ b,c\ne a}} \int \psi\langle \partial_t\Gamma^a v, N(\Gamma^b v,\Gamma^c v) + 2N(\Gamma^b v,\Lambda^c w)\rangle \mathrm{d}x\\
&+ \sum_{\substack{|a|\le\nu-1\\ b+c=a\\ b,c\ne a}} \int (1-\psi)\langle \partial_t\Gamma^a v, N(\Gamma^b v,\Gamma^c v) + 2N(\Gamma^b v,\Lambda^c w)\rangle \mathrm{d}x\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned} \tag{5.3}
$$

where ψ is a cut-off function defined as follows:

$$
\psi(t,x) = \zeta\Big(\frac{4r}{\langle c_2 t\rangle}\Big) \quad \text{and} \quad \zeta(\theta) = \begin{cases} 1, & \text{if} \quad \theta < 1,\\ 0, & \text{if} \quad \theta > 2.\end{cases}
$$

By Hölder inequality, inequality (2.3) and Lemma 4.3, we estimate term I as follows:

$$
\begin{aligned}
\mathrm{I} &\le C\sum_{|a|\le\nu-1}\int \psi|\nabla\Gamma^a v||\nabla\Gamma^a v||\partial\nabla v|\mathrm{d}x + C\sum_{|a|\le\nu-1}\int \psi|\nabla\Gamma^a v||\nabla\Lambda^a w||\partial\nabla v|\mathrm{d}x\\
&\le C(1+t)^{-\frac32}(\|\nabla w\|_{\Lambda,\nu-1} + E_\nu^{\frac12}(v(t)))\\
&\quad\times \sum_{\substack{|a|\le\nu-1\\ \alpha,\beta}} (\|\langle c_\alpha t - r\rangle P_\alpha \partial\nabla v\|_{L^2}\|\langle c_\beta t - r\rangle^{\frac12} P_\beta\nabla\Gamma^a v\|_{L^\infty})\\
&\le C(1+t)^{-\frac32}(\|\nabla w\|_{\Lambda,\nu-1} + E_\nu^{\frac12}(v(t)))E_\nu^{\frac12}(v(t))E_k^{\frac12}(v(t)).
\end{aligned} \tag{5.4}
$$

Now let us estimate III, provided that $|a|\le\nu-1$, $|b|+|c|=|a|$ and $|b|,|c|\le\nu-2$. We separate it into two cases:

$$
\begin{aligned}
&\int \psi\langle\partial_t\Gamma^a v, N(\Gamma^b v,\Gamma^c v)\rangle\mathrm{d}x\\
&\le C(1+t)^{-\frac32}\|\partial\Gamma^a v\|_{L^2}\sum_{\alpha,\beta}(\|\langle c_\alpha t - r\rangle P_\alpha\nabla^2\Gamma^b v\|_{L^2}\|\langle c_\beta t - r\rangle^{\frac12}P_\beta\nabla\Gamma^c v\|_{L^\infty})\\
&\le C(1+t)^{-\frac32}E_\nu(v(t))E_k^{\frac12}(v(t)),
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
&\int \psi\langle\partial_t\Gamma^a v, N(\Gamma^b v,\Lambda^c w)\rangle\mathrm{d}x\\
&\le C\int\psi|\partial\Gamma^a v||\nabla\Gamma^b v||\nabla^2\Lambda^c w|\mathrm{d}x + C\int\psi|\partial\Gamma^a v||\nabla^2\Gamma^b v||\nabla\Lambda^c w|\mathrm{d}x.
\end{aligned} \tag{5.6}
$$

Then we estimate the right-hand side of inequality (5.6). By Hölder inequality, Hardy inequality, Lemma 3.2, inequalities (2.3)–(2.4) and Lemma 4.3, since the first order partial derivatives of $\psi(t,x)$ are bounded by $C(1+t)^{-1}$, we have

$$
\begin{aligned}
&\int\psi|\partial\Gamma^a v\nabla\Gamma^b v||\nabla^2\Lambda^c w|\mathrm{d}x\\
&\le C\|r\nabla^2\Lambda^c w\|_{L^2}\|\nabla(\psi\partial\Gamma^a v\nabla\Gamma^b v)\|_{L^2}\\
&\le C(1+t)^{-\frac32}\|\nabla w\|_{\Lambda,\nu-1}\Big(\sum_\alpha\|\langle c_\alpha t - r\rangle^{\frac12}P_\alpha\partial\Gamma^a v\|_{L^\infty}\|\nabla\Gamma^b v\|_{L^2}\\
&\quad + \sum_{\alpha,\beta}(\|\langle c_\alpha t - r\rangle^{\frac12}P_\alpha\partial\Gamma^a v\|_{L^\infty}\|\langle c_\beta t - r\rangle P_\beta\nabla^2\Gamma^b v\|_{L^2})
\end{aligned}
$$

$$+\sum_{\alpha}\Big(\|\langle r\rangle\langle c_\alpha t-r\rangle P_\alpha\partial\nabla\Gamma^a v\|_{L^\infty}\Big\|\frac{\tau(t,r)}{r}\nabla\Gamma^b v\Big\|_{L^2}\Big)\Big)$$
$$\le C(1+t)^{-\frac{3}{2}}\|\nabla w\|_{\Lambda,\nu-1}E_\nu^{\frac{1}{2}}(v(t))E_k^{\frac{1}{2}}(v(t)). \tag{5.7}$$

By Hölder inequality, inequality (2.3) and Lemma 4.3, we have

$$\int\psi|\partial\Gamma^a v||\nabla^2\Gamma^b v||\nabla\Lambda^c w|\mathrm{d}x$$
$$\le C(1+t)^{-\frac{3}{2}}\|\nabla\Lambda^c w\|_{L^2}\sum_{\alpha,\beta}(\|\langle c_\alpha t-r\rangle^{\frac{1}{2}}P_\alpha\partial\Gamma^a v\|_{L^\infty}\|\langle c_\beta t-r\rangle P_\beta\nabla^2\Gamma^b v\|_{L^2})$$
$$\le C(1+t)^{-\frac{3}{2}}\|\nabla w\|_{\Lambda,\nu-1}E_\nu^{\frac{1}{2}}(v(t))E_k^{\frac{1}{2}}(v(t)). \tag{5.8}$$

The combination of estimates (5.5)–(5.8) shows that

$$\mathrm{III}\le C(1+t)^{-\frac{3}{2}}(\|\nabla w\|_{\Lambda,\nu-1}+E_\nu^{\frac{1}{2}}(v(t)))E_\nu^{\frac{1}{2}}(v(t))E_k^{\frac{1}{2}}(v(t)). \tag{5.9}$$

Next we estimate II. By Lemma 2.3, we estimate the first term of II as follows:

$$\int(1-\psi)\widetilde{N}(\Gamma^a v,\Gamma^a v,\partial_t v)\mathrm{d}x$$
$$\le C(1+t)^{-1}\sum_{|d|\le 1}\int(1-\psi)(|\nabla\Gamma^a v||\nabla\Gamma^a v||\widetilde{\Omega}^d\partial_t v|+|\nabla\Gamma^a v||\widetilde{\Omega}^d\Gamma^a v||\nabla\partial_t v|)\mathrm{d}x$$
$$+C\sum_{\mathcal{N}}\int(1-\psi)|P_\alpha\nabla\Gamma^a v||P_\beta\nabla\Gamma^a v||P_\gamma\partial\nabla v|\mathrm{d}x$$
$$\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))\sum_{|d|\le 1}(\|r\partial_t\widetilde{\Omega}^d v\|_{L^\infty}+\|r^{\frac{1}{2}}\widetilde{\Omega}^d\Gamma^a v\|_{L^\infty})$$
$$+C\sum_{\mathcal{N}}\int(1-\psi)|P_\alpha\nabla\Gamma^a v||P_\beta\nabla\Gamma^a v||P_\gamma\partial\nabla v|\mathrm{d}x. \tag{5.10}$$

To estimate the nonresonant terms, if $\beta\ne\gamma$, then

$$\int(1-\psi)|P_\alpha\nabla\Gamma^a v||P_\beta\nabla\Gamma^a v||P_\gamma\partial\nabla v|\mathrm{d}x$$
$$\le C(1+t)^{-\frac{3}{2}}\int|P_\alpha\nabla\Gamma^a v||\langle r\rangle\langle c_\beta t-r\rangle^{\frac{1}{2}}P_\beta\nabla\Gamma^a v||\langle c_\gamma t-r\rangle P_\gamma\partial\nabla v|\mathrm{d}x$$
$$\le C(1+t)^{-\frac{3}{2}}\|\nabla\Gamma^a v\|_{L^2}\|\langle r\rangle\langle c_\beta t-r\rangle^{\frac{1}{2}}P_\beta\nabla\Gamma^a v\|_{L^\infty}\|\langle c_\gamma t-r\rangle P_\gamma\partial\nabla v\|_{L^2}$$
$$\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))E_k^{\frac{1}{2}}(v(t)), \tag{5.11}$$

otherwise, if $\alpha\ne\beta$,

$$\int(1-\psi)|P_\alpha\nabla\Gamma^a v||P_\beta\nabla\Gamma^a v||P_\gamma\partial\nabla v|\mathrm{d}x$$
$$\le C(1+t)^{-\frac{3}{2}}\Big(\int|P_\alpha\nabla\Gamma^a v||\langle r\rangle\langle c_\beta t-r\rangle^{\frac{1}{2}}P_\beta\nabla\Gamma^a v||P_\gamma\partial\nabla v|\mathrm{d}x$$
$$+\int|\langle r\rangle\langle c_\alpha t-r\rangle^{\frac{1}{2}}P_\alpha\nabla\Gamma^a v||P_\beta\nabla\Gamma^a v||P_\gamma\partial\nabla v|\mathrm{d}x\Big)$$
$$\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))E_k^{\frac{1}{2}}(v(t)). \tag{5.12}$$

By Lemma 3.3, the second term of II is estimated as follows:

$$\int (1-\psi)\widetilde{N}(\Gamma^a v, \Lambda^a w, \partial_t v)\mathrm{d}x \le C(1+t)^{-\frac{3}{2}}\|\nabla\Gamma^a v\|_{L^2}\|r^{\frac{3}{2}}\nabla\Lambda^a w\|_{L^\infty}\|\partial\nabla v\|_{L^2}$$
$$\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))\|\nabla w\|_{\Lambda,k-1}. \tag{5.13}$$

The combination of estimates (5.10)–(5.13) shows

$$\mathrm{II} \le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))(E_k^{\frac{1}{2}}(v(t)) + \|\nabla w\|_{\Lambda,k-1}). \tag{5.14}$$

Now it remains to estimate IV. Let $|a| \le \nu - 1$, $|b| + |c| = |a|$ and $|b|, |c| \le \nu - 2$. For the first term of IV, we apply Lemma 2.2. This leads to

$$\begin{aligned}
&\int (1-\psi)\langle \partial_t \Gamma^a v, N(\Gamma^b v, \Gamma^c v)\rangle \mathrm{d}x \\
&\le C(1+t)^{-1}\|\partial_t\Gamma^a v\|_{L^2}\sum_{|d|,|e|\le 1}\|(1-\psi)\nabla\Gamma^{b+d}v\Gamma^{c+e}v\|_{L^2} \\
&\quad + C\sum_{\mathcal{N}}\int (1-\psi)|P_\alpha\partial_t\Gamma^a v||P_\beta\nabla^2\Gamma^b v||P_\gamma\nabla\Gamma^c v|\mathrm{d}x \\
&\le C(1+t)^{-\frac{3}{2}}\|\partial_t\Gamma^a v\|_{L^2}\sum_{|d|,|e|\le 1}\|\nabla\Gamma^{b+d}v\|_{L^2}\|r^{\frac{1}{2}}\Gamma^{c+e}v\|_{L^\infty} \\
&\quad + C\sum_{\mathcal{N}}\int (1-\psi)|P_\alpha\partial\Gamma^a v||P_\beta\nabla^2\Gamma^b v||P_\gamma\nabla\Gamma^c v|\mathrm{d}x.
\end{aligned} \tag{5.15}$$

Now we estimate the nonresonant terms. If $\alpha \ne \beta$, then

$$\begin{aligned}
&\int (1-\psi)|P_\alpha\partial\Gamma^a v||P_\beta\nabla^2\Gamma^b v||P_\gamma\nabla\Gamma^c v|\mathrm{d}x \\
&\le C(1+t)^{-\frac{3}{2}}\int |\langle r\rangle\langle c_\alpha t - r\rangle^{\frac{1}{2}}P_\alpha\partial\Gamma^a v||\langle c_\beta t - r\rangle P_\beta\nabla^2\Gamma^b v||P_\gamma\nabla\Gamma^c v|\mathrm{d}x \\
&\le C(1+t)^{-\frac{3}{2}}\|\langle r\rangle\langle c_\alpha t - r\rangle^{\frac{1}{2}}P_\alpha\partial\Gamma^a v\|_{L^\infty}\|\langle c_\beta t - r\rangle P_\beta\nabla^2\Gamma^b v\|_{L^2}\|\nabla\Gamma^c v\|_{L^2} \\
&\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))E_k^{\frac{1}{2}}(v(t)),
\end{aligned} \tag{5.16}$$

otherwise, if $\alpha \ne \gamma$, then

$$\begin{aligned}
&\int (1-\psi)|P_\alpha\partial\Gamma^a v||P_\beta\nabla^2\Gamma^b v||P_\gamma\nabla\Gamma^c v|\mathrm{d}x \\
&\le C(1+t)^{-\frac{3}{2}}\Big(\int |\langle r\rangle\langle c_\alpha t - r\rangle^{\frac{1}{2}}P_\alpha\partial\Gamma^a v||P_\beta\nabla^2\Gamma^b v||P_\gamma\nabla\Gamma^c v|\mathrm{d}x \\
&\quad + \int |P_\alpha\partial\Gamma^a v||P_\beta\nabla^2\Gamma^b v||\langle r\rangle\langle c_\gamma t - r\rangle^{\frac{1}{2}}P_\gamma\nabla\Gamma^c v|\mathrm{d}x\Big) \\
&\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))E_k^{\frac{1}{2}}(v(t)).
\end{aligned} \tag{5.17}$$

The combination of estimates (5.15)–(5.17) shows

$$\int (1-\psi)\langle \partial_t\Gamma^a v, N(\Gamma^b v, \Gamma^c v)\rangle\mathrm{d}x \le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))E_k^{\frac{1}{2}}(v(t)). \tag{5.18}$$

By Hölder inequality and Lemma 3.3, the second term of IV is estimated as follows:

$$\begin{aligned}
&\int(1-\psi)\langle\partial_t\Gamma^a v, N(\Gamma^b v,\Lambda^c w)\rangle \mathrm{d}x\\
&\le C\|\partial\Gamma^a v\|_{L^2}(\|(1-\psi)\nabla^2\Gamma^b v\nabla\Lambda^c w\|_{L^2}+\|(1-\psi)\nabla\Gamma^b v\nabla^2\Lambda^c w\|_{L^2})\\
&\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))(\|r^{\frac{3}{2}}\nabla\Lambda^c w\|_{L^\infty}+\|r^{\frac{3}{2}}\nabla^2\Lambda^c w\|_{L^\infty})\\
&\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))\|\nabla w\|_{\Lambda,k-1}.
\end{aligned}\tag{5.19}$$

By inequality (5.18) and inequality (5.19), we have

$$\mathrm{IV}\le C(1+t)^{-\frac{3}{2}}E_\nu(v(t))(\|\nabla w\|_{\Lambda,k-1}+E_k^{\frac{1}{2}}(v(t))).\tag{5.20}$$

Combining (5.3)–(5.4), (5.9), (5.14), (5.20), we have

$$\widetilde{E}_\nu'(v(t))\le C(1+t)^{-\frac{3}{2}}E_\nu^{\frac{1}{2}}(v(t))E_k^{\frac{1}{2}}(v(t))(\|\nabla w\|_{\Lambda,k-1}+E_k^{\frac{1}{2}}(v(t))),$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_\nu^{\frac{1}{2}}(v(t))\le C(1+t)^{-\frac{3}{2}}\widetilde{E}_k^{\frac{1}{2}}(v(t))(\|\nabla w\|_{\Lambda,k-1}+\widetilde{E}_k^{\frac{1}{2}}(v(t))).\tag{5.21}$$

Now let us complete the proof of Theorem 1.2. By the higher-order energy estimates (5.2), there exists a positive constant $C(\widetilde{C})$ depending only on $\widetilde{C}$, such that

$$\widetilde{E}_k(v(t))\le\widetilde{E}_k(v(0))(1+t)^{C(\widetilde{C})\epsilon}.\tag{5.22}$$

The combination of inequality (5.21) and inequality (5.22) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_\nu^{\frac{1}{2}}(v(t))\le C(1+t)^{-\frac{3}{2}}(\widetilde{E}_k(v(0))(1+t)^{C(\widetilde{C})\epsilon}+\|\nabla w\|_{\Lambda,k-1}^2),$$

which implies

$$\begin{aligned}
\frac{1}{\sqrt{2}}E_\nu^{\frac{1}{2}}(v(t))&\le\widetilde{E}_\nu^{\frac{1}{2}}(v(t))\le\widetilde{E}_\nu^{\frac{1}{2}}(v(0))+C(\widetilde{E}_k(v(0))+\|\nabla w\|_{\Lambda,k-1}^2)\\
&\le\sqrt{2}E_\nu^{\frac{1}{2}}(v(0))+C\widetilde{C}^2\epsilon^2,
\end{aligned}\tag{5.23}$$

provided that ϵ is small enough, such that $C(\widetilde{C})\epsilon\le\frac{1}{4}$.

By inequality (5.23), we have $E_\nu^{\frac{1}{2}}(v(t))\le 2E_\nu^{\frac{1}{2}}(v(0))+2\widetilde{C}\epsilon$, provided that ϵ is sufficiently small. Consequently

$$\sup_{0\le t<T}E_\nu^{\frac{1}{2}}(v(t))\le 4\widetilde{C}\epsilon\quad\text{ for any } T>0.$$

This completes the proof of Theorem 1.2.

References

[1] Agemi, R., Global existence of nonlinear elastic waves, *Invent. Math.*, **142**, 2000, 225–250.

[2] Hidano, K., An elementary proof of global or almost global existence for quasi-linear wave equations, *Tohoku Math. J.* (2), **56**(2), 2004, 271–287.

[3] John, F., Formation of singularities in elastic waves, Lecture Notes in Physics, Vol. 195, Springer-Verlag, New York, 1984, 194–210.

[4] John, F., Almost global existence of elastic waves of finite amplitude arising from small initial disturbances, *Comm. Pure Appl. Math.*, **41**, 1988, 615–666.

[5] Klainerman, S. and Sideris, T. C., On almost global existence for nonrelativistic wave equations in 3D, *Comm. Pure Appl. Math.*, **49**(3), 1996, 307–321.

[6] Sideris, T. C., Nonresonance and global existence of prestressed nonlinear elastic waves, *Ann. of Math.* (2), **151**(2), 2000, 849–874.

[7] Sideris, T. C., The null condition and global existence of nonlinear elastic waves, *Invent. Math.*, **123**, 1996, 323–342.

[8] Tahvildar-Zadeh, A. S., Relativistic and nonrelativistic elastodynamics with small shear strains, *Ann. Inst. H. Poincaré-Phys. Théor.*, **69**, 1998, 275–307.

[9] Zhou, Y. and Xu, W., Almost global existence for quasilinear wave equations with inhomogeneous terms in 3D, *Forum Math.*, **23**, 2011, 1113–1134.